An Overview of Numeristics and Equinfinitesimal Analysis

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OVERVIEW

This paper gives an overview of four papers written by the author on ***numeristics*** and ***equinfinitesimal analysis***. Numeristics is a number-based foundational theory of mathematics. Equinfinitesimal analysis is a system of calculus based on numeristic principles.

Each of the four source papers is monograph length. The present paper summarizes key findings. Definitions, proofs, and secondary findings are found in the source papers.

The four source papers are:

- **Numeristics: A number-based foundational theory of mathematics** ([CN]). Basic principles and concepts of numeristics. See the numeristics section below.
- **Equinfinitesimal analysis: A numeristic approach to calculus** ([CE]). A system of calculus and analysis using numeristic principles and an extended number system. See the equinfinitesimal analysis section below.
- **Divergent series: A numeristic approach** ([CD]). An alternative approach to the theory of divergent series using numeristic principles. See the divergent series section below.
- **Repeating decimals: A numerisitic approach** ([CN]). Theorems and proofs about repeating decimals, and an extension of repeating decimals using numeristic principles. See the repeating decimals section below.

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NUMERISTICS

The first of the source papers on numeristics is *Numeristics: A number-based foundational theory of mathematics* [CN]. This paper contains basic principles and concepts of numeristics. Key principles and results include the following.

Numeristics bases mathmatics purely on number. Numeristics starts with three numeric ***ultraprimitives*** or alternate ways of expressing the ultimate value of mathematics:

Infinity Number, as with everything else, ultimately starts from the infinite. The infinite is inexhaustible and therefore only partly conceptualizable.

Unity Unity, the number 1, is the first mathematical manifestation. It expresses the unified nature of infinity.

Zero The number 0 represents the unmanifest quality of pure consciousness. Zero is called the absolute number, because it is the unmanifest state from which all manifestation begins.

In its multiple step phase, numeristics starts with the following ***primitives***, which function somewhat as axioms. These primitives are generally those which existed in "classical mathematics," by which we mean mathematics as it historically existed before the development of set theory, and which is currently taught at the primary, secondary, and upper division undergraduate levels.

- Natural, integral, rational, real, and complex numbers
- Addition, multiplication, exponentiation of these numbers and their inverses
- Usual commutative, associative, and distributive properties of these numbers
- Elementary equality and order relations
- Euclidan geometry
- Ordinary classical logic with quantifiers (first order logic)

Numeristics does not use sets. Instead, a numeristic *class* is a potentially multivalued number or number-like construction. Numeristic classes have a flat structure: Every number is a single valued class; a class containing a single number is identical to the number: $a = \{a\}$. A class

containing multiple numbers distributes operations on it and statements about it over each constituent number: $\{2,3\} + 1 = \{3,4\}$.

The ***empty class*** or ***null class***, denoted $\emptyset$ or $\{\}$, is a class with no values. For example, $1 \cap 2 = \{\}$.

One important consequence of the flat structure of numeristic classes is that the inverse of a function is always itself a function. For instance, the inverse of $f(x) = x^2$ is $f^{-1}(x) = \pm\sqrt{x}$. The inverse of a conventionally indeterminate form such as $\frac{0}{0}$ is the *full class*, the class of all numeric values. An inverse that has no values for a given point is well defined at that point as the empty class.

Numeristics uses several methods of adding infinite elements to the real numbers, the primary being the ***projectively extended*** real numbers (which is known to conventional mathematics). The projectively extended real numbers add one infinite element, ∞, to the real numbers. Thus $\frac{1}{0} = \frac{2}{0} = \infty = -\infty$.

All numeric operations are thus well defined. This includes infinite sums, which are not defined as limits but as sums with an actual infinite number of terms.

EQUINFINITESIMAL ANALYSIS

The second of the source papers on numeristics is *Equinfinitesimal analysis: A numeristic approach to calculus* [CE]. Equinfinitesimal analysis is a system of calculus and analysis using numeristic principles and an extended number system. Key concepts and results include the following.

We define multiple ***levels of sensitivity***. The real numbers are at the most coarse level of sensitivity and are called the ***folded numbers***. Then we ***unfold*** each real number to a finer sensitivity level. Each unfolded number may be unfolded again to an even finer sensitivity level.

When we unfold a number, what is a single value at the coarser level becomes a multivalued class at the finer level. The unfolded members of 0 are a class of ***infinitesimals***. The reciprocals of infinitesimals, which are the unfolded members of ∞, are a class of infinite numbers.

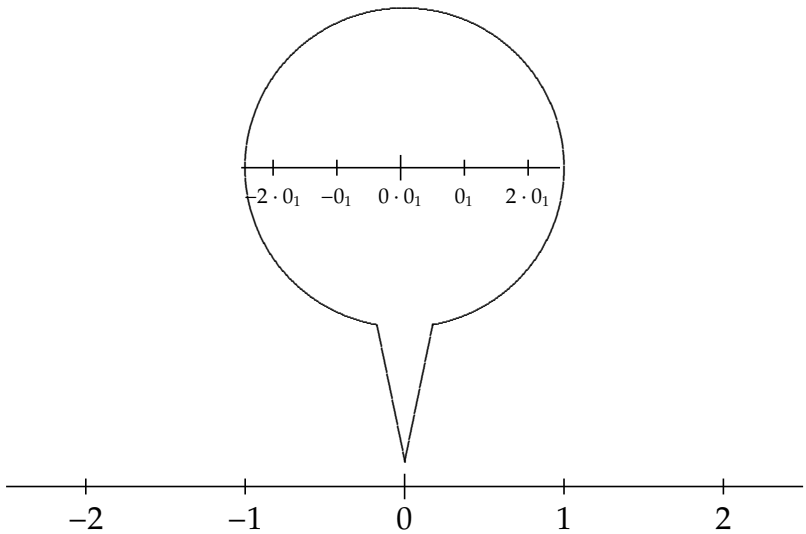

FIG. 1:
Real number line with
microscope view of unfolded 0

Figure 1 is a diagram of the infinitesimals, through an infinitely magnifying ***microscope*** view.

We pick one infinitesimal which is not at the origin of the unfolded class and denote it $0'$, and we similarly pick one unfolded infinite and denote it ∞'.

The extension of unfolded numbers allows direct arithmetic with infinitesimals and infinite numbers. For example, the derivative is defined as an actual ratio of infinitesimals:

$$\frac{df(x)}{dx} := \frac{f(x+0') - f(x)}{0'}.$$

Using this definition, the derivative of x^2 is calculated as follows:

$$\begin{aligned}
f(x) &= x^2 \\
\frac{df(x)}{dx} &\equiv \frac{(x+0')^2 - x^2}{0'} \\
&\equiv \frac{x^2 + 2 \cdot 0'x + 0'^2 - x^2}{0'} \\
&\equiv \frac{2 \cdot 0'x}{0'} \\
&= 2x.
\end{aligned}$$

Similarly, an integral is defined as an actual sum of an infinite number of infinitesimals:

$$\int_a^b f(x)dx := \sum_{k=1}^{\infty'} f\left(a + \frac{k(b-a)}{\infty'}\right) \frac{b-a}{\infty'}.$$

The integral of x^2 is calculated as follows:

$$\begin{aligned}
\int_0^u 2x dx &\equiv \sum_{k=1}^{\infty'} 2\frac{ku}{\infty'}\frac{u}{\infty'} \\
&\equiv \frac{2u^2}{\infty'^2} \sum_{k=1}^{\infty'} k \\
&\equiv \frac{2u^2}{\infty'^2} \infty' \frac{\infty' + 1}{2} \\
&\equiv u^2 \left(1 + \frac{1}{\infty'}\right) \\
&\equiv u^2(1 + 0') \\
&= u^2.
\end{aligned}$$

The direct arithmetic of unfolded numbers allows us to formulate simpler proofs. For example, the proof of the first fundamental theorem of

calculus does not require the mean value theorem:

$$\begin{aligned}
\int_a^b \frac{df(x)}{dx}dx &\equiv \int_a^b \frac{f\left(x-\frac{b-a}{\infty'}\right)-f(x)}{\frac{b-a}{\infty'}}dx \\
&\equiv \frac{b-a}{\infty'}\frac{\infty'}{b-a}\sum_{k=1}^{\infty'} f\left(a+\frac{k(b-a)}{\infty'}+\frac{b-a}{\infty'}\right)-f\left(a+\frac{k(b-a)}{\infty'}\right) \\
&\equiv \sum_{k=1}^{\infty'} f\left(a+(k+1)\frac{b-a}{\infty'}\right)-f\left(a+k\frac{b-a}{\infty'}\right) \\
&\equiv f\left(a+(\infty'+1)\frac{b-a}{\infty'}\right)-f\left(a+\frac{b-a}{\infty'}\right) \\
&\equiv f\left(a+(b-a)+0'\right)-f\left(a+0'\right) \\
&= f(b)-f(a)
\end{aligned}$$

The equinfinitesimal approach allows us to easily discover that the natural logarithm can be expressed as a polynomial of order $0'$:

$$\ln y = \frac{y^{0'}-y^{0'^2}}{0'},$$

which shows that $\int_1^t t^{-1}dt = \ln t$ is an instance of the general law $\int_0^t t^n dt = \frac{t^{n+1}}{n+1}$ and not an exception.

The Dirac delta function $\delta(x)$ can be defined as the derivative of the Heaviside step function. It is a true function in equinfinitesimal analysis, but it is defined only in unfolded arithmetic, i.e. it is not the unfolding of any folded function.

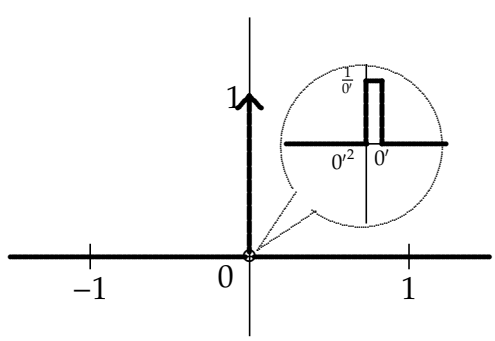

FIG. 2:
Dirac delta function $\delta(x)$

Figure 2 shows the infinite value at $\delta(0)$. The microscope in this figure expands infinitely in the x direction and contracts infinitely in the y direction.

Unrestricted direct arithmetic in the derivative allows us to derive direct formulas for higher derivatives:

$$f^{(n)}(x) = \frac{\sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} f(x + 0'k)}{0'^n}.$$

The indefinite integral can be derived from this formula as a negative order derivative:

$$\int f(x)dx = f^{(-1)}(x) = \sum_{k=1}^{\infty'} f(x - 0'k)\ 0'.$$

Unrestricted direct arithmetic means that the derivative of *any* function is defined, including functions that have no conventional derivative. One example is the Dirac delta function, which as mentioned above can be defined as the derivative of the Heaviside step function. Another example is the Weierstraßfunction, which is a fractal, continuous everywhere but conventionally differentiable nowhere. The Weierstraß-like function

$$W(x) := \sum_{n=0}^{\infty} \frac{\sin(2^n x)}{2^n} = \sin x + \frac{\sin 2x}{2} + \frac{\sin 4x}{4} + \dots$$

has the equinfinitesimal derivative

$$\frac{dW(x)}{dx} = \sum_{n=0}^{\infty} \cos(2^n x).$$

Complex functions which violate the Cauchy-Riemann equations are also equinfinitesimally differentiable. For example, for the function

$$M(x) := 3\,\mathrm{Re}\, z + 2\,\mathrm{Im}\, z$$

the derivative at each point is a class of complex numbers which depends on the unit of differentiation $0'$:

$$\frac{{}_{0'}d}{{}_{0'}dz} M(x) = \frac{3\cos\arg 0' + 2i\sin\arg 0'}{\mathrm{sgn}\, 0'}.$$

Equinfinitesimal analysis may improve the understanding of quantum renormalization. A very simplified example is to evaluate

$$\int_{-\infty}^{\infty} x^2 \, dx,$$

which is conventionally evaluated through limits. In the equinfinitesimal approach, we we avoid limits and directly write

$$\frac{1}{\infty'^3} \int_{-\infty'}^{\infty'} x^2 \, dx = \frac{2}{3}.$$

DIVERGENT SERIES

The third of the source papers on numeristics is *Divergent series: A numeristic approach* [CD]. This paper describes an alternative approach to the theory of divergent series using numeristic principles. Key concepts and results include the following.

The conventional theory of divergent series was pioneered by Euler and given substantial expression by Hardy, whose posthumous work on the subject is considered a classic. Euler and Hardy each start out with the following result for $|a| > 1$:

$$\sum_{k=m}^{\infty} a^k = a^m + a^{m+1} + a^{m+2} + a^{m+3} + \ldots = \frac{a^m}{1-a}.$$

The conventional theory as expressed by Hardy then uses ***methods of summation***, an approach which has several weaknesses:

1. An unworkable conception of weak equality.
2. The failure of all methods, except one expressed by Euler, to account for the above result.
3. A faulty understanding of Euler's method, including a circular definition in Hardy, and failure to realize this method as an extension of arithmetic.

The numeristic theory of divergent series firmly justifies Euler's method and does not use other methods. This solves all of the above problems. The numeristic definition of this method is:

> If a function is continuous and has continous derivatives within a region of the complex plane, and if a power series converges to the function within a second region that is contained within the first region, then we assume that the power series represents the function everywhere within the first region.

In other words, we do not reject an algebraic process simply because it deals with divergent series. If we find an algebraic process that is valid for a convergent case, then we assume that that process is also valid for the corresponding divergent case.

Moreover, using the projectively extended system for real numbers, we find that both convergent and divergent series are multivalued, with most series giving both one infinite and one finite value. For example:

$$\sum_{k=1}^{\infty} 2^{-k} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = \{1, \infty\}$$

$$\sum_{k=0}^{\infty} 2^{k} = 1 + 2 + 4 + 8 + \ldots = \{-1, \infty\}$$

$$\sum_{k=-\infty}^{\infty} 2^{k} = \ldots + \frac{1}{8} + \frac{1}{4} + \frac{1}{2} + 1 + 2 + 4 + 8 + \ldots = \{0, \infty\}$$

Other results include:

$$\sum_{k=m}^{\infty} a^{k} = a^{m} + a^{m+1} + a^{m+2} + a^{m+3} + \ldots = \left\{ \frac{a^{m}}{1-a}, \infty \right\}$$

$$\sum_{k=1}^{\infty} k = 1 + 2 + 3 + 4 + 5 + \ldots = \left\{ -\frac{1}{12}, \infty \right\}$$

$$\int_{0}^{\infty} a^{x} dx = \left\{ \frac{-1}{\ln a}, \infty \right\}$$

REPEATING DECIMALS

The fourth of the source papers on numeristics is *Repeating decimals: A numerisitic approach* [CR]. This paper contains various theorems and proofs about repeating decimals, most of which are conventional, but also with some numeristic extensions. Key concepts and results include the following.

A repeating decimal is a convergent geometric series. Based on the results of [CD], a repeating decimal in the projectively extended real numbers is multivalued:

$$\begin{aligned} 0.\overline{d_1 \ldots d_P} &= \sum_{K=1}^{\infty} d_1 \ldots d_P \cdot 10^{-PK} \\ &= \left\{ \frac{R}{10^P - 1}, \infty \right\}, \end{aligned}$$

where $R = d_1 \ldots d_P$ is the repetend and P is the period.

For example,

$$0.333\ldots = 0.\overline{3} = \sum_{K=1}^{\infty} 3 \cdot 10^{-K} = \left\{ \frac{1}{3}, \infty \right\}$$

$$0.999\ldots = 0.\overline{9} = \{1, \infty\}$$

A ***left repeating decimal*** has an infinite number of repeating digits to the left of the decimal point. This type of decimal is a divergent geometric series, which is also multivalued:

$$\begin{aligned} \overline{d_1 \ldots d_P} &= \sum_{K=0}^{\infty} d_1 \ldots d_P \cdot 10^{PK} \\ &= \left\{ -\frac{R}{10^P - 1}, \infty \right\}, \end{aligned}$$

and

$$\ldots 333 = \overline{3} = \sum_{K=0}^{\infty} 3 \cdot 10^{K} = \left\{ -\frac{1}{3}, \infty \right\}$$

$$\ldots 999 = \overline{9} = \{-1, \infty\}$$

Right and left repeating decimals with the same repetend are negatives:

$$\overline{d_1 \ldots d_P}.\overline{d_1 \ldots d_P} = \sum_{K=-\infty}^{\infty} d_1 \ldots d_P \cdot 10^{-PK}$$
$$= \{0, \infty\}$$
$$\overline{3}.\overline{3} = \overline{9}.\overline{9} = \{0, \infty\}$$

When we allow infinite left decimals, every real number has an infinite number of representations.

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