

# REPEATING DECIMALS

## A NUMERISTIC APPROACH

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*C'est de l'Inde que nous vient l'ingénieuse méthode d'exprimer tous les nombres avec dix caractères, en leur donnant à la fois une valeur absolue et une valeur de position, idée fine et importante, qui nous paraît maintenant si simple que nous en sentons à peine le mérite. Mais cette simplicité même et l'extrême facilité qui en résulte pour tous les calculs placent notre système d'Arithmétique au premier rang des inventions utiles, et l'on appréciera la difficulté d'y parvenir, si l'on considère qu'il a échappé au génie d'Archimède et d'Apollonius, deux des plus grands hommes dont l'antiquité s'honore.*

It is from India that the ingenious method comes from expressing all numbers with ten characters, giving them both an absolute value and a position value, a fine and important idea, which now seems so simple to us that we hardly feel the merit. But this very simplicity and the extreme ease which results from it for all calculations place our Arithmetic system at the forefront of useful inventions, and we will appreciate the difficulty of achieving this, if we consider that it has escaped the genius of Archimedes and Apollonius, two of the greatest and most honored men of antiquity.—Pierre-Simon Laplace, [Lp, p. 404–405]



## SUMMARY

This document derives several theorems on terminating and repeating decimals, along with illustrative examples. It is shown that every rational number has a unique decimal representation, except that some rational numbers have two representations, such as  $0.999\dots = 1$ .

It is also shown which rational numbers result in terminating, pure repeating, and mixed repeating decimals. Some results for the lengths of terminating and repeating portions are proved.

Using numeristics, developed in a separate document, an extension to the usual decimal scheme is developed here: infinite decimals on the left side of the decimal point, such as  $\dots999 = -1$ .

Other number bases are briefly considered.

Another extension to the decimal scheme, equipoint analysis, is briefly introduced here. This theory uses multiple levels of sensitivity to extend decimal arithmetic to infinite and infinitesimal numbers, including infinite and infinitesimal integers and rational numbers.

## DEFINITIONS AND NOTATION

A *terminating decimal* is a fraction whose decimal representation has a finite number of digits. Examples include 0.25 and 3.728.

A *repeating decimal* (also called a *recurring* or a *periodic* decimal) has an infinite number of digits, and the digits are periodic after a certain point. Examples include  $0.333\dots = 0.\overline{3}$  and  $0.81090909\dots = 0.81\overline{09}$ .

A terminating decimal can be considered a special case of a repeating decimal, since, for instance,  $0.25 = 0.25000\dots = 0.25\overline{0}$ . However, as we use the term, *repeating decimal* means only a *non-terminating* repeating decimal, so that terminating and repeating decimals are distinct classes.

A *pure repeating decimal* is a repeating decimal in which all the digits are periodic, i.e. the periodicity starts at the decimal point. A *mixed repeating decimal* is any repeating decimal which is not pure, i.e. the digits after the decimal point consist of a nonperiodic portion followed by a periodic portion.

A *non-repeating infinite decimal* has an infinite number of digits to the right of the decimal point, but the digits never become periodic.

A *ten-pure* or *regular* number is a positive integer whose prime factors include only 2 or 5 or both. The first few ten-pure numbers are 2, 4, 5, 8, 10, 16, 20, and 25.

A *ten-free* number is a positive integer whose prime factors are all different from 2 or 5. The first few ten-free numbers are 3, 7, 9, 11, 13, 17, 19, and 21.

A *ten-mixed* number is a positive integer whose prime factors include 2 or 5 or both, and other prime factors. The first few ten-mixed numbers are 6, 12, 14, 15, 18, 22, 24, and 26.

In the following, uppercase letters denote integers, e.g.  $A$ ,  $B$ ,  $N$ , which are positive unless otherwise noted.

Whenever we write a fraction  $\frac{M}{N}$ , we assume  $0 < M < N$  and  $N \geq 2$ , unless otherwise noted.



Subscripted lowercase letters denote decimal digits, e.g.  $d_1$  or  $d_n$ . A string such as  $d_1d_2d_3 \dots d_n$  denotes an integer or decimal represented by those digits. For example, if  $d_1 = 1$ ,  $d_2 = 2$ , and  $d_3 = 5$ , then  $d_1d_2d_3 = 125$ , and  $0.d_1d_2d_3 = 0.125$ .

For any terminating decimal  $0.d_1d_2d_3 \dots d_n$ , the integer  $d_1d_2d_3 \dots d_n$  is called the *terminant*. For example, 125 is the terminant of 0.125.

Similarly, for any repeating decimal  $0.d_1d_2d_3 \dots d_n\overline{e_1e_2e_3 \dots e_m}$ , the integer  $e_1e_2e_3 \dots e_m$  is called the *repetend*. (Articles in MathWorld, e.g. [WRD], use the word “reptend” instead, but this almost certainly originated as a typo, since no standard references published prior to MathWorld used this spelling.)

When we consider fractions  $\frac{M}{N}$ , we usually consider only those that are in lowest terms, i.e. in which  $M$  and  $N$  are relatively prime, also known as *coprime*. We call such fractions *reduced*.

We should note that if  $\frac{M}{N}$  is *not* reduced and the denominator  $N$  is ten-free, then in the reduced form  $\frac{J}{K}$ , the denominator  $K$  will still be ten-free, since any prime factor in  $K$  must also be in  $N$ , and  $N$  contains no ten-pure prime factors. Similarly, if  $\frac{M}{N}$  reduces to  $\frac{J}{K}$  and  $N$  is ten-pure, then  $K$  is ten-pure. However, if  $\frac{M}{N}$  reduces to  $\frac{J}{K}$  and  $N$  is ten-mixed, then  $K$  may be ten-mixed, ten-pure, or ten-free, depending on which prime factors  $M$  and  $N$  have in common.

The *stint* of a fraction  $\frac{M}{N}$  is the number of digits between the decimal point and the end of the non-repeating portion of the decimal representation of  $\frac{M}{N}$ . The stint is the number of digits in the terminant, including leading zeros but not trailing zeros, unless they are followed by a nonzero repetend. For example, the stint of  $\frac{3}{4}$  is 2, since  $\frac{3}{4} = 0.75$  and there are two non-repeating places in the decimal; it is not 3 even though  $\frac{3}{4} = 0.750$ . The stint of  $\frac{1}{12}$  is 2 because  $\frac{1}{12} = 0.08\overline{3}$  and there are two non-repeating digits 08. We denote the stint of  $\frac{1}{N}$  as  $\Lambda(N)$ .

The *period* of a fraction  $\frac{M}{N}$ , is the number of digits in the repeating portion of the decimal representation of  $\frac{M}{N}$ . The period is the number of digits in the repetend, including leading and trailing zeros in the repeating portion. For example, the period of  $\frac{3}{7} = 6$ , since  $\frac{3}{7} = 0.\overline{428571}$ . The period of  $\frac{1}{11}$  is 2 because  $\frac{1}{11} = 0.\overline{09}$ . The period is always derived from the *minimum* number of

decimal places necessary to represent  $\frac{M}{N}$ . For example, the period of  $\frac{1}{3}$  is 1 even though  $\frac{1}{3} = 0.\overline{33}$ , and the period of  $\frac{4}{5}$  is 0 even though  $\frac{4}{5} = 0.8\overline{0}$ . We denote the period of  $\frac{1}{N}$  as  $\lambda(N)$ .

# FUNDAMENTAL THEOREM

**The decimal expansion of any fraction  $\frac{M}{N}$  is either a terminating decimal or a repeating decimal.**

PROOF.  $\frac{M}{N}$  is converted to a decimal expansion by long division of  $M$  by  $N$ . In long division by  $N$ , each remainder is less than  $N$ , and the first remainder is  $M$ . The remainder at any point determines all digits of the quotient that follow. At some point, the remainder will either be 0, in which case the expansion terminates, or a nonzero number which has occurred before, in which case the expansion repeats periodically. In the latter case, the period is the number of steps between the recurring remainders and is less than  $N$ , since only remainders from 1 to  $N - 1$  are possible.  $\square$

We will see below that the converse is also true: any terminating or repeating decimal represents some fraction  $\frac{M}{N}$ .

Figure 1 shows a typical long division yielding a repeating decimal. The final remainder of 1 is the same as the first remainder (the original dividend  $M$ ), so continued division will repeat the digits of the quotient periodically.

$$\begin{array}{r} 0.142857 \\ 7 \overline{) 1.000000} \\ \underline{7} \phantom{000000} \\ 30 \phantom{00000} \\ \underline{28} \phantom{00000} \\ 20 \phantom{00000} \\ \underline{14} \phantom{00000} \\ 60 \phantom{0000} \\ \underline{56} \phantom{0000} \\ 40 \phantom{0000} \\ \underline{35} \phantom{0000} \\ 50 \phantom{0000} \\ \underline{49} \phantom{0000} \\ 1 \phantom{0000} \end{array}$$

FIG. 1:  
Computation of repetend of  $\frac{1}{7}$

Table 2 shows stints and periods for the first unit fractions.

$N$	$\Lambda(N)$	$\lambda(N)$	$\frac{1}{N}$
2	1	0	0.5
3	0	1	$0.\overline{3}$
4	2	0	0.25
5	1	0	0.2
6	1	1	$0.1\overline{6}$
7	0	6	$0.\overline{142857}$
8	3	0	0.125
9	0	1	$0.\overline{1}$
10	1	0	0.1
11	0	2	$0.\overline{09}$
12	1	1	$0.08\overline{3}$
13	0	6	$0.\overline{076923}$
14	1	6	$0.0\overline{714285}$
15	1	1	$0.0\overline{6}$
16	4	0	0.0625
17	0	16	$0.\overline{0588235294117647}$
18	1	1	$0.0\overline{5}$
19	0	18	$0.\overline{052631578947368421}$
20	2	0	0.05

TABLE 2: Stints and periods of

$\frac{1}{N}$  for  $N = 1$  to 20, where

$$\Lambda(N) = \text{stint of } \frac{1}{N},$$

$$\lambda(N) = \text{period of } \frac{1}{N}$$

## TERMINATING DECIMALS

**The decimal representation of a fraction  $\frac{M}{N}$  is terminating if and only if  $N$  is ten-pure.**

PROOF. Let  $\frac{M}{N} = 0.d_1 \dots d_Q$  for some  $Q$  and  $T = d_1 \dots d_Q$ . Then  $\frac{M}{N} = \frac{T}{10^Q}$ , and  $N \mid 10^Q$ . Thus  $N$  is ten-pure. The converse follows by reversing these steps.  $\square$

**The stint of  $\frac{1}{N}$ ,  $\Lambda(N)$ , where  $N$  is ten-pure and  $N = 2^A 5^B$ , is  $\max(A, B)$ .**

PROOF.  $\max(A, B)$  is the minimum  $Q$  such that  $2^A 5^B \mid 10^Q$ .  $\square$

**Given any reduced fraction  $\frac{M}{N}$ , the length of the terminant is independent of  $M$  and is thus equal to  $\Lambda(N)$ , and the terminant of  $\frac{M}{N}$  is  $M$  times the terminant of  $\frac{1}{N}$ .**

PROOF. Let  $P =$  the stint of  $\frac{1}{N}$  and  $T$  its terminant, and let  $Q =$  the stint of  $\frac{M}{N}$  and  $U$  its terminant. Then  $T = \frac{10^P}{N}$  and  $U = \frac{M10^Q}{N}$ .  $P$  is the minimum value for which  $N \mid 10^P$  and likewise  $Q$  is the minimum value for which  $N \mid M10^Q$ . Since  $M$  and  $N$  are coprime, they have no factors in common, so  $N \mid 10^Q$ .  $Q$  must be the minimum value for which this holds, since if we had  $S < Q$  and  $N \mid 10^S$ , then we would have  $N \mid M10^S$  and  $Q$  would not be the minimum. But  $P$  is also the minimum value for which  $N \mid 10^P$ . Hence  $P = Q$  and  $U = MT$ .

**For ten-pure numbers  $H$  and  $N$ ,  $\Lambda(HN) \leq \Lambda(H) + \Lambda(N)$ .**

PROOF. If  $H$  and  $N$  are ten-pure, they are of the form  $H = 2^A 5^B$  and  $N = 2^C 5^D$ . We then have  $\Lambda(HN) = \Lambda(2^{A+C} 5^{B+D}) = \max(A + C, B + D) \leq \max(A, B) + \max(C, D) = \Lambda(H) + \Lambda(N)$ .  $\square$

For ten-pure  $N$ ,  $\Lambda(N^K) = K \Lambda(N)$ .

PROOF. As before,  $N = 2^C 5^D$ . Then  $\Lambda(H^K) = \Lambda(2^{KC} 5^{KD}) = \max(KC, KD) = K \max(C, D) = K \Lambda(N)$ .  $\square$

Table 3 demonstrates this last theorem for  $N = 2$  and  $K$  from 1 to 10. The terminants are powers of 5, since  $\frac{2^N 5^N}{2^N} = 5^N$ .

$K$	$2^K$	$\frac{1}{2^K}$	$\Lambda(2^K)$
1	2	0.5	1
2	4	0.25	2
3	8	0.125	3
4	16	0.0625	4
5	32	0.03125	5
6	64	0.015625	6
7	128	0.0078125	7
8	256	0.00390625	8
9	512	0.001953125	9
10	1024	0.0009765625	10

TABLE 3: Decimals and stints of  $\frac{1}{2^K}$  for  $K$  from 1 to 10

## REPEATING DECIMALS ARE MULTIVALUED

A repeating decimal is a convergent geometric series. A finite geometric series

$$x = \sum_{K=M}^N a^K = a^M + a^{M+1} + a^{M+2} + a^{M+3} + \dots + a^N$$

can easily be summed through a recurrence formula. When we multiply both sides by  $a$ :

$$xa = a^{M+1} + a^{M+2} + a^{M+3} + a^{M+4} + \dots + a^{N+1},$$

we observe that  $xa + a^M = x + a^{N+1}$ . We therefore have  $x - xa = a^M - a^{N+1}$ , which yields

$$x = \frac{a^M - a^{N+1}}{1 - a},$$

the well-known result

$$\sum_{K=m}^N a^K = a^m + a^{m+1} + a^{m+2} + a^{m+3} + \dots + a^N = \frac{a^m - a^{N+1}}{1 - a}.$$

For the case of an infinite geometric series, the numeric theory of infinite series postulates adding a single infinite element  $\infty$  to the real numbers. The resulting number system is called the *projectively extended real numbers* and is described in detail in [CN].  $\infty$  is simultaneously greater and less than all finite real numbers. We also have:

$$\infty + k = \infty, \text{ where } k \text{ is any finite number}$$

$$k\infty = \frac{\infty}{k} = \infty, \text{ where } k \text{ is any nonzero finite number}$$

$$\frac{1}{0} = \infty$$

$$\frac{1}{\infty} = 0$$

$$\infty = -\infty$$

Since  $+\infty = -\infty$ ,  $a^\infty$  has two real values for most  $a$ ,  $a^{-\infty} = 0$  and  $a^{+\infty} = \infty$ . This in turn means that most convergent series have two sums, one finite and one infinite:

$$\sum_{K=M}^{\infty} a^K = a^M + a^{M+1} + a^{M+2} + a^{M+3} + \dots = \left\{ \infty, \frac{a^M}{1 - a} \right\}.$$

For example,

$$\begin{aligned}0.\bar{3} &= 3 \sum_{N=1}^{\infty} 10^{-N} \\ &= 3 \frac{0.1 - 10^{\infty+1}}{1 - 0.1} \\ &= 3 \left\{ \frac{0.1 - \infty}{0.9}, \frac{0.1 - 0}{0.9} \right\} \\ &= 3 \left\{ \infty, \frac{1}{9} \right\} \\ &= \left\{ \infty, \frac{1}{3} \right\}\end{aligned}$$

The infinite value can also be justified by comparison with  $\infty \cdot 0 = \sum_{N=1}^{\infty} 0$ , which may take any finite or infinite value, and yet each term of which is infinitely smaller than the corresponding term of any geometric series with nonzero  $a$ .

Decimals with zero repetends, called *nildecimals*, are an exception to the two-value rule:

$$0.\bar{0} = 0 \cdot 0.\bar{1} = 0 \left\{ \infty, \frac{1}{9} \right\} = \{\mathbb{R}, 0\} = \mathbb{R}.$$

See [Nildecimals and nonidecimals](#) (p. 25).

Since only finite values are usually of interest with repeating decimals, and since almost all of the proofs in this document use calculations which are valid only for finite values, we will usually ignore the infinite value of repeating decimals and use only the finite values. We also usually ignore the indeterminate nature of nildecimals and use only the single value 0.



## PURE REPEATING DECIMALS

**For any pure repeating decimal,  $0.\overline{d_1 \dots d_P} = \frac{R}{10^P - 1}$ , where  $R = d_1 \dots d_P$  is the repetend and  $P$  is the period.**

PROOF. A repeating decimal is a convergent geometric series.  $0.\overline{d_1 \dots d_P}$  is a convergent geometric series whose first term is  $0.d_1 \dots d_P$  and whose term ratio is  $10^{-P}$ . The sum is therefore

$$\frac{0.d_1 \dots d_P}{1 - 10^{-P}} = \frac{\frac{R}{10^P}}{1 - 10^{-P}} = \frac{R}{10^P - 1}. \quad \square$$

**The decimal expansion of a reduced fraction  $\frac{M}{N}$  is a pure repeating decimal if and only if  $N$  is ten-free.**

PROOF. Let  $\frac{M}{N}$  be the reduced form of  $\frac{R}{10^P - 1}$  which is given above. Since  $M$  and  $N$  have no common factors,  $N \mid 10^P - 1$ . Each prime factor of  $N$  also divides  $10^P - 1$ . But 2 and 5 can never divide  $10^P - 1$ , since the last digit of  $10^P - 1$  is always 9.  $N$  is therefore ten-free. These steps are reversible, so the converse easily follows.  $\square$

**Given any reduced fraction  $\frac{M}{N}$ , the length of the repetend is independent of  $M$  and is thus equal to  $\lambda(N)$ , and the repetend of  $\frac{M}{N}$  is  $M$  times the repetend of  $\frac{1}{N}$ .**

PROOF. Let  $P$  be the period of  $\frac{1}{N}$  and  $R$  its repetend, and let  $Q$  be the period of  $\frac{M}{N}$  and  $S$  its repetend. Then  $R = \frac{10^P - 1}{N}$  and  $S = \frac{M(10^Q - 1)}{N}$ .  $P$  is the minimum value for which  $N \mid 10^P - 1$ , and likewise  $Q$  is the minimum value for which  $N \mid M(10^Q - 1)$ . Since  $M$  and  $N$  are coprime, they have no factors in common, so  $N \mid 10^Q - 1$ .  $Q$  must be the minimum value for which this holds, since if we had  $T < Q$  and  $N \mid 10^T - 1$ , then we would have  $N \mid M(10^T - 1)$  and  $Q$  would not be the minimum. But  $P$  is also the minimum value for which  $N \mid 10^P - 1$ . Hence  $P = Q$  and  $S = MR$ .

**The period  $\lambda(N)$  of  $\frac{M}{N}$  never exceeds  $N - 1$ .**

PROOF. All remainders in the long division of  $M$  by  $N$  are in the range 1 to  $N - 1$ . Whenever any of them occurs a second time, a repetend is established. Hence the period is always within this range.  $\square$

We now define a *repnine* of length  $N$  as  $9_1 9_2 \dots 9_N = 10^N - 1$ , a *repunit* of length  $N$  as  $1_1 1_2 \dots 1_N = \frac{10^N - 1}{9}$ , and a *repdigit* of length  $N$  as  $d_1 d_2 \dots d_N = \frac{d}{9}(10^N - 1)$  for any digit  $d$ . We denote a repunit of length  $N$  as  $U(N)$ .

**Every prime  $M$ , except 2, 3, and 5, divides a repunit  $U(P)$  for some  $P < M$ .**

PROOF. Since  $\frac{1}{M} = \frac{R}{9U(P)}$  for some  $P < M$ , where  $R$  is the repetend, we have  $RM = 9U(P)$ . Since  $M \neq 3$ ,  $M \mid U(P)$ .  $\square$

For  $M = 3$ ,  $P = M$ , since  $3 \mid 111$ . Since repdigits are multiples of repunits, the same is true for every repdigit.

**If the repetend of  $\frac{M}{N}$  is  $R$  and  $\lambda(N) = N - 1$ , then the repetend of  $\frac{J}{N}$  is a cyclic permutation of  $R$ .**

PROOF. Since  $0 < M, J < N$  and  $\lambda(N) = N - 1$ , every integer from 1 to  $N - 1$ , including  $M$  and  $J$ , must occur as one of the remainders in the calculation of the decimal expansion of  $\frac{M}{N}$ . The numerator is always the first remainder in any such calculation, so  $J$  will occur as a remainder at some later point in the calculation of the digits of  $\frac{M}{N}$ . When it does, the digit in  $\frac{M}{N}$  at that point will be the first digit of  $\frac{J}{N}$ . Since subsequent digits depend only on the current remainder, the digits of  $\frac{J}{N}$  will repeat those of  $\frac{M}{N}$  but starting at a different point, i.e. the digits of  $\frac{J}{N}$  will be a cyclic permutation of the digits of  $\frac{M}{N}$ .  $\square$

**If  $\lambda(N) = N - 1$ , then  $N$  is prime.**

PROOF. If  $N$  were composite, then  $\frac{M}{N}$  would reduce to  $\frac{I}{K}$  for some  $M$  and  $K < N$ . Since  $\lambda(N)$  can never exceed  $N - 1$ ,  $\lambda(K)$  could not exceed  $K - 1$ , which is less than  $N - 1$ . But by the previous theorem,  $\lambda(K) = \lambda(N) = N - 1$ . Hence  $N$  is prime.  $\square$

The converse of this theorem is not true. Two counterexamples are  $\lambda(3) = 1$  and  $\lambda(11) = 2$ .

If  $\lambda(N) < N - 1$ , then there may be multiple cycles of length  $\lambda(N)$ . This can be expressed more precisely as follows.

**$\lambda(N) \mid \phi(N)$ , where  $\phi(N)$  is the Euler totient function.**

PROOF. The Euler totient function  $\phi(N)$  is the number of integers that are relatively prime to  $N$ . It is thus the number of integers  $M < N$  such that  $\frac{M}{N}$  is reduced. The number of cycles of length  $\lambda(N)$  is  $\frac{\phi(N)}{\lambda(N)}$ .  $\square$

Tables 4 through 7 show the example  $N = 21$ , where  $\phi(N) = 12$  and  $\lambda = 6$ .

$M$	$\frac{M}{21}$	$M$	$\frac{M}{21}$	$M$	$\frac{M}{21}$
1	$0.\overline{047619}$	8	$0.\overline{380952}$	15	$0.\overline{714285}$
2	$0.\overline{095238}$	9	$0.\overline{428571}$	16	$0.\overline{761904}$
3	$0.\overline{142857}$	10	$0.\overline{476190}$	17	$0.\overline{809523}$
4	$0.\overline{190476}$	11	$0.\overline{523809}$	18	$0.\overline{857142}$
5	$0.\overline{238095}$	12	$0.\overline{571428}$	19	$0.\overline{904761}$
6	$0.\overline{285174}$	13	$0.\overline{619048}$	20	$0.\overline{952380}$
7	$0.\overline{333333}$	14	$0.\overline{666666}$	21	$0.\overline{999999}$

TABLE 4: Decimal representations of all  $\frac{M}{21}$

$M$	$\frac{M}{21}$	$M$	$\frac{M}{21}$
1	$0.\overline{047619}$	2	$0.\overline{095238}$
10	$0.\overline{476190}$	20	$0.\overline{952380}$
16	$0.\overline{761904}$	11	$0.\overline{523809}$
13	$0.\overline{619048}$	5	$0.\overline{238095}$
4	$0.\overline{190476}$	8	$0.\overline{380952}$
19	$0.\overline{904761}$	17	$0.\overline{809523}$

TABLE 5: The two cycles for which  $\frac{M}{21}$  is reduced

$M$	$\frac{M}{21}$
3	$0.\overline{142857}$
9	$0.\overline{428571}$
6	$0.\overline{285174}$
18	$0.\overline{857142}$
12	$0.\overline{571428}$
15	$0.\overline{714285}$

TABLE 6: The cycle for which  $\frac{M}{21}$  reduces to  $\frac{M}{7}$

$M$	$\frac{M}{21}$
7	$0.\overline{3}$
14	$0.\overline{6}$
21	$0.\overline{9}$

TABLE 7: The cycle for which  $\frac{M}{21}$  reduces to  $\frac{M}{3}$

**$\lambda$  is not multiplicative for ten-free integers, i.e. for ten-free  $H$  and  $N$ ,  $\lambda(HN)$  is not necessarily equal to  $\lambda(H)\lambda(N)$ , and  $\lambda(N^K)$  is not necessarily equal to  $K\lambda(N)$ .**

PROOF. Tables 8 and 9 show some examples where multiplicativity does not hold, and some where it does.  $\square$

		$\lambda(H) \cdot$							
$H$	$N$	$\frac{1}{H}$	$\frac{1}{N}$	$\lambda(H)$	$\lambda(N)$	$\lambda(N)$	$\lambda(HN)$	$\frac{1}{HN}$	$HN$
3	11	$0.\overline{3}$	$0.\overline{09}$	1	2	2	2	$0.\overline{03}$	33
3	13	$0.\overline{3}$	$0.\overline{076923}$	1	6	6	6	$0.\overline{025641}$	39
7	11	$0.\overline{142857}$	$0.\overline{09}$	6	2	12	6	$0.\overline{012987}$	77
11	13	$0.\overline{09}$	$0.\overline{076923}$	2	6	12	6	$0.\overline{006993}$	143

TABLE 8:  $\lambda(H)\lambda(N)$  compared to  $\lambda(HN)$   
for select  $H$  and  $N$

		$K \cdot$						
$N$	$\frac{1}{N}$	$\lambda(N)$	$K$	$\lambda(N)$	$\lambda(N^K)$	$\frac{1}{N^K}$	$N^K$	
3	$0.\overline{3}$	1	2	2	1	$0.\overline{1}$	9	
3	$0.\overline{3}$	1	3	3	3	$0.\overline{037}$	27	
7	$0.\overline{142857}$	6	2	12	42	$0.\overline{0204081632653061224489795}$ $91836734693877551$	49	
3	$0.\overline{3}$	1	4	4	10	$0.\overline{0123456789}$	81	
11	$0.\overline{09}$	2	2	4	22	$0.\overline{0082644628099173553719}$	121	
13	$0.\overline{076923}$	6	2	12	78	$0.\overline{0059171597633136094674556}$ $2130177514792899408284023}$ $6686390532544378698224852}$ $071$	169	

TABLE 9:  $\lambda(N^K)$  compared to  $K\lambda(N)$   
for select  $N$  and  $K$

**In the decimal expansion of  $\frac{M}{N}$ , if  $N$  is a ten-free prime and  $\lambda(N)$  is even, then the sum of the two halves of the repetend is a repine of the length of the halves, e.g.  $\frac{1}{7} = 0.\overline{142857}$  and  $142 + 857 = 999$ .**

PROOF. Let  $L = \frac{1}{2}\lambda(N)$  be the length of each half, and let  $R$  be the repetend. The decimal expansion is a pure repetend since  $N$  is ten-free.  $\frac{M}{N}$  is reduced for all  $M$  since  $N$  is prime. We then have  $\frac{M}{N} = \frac{R}{10^{2L}-1}$ , and  $N \mid 10^{2L}-1 = (10^L-1)(10^L+1)$ . Since  $N$  is prime, either  $N \mid 10^L-1$  or  $N \mid 10^L+1$ . But if  $N \mid 10^L-1$ , then  $\lambda(N)$  would be  $L$ , whereas we have assumed  $\lambda(N) = 2L$ . So  $N \mid 10^L+1$ .

Compute  $A = \frac{M}{N}(10^L+1)-1$ , which is an integer in the range 0 to  $10^L-1$ , since  $N \mid 10^L+1$  and  $M < N$ .  $A$  then has at most  $L$  digits. If it has fewer than  $L$  digits, left pad it with zeros to  $L$  places.  $A$  then has exactly  $L$  digits in all cases. Denote  $A$  as  $a_1a_2 \dots a_L$ .

Next compute  $B = 10^L-1-A$ , which is also an integer in the range 0 to  $10^L-1$ , and denote it  $b_1b_2 \dots b_L$ .

Finally compute  $a_1a_2 \dots a_Lb_1b_2 \dots b_L = A10^L+B = \left[\frac{M}{N}(10^{2L}+1)-1\right]10^L+10^L-1 - \left[\frac{M}{N}(10^L+1)-1\right] = \frac{M}{N}(10^L+1)10^L - \frac{M}{N}(10^L-1) = \frac{M}{N}(10^L+1)(10^L-1) = \frac{M}{N}(10^{2L}-1) = R$ .

$A$  and  $B$  are thus the two halves of the repetend, and their sum  $A+B = 10^L-1 = 9U(L)$ .  $\square$

## MIXED REPEATING DECIMALS

For any fraction  $\frac{M}{N}$ , we have shown that the decimal expansion is a terminating decimal if and only if  $N$  is ten-pure, and a pure repeating decimal if and only if  $N$  is ten-free. By elimination then, we have established that the decimal expansion is a mixed repeating decimal if and only if  $N$  is ten-mixed. We will show this more positively below.

For a ten-mixed integer  $N$ , we will find it helpful to factor  $N$  into ten-pure and ten-free components. We define the *ten-pure part* of a ten-mixed integer  $N$  as the product of all the ten-pure prime factors of  $N$  (the 2's and 5's in the prime factorization of  $N$ ), and similarly the *ten-free part* of a ten-mixed integer  $N$  as the the product of all the ten-free prime factors of  $N$  (everything but the 2's and 5's in the prime factorization of  $N$ ). For example, the ten-pure part of 12 is 4, and the ten-free part of 12 is 3. Obviously, for any ten-mixed  $N$ , if  $P$  is the ten-pure part and  $F$  is ten-free part, then  $N = PF$ .

**The decimal expansion of a fraction  $\frac{M}{N}$  is a mixed repeating decimal if and only if  $N$  is ten-mixed.**

PROOF. We begin by factoring  $N$  into its ten-pure part  $P$  and ten-free part  $F$ . Since  $P$  and  $F$  have no factors in common, they are coprime.

We can express  $\frac{M}{N}$  as the sum of a pure repeating decimal and a terminating decimal by setting  $\frac{M}{N} = \frac{A}{P} + \frac{B}{F}$  and solving for  $A$  and  $B$ . We then have  $\frac{A}{P} + \frac{B}{F} = \frac{AF+BP}{N}$ , so  $M = AF + BP$ . From elementary number theory it is known that if  $P$  and  $F$  are coprime, then there exist nonzero integers  $C$  and  $D$  such that  $CF + DP = 1$ . Then  $A = MC$ ,  $B = MD$ , and  $\frac{M}{N} = \frac{MC}{P} + \frac{MD}{F}$ .

$\frac{MC}{P}$  can be represented by a terminating decimal, plus an integer if  $|MC| \geq |P|$ .  $\frac{MD}{F}$  can be represented by a pure repeating decimal, plus an integer if  $|MD| \geq |F|$ . The sum is a mixed repeating decimal. Since these steps can easily be reversed, the converse follows.  $\square$

A few examples of this process:

$$\frac{1}{6} = \frac{2}{3} - \frac{1}{2} = 0.666\dots - 0.5 = 0.166\dots = 0.1\bar{6}$$

$$\begin{aligned}\frac{5}{6} &= \frac{1}{3} + \frac{1}{2} = 0.333\dots + 0.5 = 0.833\dots = 0.8\bar{3} \\ \frac{1}{12} &= \frac{1}{3} - \frac{1}{4} = 0.333\dots - 0.25 = 0.083\dots = 0.08\bar{3} \\ \frac{11}{12} &= \frac{2}{3} + \frac{1}{4} = 0.666\dots + 0.25 = 0.916\dots = 0.91\bar{6}\end{aligned}$$

**For any ten-mixed  $N$ ,  $\Lambda(N) = \Lambda(P)$  and  $\lambda(N) = \lambda(F)$ , where  $P$  is the ten-pure part of  $N$  and  $F$  is the ten-free part of  $N$ .**

PROOF. We compute the decimal representation of an arbitrary reduced fraction  $\frac{M}{N}$ . Let  $C = 10^{\Lambda(P)}$ . Since  $\frac{C}{P}$  is the terminant of  $P$ , it is an integer,  $P \mid C$ , and  $\frac{MC}{P}$  is an integer. Since  $\frac{M}{N}$  is reduced,  $M$  and  $N$  have no common factors, and since  $C$  and  $P$  are ten-pure,  $\frac{MC}{P}$  and  $F$  have no common factors. Therefore,  $\frac{MC/P}{F} = \frac{MC}{NF}$  is a reduced fraction. Since  $F$  is ten-free, the decimal representation of  $\frac{MC}{NF}$  is a pure repeating decimal whose period is  $\lambda(F)$ . But since  $\frac{MC}{N} = C \frac{M}{N}$  and  $C$  is the  $\Lambda(P)$  power of 10, the decimal representation of  $\frac{MC}{N}$  is derived from the decimal representation of  $\frac{M}{N}$  by shifting the decimal point left by  $\Lambda(P)$  places. The integer to the left of the decimal point is the terminant of  $\frac{M}{N}$ , and the pure repeating decimal to the right of the decimal point is the repetend of  $\frac{M}{N}$ . Therefore,  $\Lambda(N) = \Lambda(P)$  and  $\lambda(N) = \lambda(F)$ .  $\square$



## NILDECIMALS AND NONIDECIMALS

We define the *nilrepetend* as the repetend 0, and a *nildecimal* as any decimal representation with an explicit nilrepetend. An example of a nildecimal is  $0.5\bar{0}$ .

Nildecimals and decimal representations ending with a finite number of trailing zeros, while they can be considered as distinct, duplicate representations of the decimal without the trailing zeros, are such trivial duplications that we usually ignore trailing zeros and do not consider decimals with them to be distinct.

We define the *nonirepetend* (the Latin prefix *noni-* means 9) as the repetend 9, and a *nonidecimal* as any decimal representation with the nonirepetend. Nonidecimals are special since they are nontrivial duplicate representations, as we shall soon see.

There is one pure nildecimal,  $0.000\dots = 0.\bar{0}$ , and one pure nonidecimal,  $0.999\dots = 0.\bar{9}$ . There are many mixed nildecimals and nonidecimals, e.g. e.g.  $0.35000\dots = 0.35\bar{0}$  and  $0.24999\dots = 0.24\bar{9}$ .

Nildecimals are obviously duplicates of terminating decimals, but the following establishes a similar result for nonidecimals.

$$0.\bar{9} = 1.$$

$$\text{PROOF. } 0.\bar{9} = \frac{0.9}{1-0.1} = 1. \quad \square$$

$$0.0_10_2\dots 0_P\bar{9} = 10^{-P}.$$

$$\text{PROOF. } 0.0_10_2\dots 0_P\bar{9} = \frac{0.0_10_2\dots 0_P9}{1-10^{-P}} = \frac{0.9 \cdot 10^{-P}}{0.9} = 10^{-P}. \quad \square$$

These theorems mean that nonidecimals are nontrivial duplicate decimal representations of nildecimals and terminating decimal representations, e.g.  $3.\bar{9} = 4$  and  $0.34\bar{9} = 0.34 + 0.00\bar{9} = 0.34 + 0.01 = 0.35$ .

The second theorem implies that we can convert a nonidecimal to a terminating decimal by adding 1 to the terminant and deleting the nonirepetend, and reverse it by subtracting 1 from the terminant and adding the nonirepetend.

Nonidecimals may occur through certain operations on decimals, e.g. when we multiply the decimal representation of  $\frac{1}{N}$  by  $N$  for ten-free or ten-mixed  $N$ . Examples:

$$\begin{aligned} 3 \cdot 0.\overline{3} &= 0.\overline{9} = 1 \\ 7 \cdot 0.\overline{142857} &= 0.\overline{999999} = 0.\overline{9} = 1 \\ 12 \cdot 0.08\overline{3} &= 0.96 + 0.03 + 0.00\overline{9} = 0.99\overline{9} = 0.\overline{9} = 1 \end{aligned}$$

**Terminating decimals and corresponding nonidecimals are the only nontrivial duplicate forms among all decimal representations.**

PROOF. A duplicate can occur only if adding or subtracting zero changes the digits in a nontrivial way. We will show the conditions under which this occurs by adding or subtracting zero in the form of  $0.\overline{9} - 1$ . First we look at some examples.

$$\begin{aligned} 0.5 + 0.\overline{9} - 1 &= 1.4\overline{9} - 1 = 0.4\overline{9} \\ 0.4\overline{9} + 1 - 0.\overline{9} &= 1.4\overline{9} - 0.\overline{9} = 0.5 \end{aligned}$$

Next consider  $\frac{3}{11} = 0.\overline{27}$ . We have

$$\begin{aligned} 0.\overline{27} + 0.99 &= 1.26\overline{27} \\ 0.\overline{27} + 0.9999 &= 1.2726\overline{27} \\ 0.\overline{27} + 0.999999 &= 1.272726\overline{27} \\ &\dots \\ 0.\overline{27} + 0.\overline{99} &= 1.\overline{27} \end{aligned}$$

and so

$$\begin{aligned} \frac{3}{11} &= 0.\overline{27} \\ &= 0.\overline{27} + 0.\overline{9} - 1 \\ &= 0.\overline{27} + 0.\overline{99} - 1 \end{aligned}$$

$$\begin{aligned}
&= 1.\overline{27} - 1 \\
&= 0.\overline{27}
\end{aligned}$$

Now consider  $\pi = 3.14159 \dots 502 \dots$

$$\begin{aligned}
3.14159 \dots 502 \dots + 0.9 &= 4.04159 \dots 502 \dots \\
3.14159 \dots 502 \dots + 0.99 &= 4.13159 \dots 502 \dots \\
3.14159 \dots 502 \dots + 0.999 &= 4.14059 \dots 502 \dots \\
3.14159 \dots 502 \dots + 0.9999 &= 4.14149 \dots 502 \dots \\
3.14159 \dots 502 \dots + 0.99999 &= 4.14158 \dots 502 \dots \\
3.14159 \dots 502 \dots + 0.99999 \dots 9 &= 4.14159 \dots 402 \dots \\
3.14159 \dots 502 \dots + 0.99999 \dots 99 &= 4.14159 \dots 492 \dots \\
3.14159 \dots 502 \dots + 0.99999 \dots 999 &= 4.14159 \dots 501 \dots
\end{aligned}$$

so

$$\begin{aligned}
\pi &= 3.14159 \dots 502 \dots \\
&= 3.14159 \dots 502 \dots + 0.\overline{9} - 1 \\
&= 4.14159 \dots 502 \dots - 1 \\
&= 3.14159 \dots 502 \dots
\end{aligned}$$

These examples should make it clear that only when we start with a terminating decimal, i.e. when we can append the nilrepetend and thus have an infinite number of consecutive zeros, will the digits be nontrivially affected by the subtraction of zero.

0 is the only terminating decimal representation which does not have a nonidecimal duplicate, as  $0 + 0.\overline{9} - 1 = 0.\overline{9} - 1 = -0$ . The minus form is the duplicate in this case.  $\square$

## INFINITE LEFT DECIMALS

We now explore extending decimal places infinitely far to the left of the decimal point. We define an *infinite left decimal* as a decimal expansion extending infinitely far to the left. More generally, we use the term *left* along with any of our previous terms to denote a corresponding decimal representation to the left of the decimal point. For example, a *left repeating decimal* has a repetend which extends infinitely far to the left, such as  $\dots 333 = \bar{3}$ . Naturally, we use the term *right* to refer to the decimal representations extending to the right that we considered above. For example, we now call a decimal representation such as  $0.333\dots = 0.\bar{3}$  a *right repeating decimal*.

Just as a right repeating decimal is a convergent geometric series, so a left repeating decimal is a divergent geometric series. Divergent series in general are sometimes controversial, but numeristics also provides a coherent framework for understanding them, which is explored in detail in [CD]. As we did with right repeating decimals, for left repeating decimals, we use the projectively extended real numbers and the identities

$$\begin{aligned}
 +\infty &= -\infty \\
 a^\infty &= \{\infty, 0\} \text{ for } a \neq 0, 1, \infty \\
 \sum_{K=M}^N a^K &= a^M + a^{M+1} + a^{M+2} + a^{M+3} + \dots + a^N = \frac{a^M - a^{N+1}}{1 - a} \\
 \sum_{K=M}^{\infty} a^K &= a^M + a^{M+1} + a^{M+2} + a^{M+3} + \dots = \left\{ \infty, \frac{a^M}{1 - a} \right\}
 \end{aligned}$$

A left repeating decimal therefore has two values, one infinite and one finite. While the infinite value may seem more natural for an infinite left decimal and a finite value for an infinite right decimal, both values are valid for left and right. Following this principle leads to an easy and consistent theory.

However, as we did with infinite right decimals, in the theorems and proofs below for infinite left decimals, we will usually not be interested in the infinite value, and the processes we use will usually be valid only for finite values. Therefore, we will usually ignore the infinite value and work only with the finite values.

$$\dots 999 = -1.$$

PROOF.  $\dots 999 = \overline{9} = 9(1 + 10 + 100 + 1000 + \dots) = 9(10^0 + 10^1 + 10^2 + 10^3 + \dots) = \frac{9}{1-10} = -\frac{9}{9} = -1. \square$

Left nonidecimals can be used as an alternative to right decimal representations:

$$\begin{aligned}\overline{9}8 &= \overline{9} - 1 = -2 \\ \overline{9}27.5 &= \overline{9} - 71.5 = -72.5\end{aligned}$$

Other left repetends produce negative numbers:  $\overline{3} = \dots 333 = 3 \cdot 10^1 + 3 \cdot 10^2 + 3 \cdot 10^3 + \dots = \frac{3}{1-10} = -\frac{1}{3}$ . This is the negative of the corresponding right repeating decimal  $0.\overline{3} = \frac{1}{3}$ . This generalizes as follows.

**A pure repeating decimal consisting of the same repetend  $R$  on both the left and the right is always zero, e.g.  $\overline{3}.\overline{3} = 0$ .**

PROOF. Let  $Q$  be the period. The first term of the left repeating decimal is  $R$ , and the first term of the right repeating decimal is  $\frac{R}{10^Q}$ . The value of the left repeating decimal is  $\frac{R}{1-10^Q}$ , and the value of the right repeating decimal is  $\frac{R/10^Q}{1-10^{-Q}} = \frac{R}{10^Q-1}$ . The value of the total decimal is therefore zero.  $\square$

**Every real number has an infinite number of decimal representations with left repetends.**

PROOF. If we choose *any* repetend  $R$  and use it in a pure repeating decimal on both the left and right, the value is zero. Zero thus has an infinite number of decimal representations with left repetends, for example  $\overline{3}.\overline{3} = \overline{147}.\overline{147} = \overline{09}.\overline{09} = 0$ . If we add the usual decimal representation of a real number  $r$  to any of these forms of zero, then we obtain an alternate decimal representation of  $r$ . Since there are an infinite number of possible repetends, there are an infinite number of alternate representations for  $r$ .  $\square$

Some examples:

$$\begin{aligned}\frac{1}{2} &= 0.5 = 0.5 + \overline{1}.\overline{1} = \overline{1}.\overline{61} \\ &= \overline{3}.\overline{83} = \overline{09}.\overline{590} = \dots\end{aligned}$$

$$\begin{aligned}
\frac{1}{3} &= 0.\overline{3} = 0.\overline{3} + \overline{1.1} = \overline{1.4} \\
&= \overline{3.6} = \overline{09.42} = \overline{142857.476190} = \dots \\
2 &= 2 + \overline{1.1} = \overline{13.1} \\
&= \overline{35.3} = \overline{0911.09} = \dots \\
\pi &= 3.14145265\dots + \overline{1.1} = \overline{14.25270376\dots} = \dots
\end{aligned}$$

The *signum* of a decimal representation is the minus or plus symbol that indicates sign at the front of the representation. We have already seen that infinite left decimals allow us to represent some negative numbers without a signum. We will find that this is true of all negative numbers.

The remaining theorems in this chapter illustrate the similarities and differences between left and right decimals. Each theorem has two parts: a result for right decimals that we have demonstrated in previous chapters, followed by a result for left decimals which we prove here.

**A right terminating decimal without a signum (which uses only a right terminant) can represent any positive fraction of the form  $\frac{M}{N}$ , where  $0 < M < N$ , and  $N$  is ten-pure. This representation is unique.**

**A left terminating decimal without a signum (which uses only a left terminant) can represent any positive integer. This representation is unique.**

PROOF. A left terminating decimal is simply the conventional representation of a positive integer.  $\square$

**A pure right repeating decimal without a signum (which uses only a right repetend) can represent any positive fraction of the form  $\frac{M}{N}$ , where  $0 \leq M \leq N$ , and  $N$  is ten-free. When  $M = 0$ , the representation is the pure right nildecimal  $0.\overline{0}$ , and when  $M = N$ , the representation is the pure right non-idecimal  $0.\overline{9}$ . This representation is unique.**

A pure left repeating decimal without a signum (which uses only a left repetend) can represent any negative fraction  $-\frac{M}{N}$ , where  $0 \leq M \leq N$ , and  $N$  is positive and ten-free. When  $M = 0$ , the representation is the pure left nildecimal  $\bar{0}$ , and when  $M = N$ , the representation is the pure left nonidecimal  $\bar{9}$ . Within this type of representation, the decimal is unique.

PROOF. The representation of the positive fraction  $\frac{M}{N}$  is a pure right repeating decimal. The same repetend on the left gives a representation of its negative.  $\square$

Examples:

$$\begin{aligned} -\frac{1}{3} &= -0.333\dots = -0.\bar{3} \\ &= \dots333 = \bar{3} \\ -\frac{2}{3} &= -0.666\dots = -0.\bar{6} \\ &= \dots666 = \bar{6} \\ -\frac{3}{3} &= -1 = -0.999\dots = -0.\bar{9} \\ &= \dots999 = \bar{9} \\ -\frac{1}{11} &= -0.090909\dots = -0.\overline{09} \\ &= \dots090909 = \overline{09} \\ -\frac{3}{11} &= -0.272727\dots = -0.\overline{27} \\ &= \dots272727 = \overline{27} \end{aligned}$$

A mixed right repeating decimal without a signum (which uses only a right terminant and a right repetend) can represent any positive fraction  $\frac{M}{N}$ , where  $0 < M < N$ , and  $N$  is ten-mixed. This representation is unique.

A mixed left repeating decimal without a signum (which uses only a left repetend and a left terminant) can represent any positive or negative fraction  $\frac{M}{N}$ , where  $N$  is ten-free. This representation is unique.

PROOF. The terminant portion of a mixed left repeating decimal represents a positive integer  $J$ . If we extend the left repetend right to the decimal point, this becomes a pure left repeating decimal, which can represent  $-\frac{L}{N}$ , where  $0 \leq L \leq N$  and  $N$  is ten-free. If we subtract the pure repeating decimal from the original mixed decimal, the repetend cancels, and we are left with an integer  $K$ , which may be positive or negative. Thus the mixed decimal is  $K - \frac{L}{N}$ . Since  $-1 \leq -\frac{L}{N} \leq 0$ , i.e. it is somewhere within the negative unit interval,  $K - \frac{L}{N}$  relocates  $-\frac{L}{N}$  to the interval between two integers  $K - 1$  and  $K$ .

Given any  $\frac{M}{N}$ , we compute the integer  $K$  such that  $K - 1 \leq \frac{M}{N} < K$  and  $L$  such that  $-1 \leq -\frac{L}{N} < 0$ , and add the left decimal representations of  $K$  and  $\frac{L}{N}$  to obtain the left decimal representation of  $\frac{M}{N}$ .

Fractions and left decimals are thus mapped uniquely and completely in both directions.  $\square$

To illustrate this process, we consider examples from the following four cases. The first two occur when  $M$  is a multiple of  $N$  and  $\frac{M}{N}$  is an integer.

- Any positive integer added to the pure left nonidecimal results in a mixed left nildecimal, i.e. a left terminant preceded by the nilrepetend. This is a trivial duplicate of the ordinary decimal representation of that integer. Examples:

$$4 = -1 + 5 = \dots 999 + 5 = \dots 0004 = \overline{04}$$

$$15 = -1 + 16 = \dots 999 + 16 = \dots 0015 = \overline{015}$$

- Any negative integer subtracted from the pure left nonidecimal results in a left terminant preceded by the nonirepetend. Examples:

$$-4 = -1 - 3 = \dots 999 - 3 = \dots 996 = \overline{96}$$

$$-15 = -1 - 14 = \dots 999 - 14 = \dots 985 = \overline{985}$$

- Any integer added to a left repeating decimal other than the left nonidecimal eventually stops carrying and leaves the repetend intact after that point. The result is a left terminant preceded by a left repetend. Examples:

$$\frac{32}{3} = 11 - \frac{1}{3} = \dots 333 + 11 = \dots 344 = \overline{344}$$

$$\frac{122}{11} = 12 - \frac{10}{11} = \dots 909090 + 12 = \dots 909102 = \overline{09102}$$



- Any integer subtracted from a left repeating decimal eventually stops borrowing and leaves the repetend intact after that point. The result is a left terminant preceded by a left repetend. Examples:

$$-\frac{32}{3} = -10 - \frac{2}{3} = \dots 666 - 10 = \dots 656 = \overline{656}$$

$$-\frac{122}{11} = -11 - \frac{1}{11} = \dots 090909 - 11 = \dots 090898 = \overline{09898}$$

**A mixed right repeating decimal plus a left terminating decimal without a signum can represent any positive fraction  $\frac{M}{N}$ , where  $N$  is ten-mixed. This representation is unique.**

**A mixed left repeating decimal plus a right terminating decimal without a signum can represent any positive or negative fraction  $\frac{M}{N}$ , where  $N$  is ten-mixed. This representation is unique.**

PROOF. We begin by factoring  $N$  into its ten-pure part  $P$  and ten-free part  $F$ . Since  $P$  and  $F$  have no factors in common, they are coprime.

We can express  $\frac{M}{N}$  as the sum of a mixed left repeating decimal and a right terminating decimal by setting  $\frac{M}{N} = \frac{A}{P} + \frac{B}{F}$  and solving for  $A$  and  $B$ . We then have  $\frac{A}{P} + \frac{B}{F} = \frac{AF+BP}{N}$ , so  $M = AF + BP$ . As before, we use the theorem that if  $P$  and  $F$  are coprime, then there exist nonzero integers  $C$  and  $D$  such that  $CF + DP = 1$ . Then  $A = MC$ ,  $B = MD$ , and  $\frac{M}{N} = \frac{MC}{P} + \frac{MD}{F}$ .

$\frac{MC}{P}$  can be represented by a right terminating decimal, plus an integer if  $|MC| \geq |P|$ .  $\frac{MD}{F}$  can be represented by a mixed left repeating decimal. The sum is a mixed left repeating decimal plus a right terminating decimal. Since these steps can easily be reversed, the converse follows.  $\square$

Some examples:

$$\begin{aligned} \frac{1}{6} &= \frac{2}{3} - \frac{1}{2} = 0.666\dots - 0.5 = 0.166\dots = 0.1\overline{6} \\ &= -\frac{1}{3} + \frac{1}{2} = \dots 333.5 = \overline{3}.5 \\ -\frac{1}{6} &= -0.1\overline{6} \\ &= -\frac{2}{3} + \frac{1}{2} = \dots 666.5 = \overline{6}.5 \end{aligned}$$

$$\begin{aligned}
\frac{5}{6} &= \frac{1}{3} + \frac{1}{2} = 0.333\dots + 0.5 = 0.833\dots = 0.8\bar{3} \\
&= -\frac{2}{3} + \frac{3}{2} = \dots 666 + 1.5 = \dots 667.5 = \bar{6}7.5 \\
-\frac{5}{6} &= -0.8\bar{3} \\
&= -\frac{1}{3} - \frac{1}{2} = \dots 333 - 0.5 = \dots 332.5 = \bar{3}2.5 \\
\frac{1}{12} &= \frac{1}{3} - \frac{1}{4} = 0.333\dots - 0.25 = 0.083\dots = 0.08\bar{3} \\
&= -\frac{2}{3} + \frac{3}{4} = \dots 666.75 = \bar{6}.75 \\
-\frac{1}{12} &= -0.08\bar{3} \\
&= -\frac{1}{3} + \frac{1}{4} = \dots 333.25 = \bar{3}.25 \\
\frac{11}{12} &= \frac{2}{3} + \frac{1}{4} = 0.666\dots + 0.25 = 0.916\dots = 0.91\bar{6} \\
&= -\frac{1}{3} + \frac{5}{4} = \dots 333 + 1.25 = \dots 334.25 = \bar{3}4.25 \\
-\frac{11}{12} &= -0.91\bar{6} \\
&= -\frac{2}{3} - \frac{1}{4} = \dots 666 - 0.25 = \dots 665.75 = \bar{6}5.75
\end{aligned}$$

To summarize:

**A left terminating decimal plus a right terminating or repeating decimal without a signum can represent any positive rational number, with a terminating decimal being duplicated by a nonidecimal.**

**A left terminating or repeating decimal plus a right terminating decimal without a signum can represent any positive or negative rational number, with no duplication.**

**A left terminating or repeating decimal plus a right terminating or repeating decimal without a signum can represent any positive or negative rational number, with an infinite number of duplicate representations for every real number.**

## OTHER BASES

Entirely analogous results ensue in any other number base, also called a *radix*. What is called a decimal point in base 10 is called a *radix point* in an arbitrary base, and the decimal representation can be called a *radix representation*.

For example, consider base 2, the familiar *binary* system. We can define a *two-pure* number as any power of 2, a *two-mixed* number as an even number that is not a power of 2, and a *two-free* number as an odd number. We also define a *unirepetend* representation as one with a repetend of 1, such as  $0.\bar{1}_2$  or  $0.0\bar{1}_2$ .

Then the radix-two representation of a fraction is terminating or unirepetend if and only if the denominator is two-pure, mixed repeating if and only if the denominator is two-mixed, and pure repeating non-unirepetend if and only if the denominator is two-free.

Examples in this base:

$$\frac{1}{2} = \frac{1}{10_2} = 0.1_2 = 0.0\bar{1}_2$$

$$\frac{1}{5} = \frac{1}{101_2} = 0.\overline{0011}_2$$

$$\lambda_2(5) = 4$$

$$101_2 \cdot 0011_2 = 5 \cdot 3 = 15 = 2^{\lambda_2(5)} - 1$$

$$\bar{1}_2 = -1$$

$$0.\bar{1}_2 = 1$$

# INFINITY AND ZERO

## Decimal representations of infinity

There is one number in our extended number system for which we have not yet developed a right or left decimal, namely  $\infty$ . To do this, we need to return to our decision to use only the finite value of  $a^\infty = a^{-\infty}$ . Now we use the infinite value.  $\infty$  can be thus represented by any right or left repeating decimal, such as:

$$\begin{aligned}\bar{1} &= 10^1 + 10^2 + 10^3 + \dots + 10^\infty \\ &= \frac{10^{\infty+1} - 10^1}{10 - 1} = \frac{10^\infty - 10}{9} = \frac{\infty - 10}{9} = \frac{\infty}{9} = \infty \\ 0.\bar{1} &= 10^{-1} + 10^{-2} + 10^{-3} + \dots + 10^{-\infty} \\ &= \frac{10^{-\infty+1} - 10^{-1}}{10^{-1} - 1} = \frac{10^{-\infty} - 0.1}{0.9} = \frac{\infty - 0.1}{9} = \frac{\infty}{0.9} = \infty\end{aligned}$$

In previous chapters, we used the projectively extended real numbers to obtain finite results for infinite left decimals. We now examine another extended real number system, the *affinely extended* real number. Both of these systems are explored in detail in [CN]. The affinely extended system adds *two* distinct infinite elements,  $+\infty$  and  $-\infty$ . In this system,  $a^{+\infty} = +\infty$  and  $a^{-\infty} = 0$ , so we do not have any choice between finite and infinite values of  $a^{\pm\infty}$ . Infinite left decimals are always equal to  $+\infty$ , and infinite right decimals are always finite, so the sum of an infinite left and an infinite right decimal is always infinite.  $+\infty$  can be represented by any infinite left decimal without a signum, and  $-\infty$  can be represented by any infinite left decimal with a minus signum.

While the affinely extended real number system may initially appear more palatable than the projectively extended system, there are two problems with using the affinely extended system: (1) it leads to a more difficult theory with various conflicting approaches, and (2) the results are not fully consistent with quantum renormalization. Renormalization, which has been repeatedly verified by physical experiment, is a mathematical procedure which uses assumptions similar to those of the projectively extended system. These considerations are examined in more detail in [CD].

## Unfolding infinity and zero

To understand infinity more deeply, we extend the number system again. This time we unfold infinity and zero into multiple infinite and infinitesimal values. To do this we first develop a way of handling multiple values simultaneously.

A numeric *class* is the simultaneous presence of zero or more values. It is potentially multivalued number or other numeric or number-like construction. Classes have a flat structure: Every number is a single valued class; a class containing a single number is identical to the number.

Classes are somewhat similar to sets, but the main difference is that classes are flat, whereas sets introduce extra structure on top of their elements. Since we don't use sets here, we borrow set theoretical notation for class concepts. For example,  $\{+1, -1\}$  is a class containing these two numbers. The flatness property of classes means that  $a = \{a\}$  for any single number  $a$ .

A class containing multiple numbers *distributes* operations on it and statements about it over each element. For example,  $\{+1, -1\} + 5 = \{+1 + 5, -1 + 5\} = \{+6, +4\}$ .

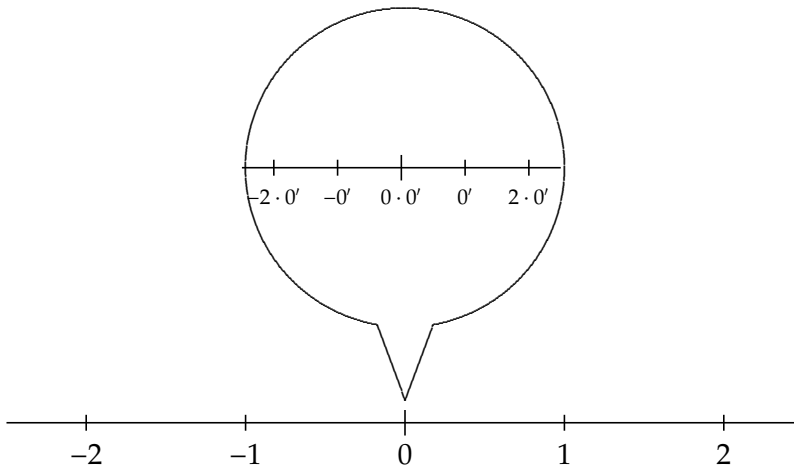


FIG. 10:  
Real number line with  
microscope view of unfolded 0

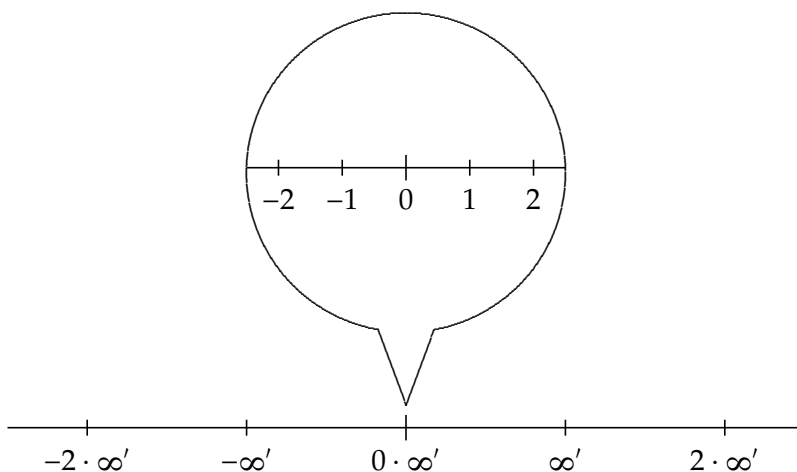


FIG. 11:  
Line of infinities with microscope view of  
real number line within  $0 \cdot \infty'$

We extend the number system into multiple infinite and infinitesimal values using *unfoldings* or *sensitivity levels*. The ordinary real numbers are the *folded* real numbers. Every number becomes a multivalued class when it is unfolded at a greater sensitivity level, and the class of all unfoldings is the *unfolded* real numbers. For 0 and  $\infty$ , we denote representative units within their unfoldings as  $0'$  and  $\infty'$ .

Figure 10 shows the real number line with the unfolding of 0 in an infinitely expanded space, where the single element 0 becomes a multivalued class of unfolded multiples of  $0'$ , the class  $\mathbb{R}0'$ . The bubble in this diagram is called a *microscope*. Figure 11 shows the unfolding of  $\infty$  into a multivalued class consisting of multiples of  $\infty'$ .

Equality is relative to the sensitivity level: if two elements  $a$  and  $b$  are identical at the unfolded level, we denote this as  $a = b$  (say " $a$  unfolded equals  $b$ " or " $a$  equals prime  $b$ "). If they are members of the unfolding of the same real number, we write  $a = b$  (" $a$  equals  $b$ " or " $a$  folded equals  $b$ "). For example:

$$\begin{aligned} 0' &= 2 \cdot 0' && \text{(folded)} \\ 0' &\neq 2 \cdot 0' && \text{(unfolded)} \\ 0'^2 &= 0' && \text{(unfolded)} \end{aligned}$$

There may be multiple unfoldings: an unfolded number  $0'$  may itself be unfolded into a second unfolding. If two expressions are identical at all unfoldings, we write  $a \equiv b$  (" $a$  is equivalent to  $b$ "), e.g.

$$(a + b)^2 \equiv a^2 + 2ab + b^2.$$

The unfolding of  $\infty$  includes both positive and negative multiples of  $\infty'$ , such as  $2 \cdot \infty'$  and  $-3 \cdot \infty'$ . At the folded level, such numbers are equal and both positive and negative, but at the unfolded level, they are distinct and are either positive or negative but not both, e.g.:

$$\begin{aligned} 2 \cdot \infty' &= -3 \cdot \infty' \\ 2 \cdot \infty' &\neq -3 \cdot \infty' \\ 2 \cdot \infty' &>' 0 \\ -3 \cdot \infty' &<' 0. \end{aligned}$$

We add to this the postulate of the projectively extended real numbers that  $+\infty = -\infty$ . Thus we have

$$\begin{aligned} -\infty' &= -\infty = \infty = \infty' \\ 10^{-\infty'} &= 10^{-\infty} = 10^{\infty} = 10^{\infty'} = \{0, \infty\} \\ \log 0' &= \log 0 = \log \infty = \log \infty' = \infty \supset' \{\infty'', -\infty''\} \end{aligned}$$

## Infinite integers and rational numbers

Given any sensitivity unit  $0'$ , the unfolded infinite numbers  $\frac{\mathbb{R}}{0'}$  contain unfolded integers, rationals, and irrationals. Similarly, the unfolded infinitesimal numbers  $\mathbb{R}0'$  contain unfolded rationals and irrationals.

If  $M$  is an infinite integer and  $n$  is a finite integer, then  $\frac{M}{n}$  is an *infinite rational*, and  $\frac{n}{M}$  is an *infinitesimal rational* or *zero rational*.

If  $M$  and  $N$  are both infinite integers, then  $\frac{M}{N}$  may be either rational or irrational, and either infinite, zero, or *perfinite* (neither infinite nor zero).  $\frac{M}{N}$  can be irrational even though it is a ratio of integers, because  $M$  and  $N$  have an infinite number of digits.

A perfinite irrational can be unfolded into a class which includes such ratios. For example,  $\sqrt{2} = 1.414\dots$ , when unfolded, includes elements of the form  $1.414\dots d(\infty')$ , where  $d(n)$  is the  $n$ -th digit:

$$\sqrt{2} = 1.414\dots d(\infty') = \frac{1414\dots d(\infty')}{10^{\infty'}} \equiv \frac{M_1}{N_1}$$

$$\sqrt{2} = 1.414\dots d(\infty' + 1) = \frac{1414\dots d(\infty' + 1)}{10^{\infty'+1}} \equiv \frac{M_2}{N_2}$$

$$\sqrt{2} = 1.414\dots d(2\infty') = \frac{1414\dots d(2\infty')}{10^{2\infty'}} \equiv \frac{M_3}{N_3}$$

A perfinite irrational can also be thought of as having a repetend of infinite length:

$$\sqrt{2} = \frac{M_1}{N_1} = 10 \cdot \overline{0.1414\dots d(\infty')}$$

$$\equiv 1.414\dots d(\infty')1414\dots d(\infty')1414\dots d(\infty')1414\dots$$

Another way of finding such infinite integers is with a continued fraction:

$$\sqrt{2} = 1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \dots}}}$$

The convergents (partial evaluations) of this continued fraction are  $\frac{1}{1}$ ,  $\frac{3}{2}$ ,  $\frac{7}{5}$ ,  $\frac{17}{12}$ ,  $\frac{41}{29}$ ,  $\frac{99}{70}$ ,  $\dots$ ,  $\frac{a_n}{b_n}$ ,  $\dots$ , where  $b_n$  are Pell numbers and  $a_n$  are one-half the companion Pell numbers or Pell-Lucas numbers. These numbers satisfy the relations

$$a_n = 2a_{n-1} + a_{n-2} = a_{n-1} + 2b_{n-1}$$

$$b_n = 2b_{n-1} + b_{n-2} = a_{n-1} + b_{n-1}$$

$$a_n = \frac{(1 + \sqrt{2})^n + (1 - \sqrt{2})^n}{2}$$

$$b_n = \frac{(1 + \sqrt{2})^n - (1 - \sqrt{2})^n}{2\sqrt{2}}$$

So for  $M$  and  $N$  we have

$$M = \frac{(1 + \sqrt{2})^{\infty'} + (1 - \sqrt{2})^{\infty'}}{2}$$

$$N = \frac{(1 + \sqrt{2})^{\infty'} - (1 - \sqrt{2})^{\infty'}}{2\sqrt{2}}$$



# DECIMAL STRUCTURE OF REAL NUMBERS

## Folded real numbers

A folded real number  $r$ , identified with a decimal expansion, has the form

$$r = \sum_{m=-\infty}^{+\infty} d_m 10^m, \text{ where } d_m \in \mathbb{Z}_{10} = 0, 1, \dots, 9.$$

For example, for

$$\sqrt{2} = 1.414\dots$$

we have

$$\begin{aligned} d_n &= 0 \text{ for } n > 0. \\ d_0 &= 1 \\ d_{-1} &= 4 \\ d_{-2} &= 1 \\ d_{-3} &= 4 \\ &\dots \end{aligned}$$

This form allows for **infinite left decimals** (p. 28). It also includes duplicates, as discussed in **Nildecimals and nonidecimals** (p. 25) and **Decimal representations of infinity** (p. 36).

## Unfolded real numbers

In general, an unfolded real number is the sum of one or more of: (1) an infinite number, (2) a perfinite number, and (3) an infinitesimal number.

In the first unfolding, an arbitrary real number  $r$  takes this form:

$$\begin{aligned} r &\equiv r_{+1} \infty' + r_0 + r_{-1} 0' \\ &\equiv \sum_{m=-\infty''}^{+\infty''-1} d_{+1,m} 10^m \infty' + \sum_{m=-\infty''}^{+\infty''-1} d_{0,m} 10^m + \sum_{m=-\infty''}^{+\infty''-1} d_{-1,m} 10^m 0', \end{aligned}$$

where  $\infty' := \frac{1}{\theta}$  and  $r_{+1}, r_0, r_{-1}$  are folded real numbers, and we choose  $\infty''$  so that this becomes a single sequence from smallest infinitesimal to largest infinite:

$$\begin{aligned} 10^{-\infty''} \infty' &\equiv 10^{+\infty''} & \text{and} & & 10^{-\infty''} &\equiv 10^{+\infty''} 0' &\equiv 10^{+\infty''} \infty'^{-1} \\ & & & & \infty' &\equiv 10^{2\infty''} \\ \infty'' &\equiv \frac{1}{2} \log_{10} \infty' &\equiv -\frac{1}{2} \log_{10} 0'. \end{aligned}$$

Then

$$\begin{aligned} r &\equiv \sum_{m=-\infty''}^{+\infty''-1} d_{+1,m} 10^{2\infty''+m} + d_{0,m} 10^m + d_{-1,m} 10^{-2\infty''+m} \\ &\equiv \sum_{m=-3\infty''}^{3\infty''-1} d_m 10^m \end{aligned}$$

$$\text{where } d_m = \begin{cases} d_{+1,m-2\infty''} & \text{for } +\infty'' \leq m \leq +3\infty'' - 1 \\ d_{0,m} & \text{for } -\infty'' \leq m \leq +\infty'' - 1 \\ d_{-1,m+2\infty''} & \text{for } -3\infty'' \leq m \leq -\infty'' - 1 \end{cases}$$

or  $d_m = d_{k,m-2k\infty''}$  for  $(2k-1)\infty'' \leq m \leq (2k+1)\infty'' - 1$  and  $k = +1, 0, -1$ .

In the second unfolding,

$$\begin{aligned} r &\equiv r_{+2} \infty'^2 + r_{+1} \infty' + r_0 + r_{-1} 0' + r_{-2} 0'^2 \\ &\equiv \sum_{m=-\infty''}^{+\infty''-1} d_{+2,m} 10^m \infty'^2 + d_{+1,m} 10^m \infty' + d_{0,m} 10^m + d_{-1,m} 10^m 0' + d_{-2,m} 10^m 0'^2 \\ &\equiv \sum_{m=-\infty''}^{+\infty''-1} d_{+2,m} 10^{4\infty''+m} + d_{+1,m} 10^{2\infty''+m} + d_{0,m} 10^m + d_{-1,m} 10^{-2\infty''+m} + d_{-2,m} 10^{-4\infty''+m} \\ &\equiv \sum_{m=-\infty''}^{+\infty''-1} \sum_{k=-2}^{+2} d_{k,m} 10^{2k\infty''+m} \\ &\equiv \sum_{m=-5\infty''}^{+5\infty''-1} d_m 10^m. \end{aligned}$$

In the  $n$ -th unfolding,

$$\begin{aligned} r &\equiv \sum_{m=-\infty''}^{+\infty''-1} \sum_{k=-n}^{+n} d_{k,m} 10^{2k\infty''+m} \\ &\equiv \sum_{m=-(2n+1)\infty''}^{+(2n+1)\infty''-1} d_m 10^m. \end{aligned}$$

This includes the *ultimate unfolding*, where  $n = \infty$ .

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