NUMERISTICS

A Number-Based Foundational Theory of Mathematics

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All things that are known have number; for without this nothing whatever could possibly be thought of or known.—Philolaus, quoted in [Cu11].

The infinite has always stirred the *emotions* of mankind more deeply than any other question; the infinite has stimulated and fertilized reason as few other *ideas* have; but also the infinite, more than any other *notion*, is in need of *clarification*.—David Hilbert, [Hi26, p. 371.] (emphasis in the original)

[T]he more distinctly the logical fabric of analysis is brought to givenness and the more deeply and completely the glance of consciousnesspenetrates it, the clearer it becomes that, given the current approach to foundational matters, every cell (so to speak) of this mighty organism is permeated by the poison of contradiction and that a thorough revision is necessary to remedy the situation.—Hermann Weyl, [W87, p. 32]

Set theoreticians are usually of the opinion that the notion of integer should be defined and that the principle of mathematical induction should be proved. But it is clear that we cannot define or prove ad infinitum; sooner or later we come to something that is not further definable or provable. Our only concern, then, should be that the initial foundations be something immediately clear, natural, and not open to question. This condition is satisfied by the notion of integer and by inductive inferences, but it is decidedly not satisfied by set-theoretic axioms of the type of Zermelo's or anything else of that kind; if we were to accept the reduction of the former notions to the latter, the set-theoretic notions would have to be simpler than mathematical induction, and reasoning with them less open to question, but this runs entirely counter to the actual state of affairs.—Thoralf Skolem, [S22, p. 299]

ऋ॒चो अ॒क्षरे॑ प॒रमे व्यो॑म॒न्
Ṛicho akṣhare parame vyoman
The eternal expressions of knowledge are located in the collapse of infinity to its point, in the transcendental field of pure consciousness.
—Rig Veda 1.164.39, Atharva Veda 9.10.18, Shvetashvatara Upanishad 4.8

SUMMARY

Numeristics is a number-based alternative foundational theory of mathematics. Numeristics is inspired in part by the recent revival of the Vedic tradition of India, as expressed by Maharishi Mahesh Yogi in his Vedic Mathematics and has antecedents in the work of Skolem and Weyl. This book does not assume familiarity with any of this material.

Numeristics aims to establish a foundation for mathematics which: is easier, more elegant, more rigorous, more natural, and more useful; defines all operations; handles the infinite numerically; and is based on an ultimate unity.

The most prevalent foundational theory for the past century, set theory, has numerous shortfalls, some of which are described in **Inadequacies of set theory** (p. 44).

The fundamental structures of numeristics are:

- **Ultraprimitives** (p. 48), a deep level of number which show the essential self-referral property of number.
- **Primitives** (p. 51), properties of numbers which are roughly equivalent to axioms.
- **Numeristic classes** (p. 53), groupings of numbers which are somewhat similar to sets but have a flat structure.
- **Infinite element extensions** (p. 64), infinite numbers which are added to the standard number systems.

Together these structures allow total, unrestricted arithmetic and provide an elegant foundational and computational framework.

HOW TO USE THIS PART

This is not a textbook. This part of the book describes a new foundational theory, numeristics, and shows the differences between it and other foundational theories. This text should therefore be used as a supplement to other mathematical texts.

At a minimum, this text assumes familiarity with secondary school algebra. Some material is aimed at a more advanced level, where set theory is commonly used. Numeristics aims to be an alternative to set theory, and so considerable attention is given here to the differences between it and set theory. Those who are not familiar with set theory can skip these sections.

To understand numeristics, it is essential to understand its unrestricted arithmetic. This is achieved through two mechanisms:

- **Classes** (p. 53), also called ***numeristic classes*** to distinguish them from set theoretic classes; and
- **Infinite elements** (p. 64), which are added to ordinary number systems to form extended number systems.

Numeristics is the basis for an alternative approach to analysis called *equipoint analysis* (p. 125–301), an **alternative theory of divergent series** (p. 301–409), and to an **enriched theory of repeating decimals** (p. 409–457).

WHY NUMERISTICS

Philosophy of numeristics

The numeristic approach to mathematics holds that mathematics has two purposes:

1. Objective: To successfully improve the outer environment through practical applications; and

2. Subjective: To successfully develop the inner environment of the practitioner of mathematics mentally, emotionally, and spritually.

These goals can and should complement each other. Subjective development helps us to solve problems more easily, with fewer mistakes and more balance. Balanced focus on the objective brings benefits to the world at large.

Excessive concern with axioms does not contribute to the fulfillment of either of these goals. Such axiomatics can leave us suspended between the objective and subjective goals without fulfilling either one, and give us a set of conditional statments, without informing us about the context of any of the premises of the axioms.

Axiomatics typically takes smaller logical steps which can bring out mistakes in reasoning. However, the risk is great that it becomes merely a display of intellect without going outside the bounds of intellect, whether into the area of objective applications in the world of the senses, or into the subjective realm of spirituality beyond the intellect.

Therefore numeristics, at least in its early stages, does not use an axiomatic approach. Instead, it uses ***primitives***, as explained below. It also offers a more abstract approach with ***ultraprimitives***, also explained below.

Numeristics aims to develop a theory which fits closely with experience, both objective and subjective. It further aims to integrate these two by expressing the connection between them.

Objective considerations

Mathematics is a science. Like other sciences, its conclusions can be regarded as valid only if they have been empirically validated through objective means. Ancient branches of mathematics have long been validated through physical application, but not all modern branches have been.

What is called mathematical proof is really derivation, a chain of logic connecting axioms and previously proven theorems to a new theorem. It cannot be considered complete proof because it assumes axioms without proof. Since the Renaissance, the presentation of mathematics as a whole has increasingly emphasized formality and neglected objective verfication. This has led to an increasingly prevalent belief among mathematicians that mathematics is a game that derives its authority from social consensus, rather than from objective validation.

Numeristics attempts to improve this situation by using only thoroughly verified principles of number and space as the foundation of mathematics.

Subjective considerations

Numeristics is also based on sound principles of consciousness. We can define consciousness as self referral. Complete self referral is pure consciousness.

As we will examine more closely below, numbers have a simple self referral nature. Numeristics thus takes number as the most basic of mathematical structures.

Handling infinity

An important feature of numeristics is that we can directly handle infinity. Here we look at three examples.

Vanishing point

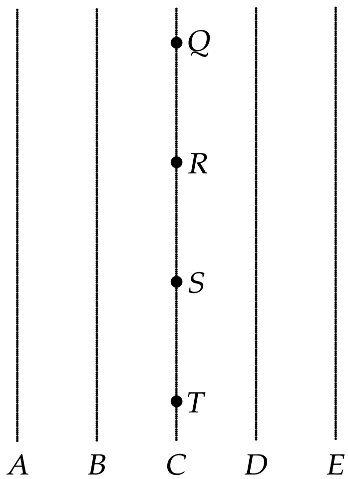

FIG. 7:
Top view of plane L

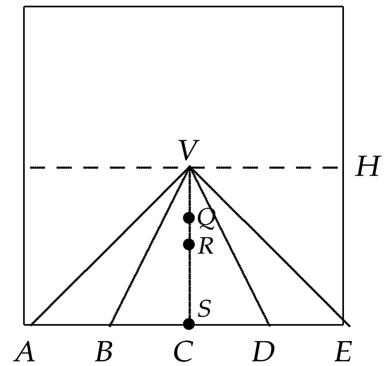

FIG. 8:
Perspective view of plane L
with horizon H and
vanishing point V

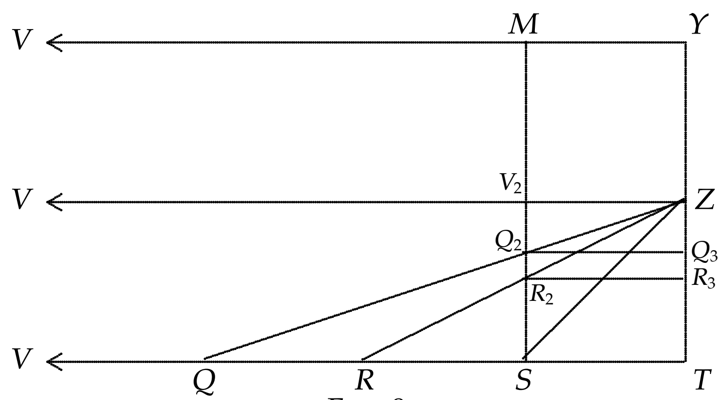

FIG. 9:
Side view of projection of
line C to perspective line MS

The first example shows how infinity comes up in the use of perspective. Figure 7 shows a top view of a plane with five parallel lines, A through

E, and four equally spaced points, Q through T, on line C. Figure 8 shows this plane in perspective through a frame such as one we would see around a drawing, painting, or photograph. The perspective is from the viewpoint of an observer who is standing at point T and whose eye is at point Z. In these figures the lines and points are on an infinite plane, not a curved surface such as the surface of the earth, which is only approximately a plane in the neighborhood of a human observer.

As the observer looks through the frame, the five lines on the plane appear to coverge at a point V infinitely far away, called the ***vanishing point***. Figure 9 shows a side view with line C running along the bottom, the observer at the far right, the line MS running through the frame, and the vanishing point V infinitely far off to the left.

Suppose we are drawing this scene and wish to compute the positions of points R, Q, and V within the frame of Figure 8. This means computing the distances R_2S, Q_2S, and V_2S along the frame line MS. By similar triangles, we have

$$R_2S = RS \cdot \frac{R_2S}{RS} = RS \cdot \frac{ZT}{RT} = RS \cdot \frac{ZR_3}{R_2R_3}$$

$$Q_2S = QS \cdot \frac{Q_2S}{QS} = QS \cdot \frac{ZT}{QT} = QS \cdot \frac{ZQ_3}{Q_2Q_3}$$

By extension, we should be able to compute

$$V_2S = VS \cdot \frac{V_2S}{VS} = VS \cdot \frac{ZT}{VT} = VS \cdot \frac{ZZ}{V_2Z}$$

where ZZ denotes a zero length line segment beginning and ending at point Z. But VS and VT have infinite length, and conventional mathematics does not allow direct calculation with infinite quantities. Instead we have to use another method, in this case observing that $V_2S = ZT$.

Unrestricted calculation

One important feature of numeristics is unrestricted calculation. All elementary functions are ***total functions,*** defined on all elements. This includes division by zero.

Division of a nonzero number by zero results in an infinite number, and division of zero by zero results in the ***full class***, a multivalued construct which is the value of an indeterminate expression. Infinite numbers are introduced in the chapter **Infinity and infinite element extensions** (p. 64), and classes are introduced in the chapter **Classes** (p. 53).

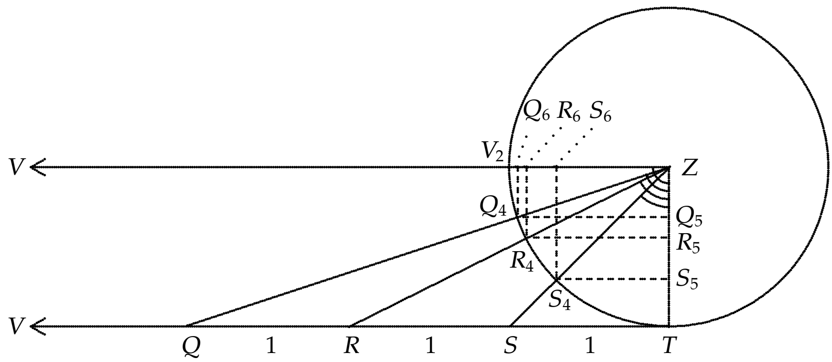

FIG. 10:
Infinite value
completing the definition
of the tangent function

Figure 10 shows how division by zero enables us to define the tangent function for all values, including $\frac{\pi}{2} \pm n\pi$.

$$\angle SZT = \tan^{-1} 1 = \tan^{-1} \frac{S_4S_5}{S_4S_6} = \tan^{-1} \frac{\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}}$$

$$\angle RZT = \tan^{-1} 2 = \tan^{-1} \frac{R_4R_5}{R_4R_6} = \tan^{-1} \frac{\frac{2}{\sqrt{5}}}{\frac{1}{\sqrt{5}}}$$

$$\angle QZT = \tan^{-1} 3 = \tan^{-1} \frac{Q_4Q_5}{Q_4Q_6} = \tan^{-1} \frac{\frac{3}{\sqrt{10}}}{\frac{1}{\sqrt{10}}}$$

$$\angle VZT = \tan^{-1} \infty = \tan^{-1} \frac{V_2Z}{V_2V_2} = \tan^{-1} \frac{1}{0}$$

Continuous compounding of interest

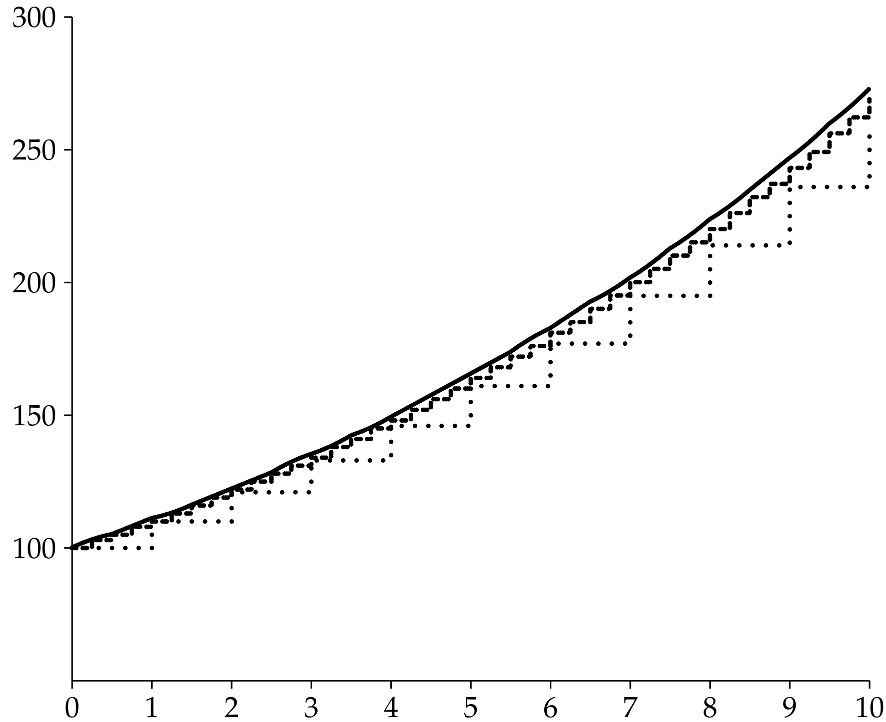

FIG. 11: An initial investment of 100 units
at 10 percent with annual (dotted),
quarterly (dashed), and
continuous (solid) compounding for 10 years

The second example of numeristics directly handling infinity is the calculation of continuously compounded interest. Interest is usually compounded with a finite compounding frequency, the number of interest payments per year. The formula for this type of compound interest is:

$$B(t) = B(0)\left(1 + \frac{r}{n}\right)^{nt},$$

where t = elapsed time

$B(t)$ = balance at time t

$B(0)$ = initial investment

r = annual interest rate

$$n = \text{compounding frequency}$$

In continuously compounded interest, the compounding frequency n is infinite, so the compounding interval is zero. In this case, the above formula becomes:

$$B(t) = B(0)\, e^{rt},$$

where e is the base of the natural logarithms and has an approximate value of 2.718.

Figure 11 shows the result of investing 100 currency units at 10 percent annual interest for 10 years, with three different compounding intervals: annual compounding (once a year) in the dotted line, quarterly compounding (4 times a year) in the dashed line, and continuous compounding in the solid line.

The numeristic theory of calculus, ***equipoint analysis*** (p. 125–301), allows us to directly define e in terms of infinity:

$$e \equiv \left(1 + \frac{1}{\infty'}\right)^{\infty'}$$

and prove theorems involving it in the same way. In conventional calculus, we must use the less direct method of limits.

Infinite series

Conventional mathematics uses limits to evaluate infinite series. For infinite series which are divergent, such as the following,

$$1 - 1 + 1 - 1 + \ldots$$
$$1 - 2 + 3 - 4 + \ldots$$
$$1 + 2 + 3 + 4 + \ldots$$

limits either cannot be used or can be used only very indirectly, often with inconsistent results.

The numeristic theory of infinite series, combined with equipoint analysis and called ***equipoint summation*** (see **Divergent Series** (p. 301–409)), allows us to directly use evaluate such sums. Equipoint summation gives the following results for the above three series:

$1 - 1 + 1 - 1 + \ldots$ can have any finite or infinite value

$1 - 2 + 3 - 4 + \ldots$ has the value $\frac{1}{4}$ or ∞

$1 + 2 + 3 + 4 + \ldots$ has only the value ∞

Inadequacies of set theory

Set theory is currently held by the vast majority of mathematicians to be a universal basis of mathematics, at least on a formal level. The basic motivation of set theory has been to provide a coherent, unified basis for all of mathematics. The author finds this motive to be admirable, but also finds its technique to be deeply unsatisfactory.

Objective inadequacies

- **No known physical verification.** The modern tendency to neglect objective verification in mathematics has had an important effect on the development of set theory. This development, starting in the late 19th century, has been purely subjective, focused on paper proofs only, and devoid of concern with objective verification. As far as this author has been able to determine, the assertions of set theory about the infinite have never been proved by physical experiment. Robin Ticciati, author of a well known reference work of the mathematics of quantum field theory [T99], when asked if he knew of any use of mathematics in quantum theory that depended on a set theoretical result, responded in the negative [T03].

- **Few mathematical properties derivable from sets.** With few exceptions, the set theoretical definitions of mathematical structures, including numbers, do not allow us to obtain the properties of those structures from their supposed definitions. The properties must instead be supplied from non-set-theoretical considerations. For this reason alone, set theory should not be considered a true foundational theory, but at best only a modeling theory.

- **Circular reasoning claim that it defines numbers.** Although it is claimed that set theory defines numbers, this reasoning is circular. Set theory and the system of logic it is built upon are implicitly dependent on numbers. Both set theory and logic assume fundamental dualities and multiplicities, such as true and false, axioms and sets, inside and oustide of sets, the multiplicity of axioms. Dualities are implicit uses of the number 2, and multiplicities are implicit uses of higher numbers. Even this consideration pales besides the implicit use of the number 1, which occurs each time we express or even think of any object of attention, and the number 0, which logically precedes all expressions and objects of attention.

- **No underlying unity.** Set theory is riddled with dualities with no underlying unity:

  - the duality of true and false values in the underlying logical system

  - the duality between the inside and outside of set

  - the duality between axioms and the objects they describe

  - the multiplicity of axioms

  Set theory is therefore incompetent to provide a proper model of unity. Without a proper model of unity, it cannot properly model any other number.

- **No abstract definition of number.** The set theoretical model of a number changes with its role, and so it has no recognition of the abstract level which underlies all of the different uses of each number. The number 1, for instance, has different set theoretical definitions depending on whether it is considered a natural number, integer, rational number, real number, complex number, or other role:

$$1_{\mathbb{N}} \equiv \{\{\ \}\}$$
$$1_{\mathbb{Z}} \equiv \{(n + 1_{\mathbb{N}}, n) \mid n \in \mathbb{N}\}$$
$$1_{\mathbb{Q}} \equiv \{(1_{\mathbb{Z}} \cdot z, z) \mid z \in \mathbb{Z} \setminus \{0_{\mathbb{Z}}\}\} = \{(z, z) \mid z \in \mathbb{Z} \setminus \{0_{\mathbb{Z}}\}\}$$
$$1_{\mathbb{R}} \equiv \left\{q \mid q < 1_{\mathbb{Q}}\right\}$$
$$1_{\mathbb{C}} \equiv \{(1_{\mathbb{R}}, 0_{\mathbb{R}})\}$$

  The set theoretical model of 1 as a natural number, $\{0\} = \{\{\ \}\}$, has no inverse, meaning that there is no "negative set" which when applied to $\{0\}$ yields $0 = \{\ \}$.

- **No self-referral.** **Russell's paradox** (p. 92) means that set theory must be constructed so that a set cannot be a member of itself. Since the only thing that sets can really do is include sets and be members of sets, this strikes a fatal blow to any aspiration of making set theory self-referent.

Subjective inadequacies

- **Difficulty and lack of naturalness.** Set theory is much harder to work with than the numbers is purports to define. As **Skolem points out** (p. 99), the foundations of a discipline should be the clearest and most natural part, but this most definitely does not hold for set theory.

- **Beset by controversy from the beginning.** Set theory met with much controversy in its early days. See the **Appendix** (p. 97) for source material that shows that Thoralf Skolem and Hermann Weyl substantially disagreed with the supposition that set theory could form a proper foundation for mathematics.

  Some of the axioms of set theory have been notoriously controversial. For instance, the axiom of infinity, which asserts the existence of an infinite set, encountered considerable controversy when it was introduced, which to this day has never been completely settled. Other axioms are even more controversial, such as the axiom of choice and the generalized continuum hypothesis.

- **Incapable of dealing with consciousness.** Consciousness is at the basis of every mathematician's and mathematics student's use of mathematics. Set theory is a hardened form of dualism that is utterly incapable of dealing with the subtleties of consciousness.

Conclusion

From the foregoing it should be clear that any system that explains number must account for the whole range of manifestation, from the subtlest thinking level to the most obvious, and it must account for both subjective and objective phenomena. It must also be clear that any such system cannot be based on intellectual values alone, since intellectual conception and expression necessarily take place in a field of multiplicity. The intellect, by itself, cannot properly account for unity and thus, by itself, is not an appropriate tool for exploring numbers.

Antecedents to numeristics

In the early 20th century, Skolem and Weyl independently anticipated some of the features of numeristics by attempting to construct foundational systems that did not use set theory.

In [S23], Skolem proves a variety of elementary number theoretical results using a system of natural numbers, standard logic, and first order recursion. This is known as primitive recursive arithmetic and was used by Gödel in the proofs of his famous incompleteness theorems in 1931. See **Skolem's recursive foundational system** (p. 99) in the Appendix.

In [W32], Weyl develops a theory of the real numbers, which he intended as an alternative to set theory as a foundation of analysis (calculus). Weyl bases his theory of the real continuum on natural numbers, basic logical operations, and primitive recursion, without transfinite set theory or proof by contradiction. See **Weyl's foundational system of the continuum** (p. 104) in the Appendix.

Some foundational theories based on **mereology** (p. 118) have developed the concept of ***fusion*** or ***sum*** which is similar to the important numeristic concept of **class** (p. 53). Numeristic classes have a flat structure which contrasts with the hierarchical structure of sets. See **Mereology** (p. 118) in the Appendix.

FOUNDATIONS OF NUMERISTICS

Ultraprimitives

Numeristics bases mathmatics purely on number. Numeristics starts with the observation that anything that can be intellectually identified can be counted. Numbers themselves can be counted. Number is therefore self sufficient and nothing is more basic than number. Any attempt to define number in terms of something else is circular.

Numeristics starts with three numeric ***ultraprimitives*** or alternate ways of expressing the ultimate value of mathematics. These three are ***infinity***, ***unity***, and ***zero***.

Infinity. Number, as with everything else, ultimately starts from the infinite. The infinite is inexhaustible and therefore only partly conceptualizable.

The infinite in its totality is beyond human conception but within the range of human experience. Vedic Mathematics shows how the infinite can be directly experienced as unbounded, pure consciousness, a fourth state of consciousness distinct from waking, dreaming, and sleeping. The Vedic tradition of India is very familiar with this state of consciousness and gives it many names, among them *samādhi* and *turīya*. It can be experienced in innumerable ways, but a systematic way of experiencing this fourth state of consciousness is through the TM (Transcendental Meditation) program. See [M96 p. 434–445]

The infinite may be visualized in ordinary space, since, even within a finite extent of space, the number of points and possible curves is infinite.

The infinite may be partially conceptualized in terms of finite numbers, by finding infinity within a number, or by finding a number within infinity.

Unity. Unity, the number 1, is the first mathematical manifestation. It expresses the unified nature of infinity.

Whenever a number is used to measure an object of experience, we can consider the number to be an attribute of the object, or we can consider the object to be an instance of the number. Since any conceivable single object is one object, everything conceivable is an instance of unity. Unity is obviously within infinity, but the infinite is also within unity.

The number one, since it is an identifiable object of attention, is an instance of itself. To put it another way, one is one number. This is the *principle of self-referral*. If we define consciousness as self-referral, then consciousness is essentially the state of unity.

Zero. The number 0 represents the unmanifest quality of pure consciousness. It is silence and balance. Whenever any mathematical object manifests, its opposite also manifests. Each positive number has a negative; each function has an inverse; each statement has a negation.

In Vedic Mathematics, zero is called the ***Absolute Number,*** because it is the unmanifest state from which all manifestation begins. See [M96 p. 611–634], [M05a], and [M05b].

Multiplicity. Zero and one observing each other give rise to the number two, and from there multiplicity comes out. Unity is found within two, since two is composed of two units. Two is found with unity, since two is one number, an instance of unity. The number two gives rise to the two values in classical logic of true and false and to all distinctions generally.

With multiple numbers comes the potential of transformation, and this manifests as functions. The values of true and false give rise to relations, including equality.

The self referral nature of number, combined with functions, gives rise to the four basic functions of addition, subtraction, multiplication, and division. Infinity can result from division by zero, which numeristics allows in a careful way, as described below.

Zero steps to multiple steps. Maharishi Vedic Mathematics is focused on the above ultraprimitives. It is a system of ***mathematics without steps***, a spontaneous knowing or cognition ([M96 p. 558–559]), as contrasted with the system of ***mathematics with steps*** in modern mathematics ([M96 p. 626–627]). Numeristics is an attempt to develop a system of stepwise mathematics that is in harmony with the ideals and practices of zero-step Maharishi Vedic Mathematics.

Knower, known, and knowing. Vedic Science identifies three divisions of knowledge: the knower, the known, and the process of knowing. It also identifies a level of unification, pure knowledge or pure awareness, that underlies and unites these three. In numeristics, we associate the three divisions with numbers/quantities (known), functions/operators (process of knowing), and relations (knower). The knowledge value itself is associated with a complete mathematical statement, which unites the three divided values of numbers, functions, and relations.

Four rigors. Numeristics has four types of rigor.

- The rigor of the knower: the inner, subjective experience of zero, unity, and infinity.
- The rigor of derivation: the intellectual connection of the knower with the known, including careful steps of calculation.
- The rigor of the known: verification of calculation by its application in the objective universe.
- The rigor of knowledge: the above three rigors working together to provide useful, systematic, and verifiable knowledge, keeping mathematical expressions in tune with observations of both inner and outer nature. The inner observation of ultraprimitive unity gives a stable platform for properly assessing the outer observations of multiplicity.

A function is an abstraction of a constant by allowing the transformation of one number into another. Functions are thus a more abstract level of numbers, but they are associated with the process of knowing since they emphasize transformation.

A relation is a function that has a logical value and is thus an abstraction of a function. Logical values are yet more subtle level of numbers since they connect the measurement level of number to the knowledge level of number, and truth values identify true and false statements. Relations are associated with the knower since logical values are much more associated with the knower than the known.

Figure 12 shows these identifications in a sample equation.

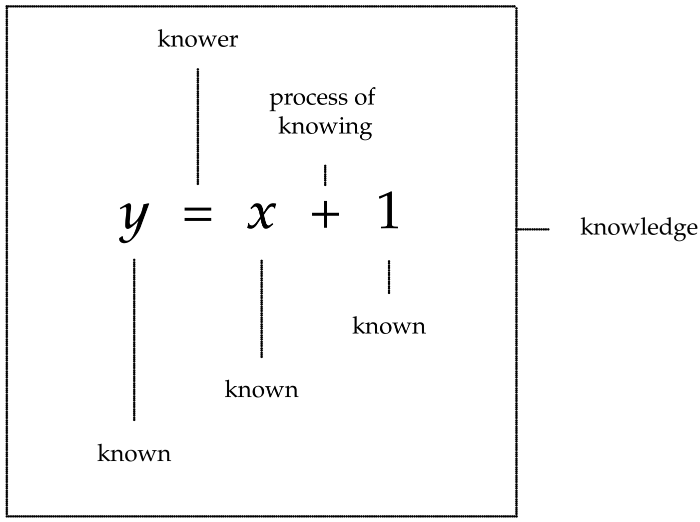

FIG. 12:
Knower, knowing, and known
in a mathematical statement

Primitives

In its multiple step phase, numeristics starts with the following ***primitives***, which function somewhat as axioms. These primitives are generally those which existed in "classical mathematics," by which we mean mathematics as it historically existed before the development of set theory, and which is currently taught at the primary, secondary, and lower division undergraduate levels.

- Natural, integral, rational, real, and complex numbers
- Addition, multiplication, exponentiation of these numbers and their inverses
- Usual commutative, associative, and distributive properties of these numbers
- Elementary equality and order relations
- Euclidan geometry
- Ordinary classical logic with quantifiers (first order logic)

We do *not* include the following.

- We do not include sets or categories.
- We do not include abstract structures, non-Euclidean geometries, or numbers beyond the complex numbers at this point. We will extend to them at a later point.
- Calculus and analysis are handled in the **second part of this book** (p. 125–301).
- Infinite series are also handled in the **third part of this book** (p. 301–409).

To the above primitives, we will soon add the following.

- Classes, which handle multiple unordered values
- Infinite element extensions, i.e. one or more infinite numeric values

We also accept the following principles.

The ***principle of freedom:*** We are free to perform any arithmetic operation, as long as we put it in correct context. This means that every numeric operation has a numeric result. No numeric operation is undefined. A numeric operation may be multivalued or empty valued, as described below.

The ***principle of reversal:*** Every operation can be reversed. This is because zero is the balance point, both a static balance point of positive and negative, and a dynamic balance point of an operation and its inverse.

The ***principle of extension:*** Number systems can be naturally extended, sometimes through reversal, and operations can be naturally extended to extended number systems.

CLASSES

Structure of classes

A numeristic ***class*** is the simultaneous presence of zero or more values. A value may be a number, function, relation, or statement.

Classes have a flat structure. Every number is a single valued class; a class containing a single number is identical to the number. Operations on and statments about a class are distributed over each constituent number.

Since numeristics and set theory do not mix but have some similar concepts, for numeristic classes we will use set theoretical notation with numeristic meanings. We denote classes with the same notation that we use for sets, either enumeration notation, as in $\{1, 2, 3\}$, or predicate notation $\{x \in \mathbb{N} \mid 1 \le x \le 3\}$. We may also put multiple functions and relations into classes, e.g. $\pm = \{+, -\}$.

Flat class structure means:

$$\text{For any element or class } c,\ \{c\} = c,$$
$$\text{For any two elements or classes } c \text{ and } d,\ \{c, d\} = c \cup d.$$

Flat structure allows arithmetic operations on classes in a simple way. For example, if $x^2 = 1$, then x is a class with the two elements $+1$ and -1, and we say $x = \pm 1 = \{+1, -1\}$, and $x + 1 = \pm 1 + 1 = \{0, 2\}$. For a list that is otherwise clearly delimited, we may drop the braces and write an expression such as $\pm 1 + 1 = 0, 2$.

We use several other notations from set theory in numeristics, including class building through enumeration and predicates. It must be emphasized that *these notations have somewhat different meanings in numeristics* from those in set theory. Below are samples of notation we can use to describe the numeristic class ± 1:

$$\begin{aligned} \pm 1 &= +1, -1 \\ &= \{1, -1\} \\ &= \{a \mid a^2 = 1\} \end{aligned}$$

$$
\begin{aligned}
&= 1 \cup -1 \\
&= \bigcup_{k=1}^{2} (-1)^k.
\end{aligned}
$$

A *subclass* is a class that is completely included in another class. We use the subset symbol for subclasses, e.g. $1 \subset \pm 1$ or $\pm 2 \subseteq \pm 2$.

We use the intersection symbol $\cap$ to denote the class of elements common to two classes, e.g. $\{1,2,3\} \cap \{3,4,5\} = 3$, the union symbol $\cup$ for the class of elements belonging to either or both of two classes, and the set subtraction symbol $\setminus$ to denote removal of elements: $c \setminus d \equiv \{a \in c \mid a \notin d\}$.

There may also be classes of functions, relations, and statements. An indefinite integral is an example of a class of functions, as explained in **Antiderivatives** (p. 182).

Elements

An ***element*** is a class which does not contain any smaller subclasses. If a is an element of b, then we use the notation $a \in b$ or $b \ni a$, e.g. $1 \in \pm 1$. Since an element is also a class, $a \in b$ implies $a \subseteq b$.

A class that is not an element is said to be ***multivalued***. An element may also be called a ***single valued*** class.

There are no general restrictions on the elements a class may contain. The context in which a class is used decides what type of elements are meaningful.

In numeristics, we do not *define* any element, whether a number, function, or relation, *as* a class. The numeristic view is that this is neither necessary nor sufficient. Rather, we define classes as collections of elements, and take numbers, functions, and relations as primitives, since both the concepts and the knowledge of how to use them emerge in a natural way from the experience of the absolute number and from objective application.

Functions

It will generally be assumed that the value of a function is a class, unless it is explicitly indicated as being single valued. This means that functions are generally multivalued.

It will also be generally assumed that a function may accept classes as arguments. This requires **distribution** (p. 55), as explained below.

A relation is a function that returns a logical value.

A ***compound*** is a special purpose function. Examples:

- An infinite sequence is a function from the positive integers to a class whose elements are terms of the sequence.
- A finite sequence is a function from $\{1, \ldots, n\}$ to a class of sequence terms.
- An ordered class is a finite sequence.
- An ordered pair is an ordered class with two elements, a function from $\{1, 2\}$ to a class of two terms.
- A matrix is a function from a class of ordered pairs (the row and column indexes) to a class of matrix entries.

Distribution

Arithmetic functions, such as $+$, $-$, and $\sqrt{}$, operate on elements. When applied to a single class, they distribute over the class elements to form another class of elements. For instance, ± 1 means $\{-1, +1\}$, and $(\pm 1)^2 = \{-1, +1\}^2 = \{(-1)^2, (+1)^2\} = \{1\} = 1$. In general, for any class c and arithmetic function f, $f(c)$ means $\{f(a) \mid a \in c\}$.

Class functions, such as $\cap$, $\cup$, $\setminus$ (class subtraction), and ^{c} (complement), operate on classes as a whole and do not distribute over class elements.

Arithmetic relations, such as $<$, $\leq$, $>$, $\geq$, and $\equiv \ldots \bmod$ (congruence), relate elements. When one of the operands is a class, the relation distributes over the class elements, yielding a class of relation statements.

An arithmetic relation distributing over a class may denote a class of statements, but such a class should usually produce a single statement by being joined with some logical connective, which may be conjunction, inclusive disjunction, or exclusive disjunction. For example, $\pm 2 < 5$ may be interpreted as $(2 < 5) \wedge (-2 < 5)$, or as $(2 < 5) \vee (-2 < 5)$, or as $(2 < 5) \veebar (-2 < 5)$.

An arithmetic relation R distributing over a class c as $R(c)$ could thus have one of four interpretations: a class of statements $\{R(a) \mid a \in c\}$, a conjunctive interpretation $(\forall a \in c)R(a)$, an inclusive disjunctive interpretation $(\exists a \in c)R(a)$, or an exclusive disjunctive interpretation $(\exists! a \in c)R(a)$. In this book, we usually assume the conjunctive interpretation.

Class relations, such as $\subset$, $\subseteq$, $\supset$, $\supseteq$, relate classes as a whole and do not distribute over class elements.

The equality relation $=$ actually has two different types, class distributed equality and statement distributed equality, and statement distributed equality has several subtypes. $(\pm 2)^2 = 4$, for instance, may mean:

- the class distributed equality $(\pm 2)^2 = \{+2, -2\}^2 = \{(+2)^2, (-2)^2\} = \{4\}$, in which the squaring operation distributes over the class ± 2 and the resulting class is compared to the class 4; or

- a statement distributed equality such as $(+2)^2 = 4 \wedge (-2)^2 = 4$ (conjuctive distribution), or $(+2)^2 = 4 \vee (-2)^2 = 4$ (disjunctive distribution), in which the whole statement distibutes over the class, yielding a class of statements that are then joined with a logical connective.

A statement distributed eqality is meaningful only when an ***implicit function*** mapping corresponding elements of the classes is clearly understood, and the logical connective is understood.

When necessary, we use the notation in Table 13 to distinguish these types.

TABLE 13: **Class distributed and statement distributed equalities and inequalities**

In this table, a and b are elements, c and d are classes, f is a function from c and d, and g is a function from d to c.

Symbol	Meaning	Implicit function	Equivalent
$c \overset{\{\}}{=} d$	$(\forall a)(a \in c \;\Leftrightarrow\; a \in d)$		$\neg\,(c \overset{\{\}}{\neq} d)$
$c \overset{\wedge}{=} d$	$(\forall a \in c)\ a = f(a)$	$f : c \to d$ bijective	$\neg\,(c \overset{\vee}{\neq} d)$
$c \overset{\vee}{=} d$	$(\exists a \in c)\ a = f(a)$ or $(\exists b \in d)\ b = g(b)$	$f : c \to d$ surjective $g : d \to c$ surjective	$\neg\,(c \overset{\wedge}{\neq} d)$
$c \overset{\{\}}{\neq} d$	$\neg\,(\forall a)(a \in c \;\Leftrightarrow\; a \in d)$		$\neg\,(c \overset{\{\}}{=} d)$
$c \overset{\wedge}{\neq} d$	$(\forall a \in c)\ a \neq f(a)$	$f : c \to d$ bijective	$\neg\,(c \overset{\vee}{=} d)$
$c \overset{\vee}{\neq} d$	$(\exists a \in c)\ a \neq f(a)$ or $(\exists b \in d)\ b \neq g(b)$	$f : c \to d$ surjective $g : d \to c$ surjective	$\neg\,(c \overset{\wedge}{=} d)$

Examples:

$$\{2,3\}^2 \overset{\{\}}{=} \{4,9\} \overset{\{\}}{=} \{9,4\} \overset{\{\}}{\neq} \{4,7\}$$

$$\{2,3\}^2 \overset{\wedge}{=} \{4,9\} \overset{\wedge}{\neq} \{3,7\}$$

$$\{2,3\}^2 \overset{\vee}{=} \{4,9\} \overset{\vee}{\neq} \{4,7\}$$

$$\pm 2 \overset{\{\}}{=} 2(\pm 1) \overset{\{\}}{=} 2(\mp 1)$$

$$\pm 2 \overset{\wedge}{=} 2(\pm 1)$$

$$\pm 2 \overset{\vee}{\neq} 2(\mp 1)$$

$$\pm 2 \overset{\vee}{=} 2$$

$$\pm 2 \overset{\vee}{\neq} 1$$

$\mathbb{Z}^* \overset{\{\}}{=} -\mathbb{Z}^*$ (where $\mathbb{Z}^*$ denotes the nonzero integers),

because the class as a whole is unchanged by negation

$$\mathbb{Z}^* \overset{\wedge}{\neq} -\mathbb{Z}^*, \text{because each element is changed by negation}$$

The flatness of classes means that no extra structure is incurred when we perform multiple distributed operations on them. For example, if we let $\sqrt{x}$ mean $\pm\sqrt{x}$,

$$\begin{aligned}
\sqrt{\left(\sqrt{4}+2\right)\frac{9}{4}+16} &= \sqrt{(\{2,-2\}+2)\frac{9}{4}+16} \\
&= \sqrt{\{4,0\}\frac{9}{4}+16} \\
&= \sqrt{\{9,0\}+16} \\
&= \sqrt{\{25,16\}} \\
&= \{5,-5,4,-4\}
\end{aligned}$$

Classes are unordered. Significance of order within class lists is only to define the implicit function. and does not indicate order within the classes. This principle governs the use of $\pm$ and $\mp$, as shown in the above examples.

This notation is only necessary to distinguish between class and statement distribution. Often these are equivalent, especially $\overset{\{\}}{=}$ and $\overset{\wedge}{=}$, or the type is understood, in which case it is sufficient to use $=$.

Distribution compared to quantification

Functions distribute over classes in the same flat way that logical quantifiers distribute connectives over quantified statements. Just as $\mathbb{N} + \frac{1}{2}$ is the same as $\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$, so $(\forall n \in \mathbb{N})\ n+1 = 1+n$ is the same as $1+1 = 1+1 \wedge 2+1 = 1+2 \wedge 3+1 = 1+3 \wedge \ldots$, and $(\exists n \in \mathbb{N})\ n+1 = 3$ is the same as $1+1=3 \vee 2+1=3 \vee 3+1=3 \vee \ldots$

Moreover, any quantified statement which operates over a singleton class is the same as a single statement, which corresponds to the principle of numeristic classes that $\{a\}$ is the same as a for any element a. For example, the quantified statement $(\forall x \in \{0\})\ x+x=x$ is the same as the single statement $0+0=0$.

In ordinary arithmetic and algebra, distribution or quantification over numbers corresponds to first order logic, and distribution or quantification over functions and relations corresponds to second or higher order logic.

Multiple distribution and threads

Expressions involving multiple classes may require more explicit disambiguation. Consider the expression $\{1,3\} + \{2,5\}$. By itself, such a formula is ambiguous: Does it represent $\{1+2, 3+5\} = \{3,8\}$ or $\{1+2, 1+5, 2+3, 3+5\} = \{3,5,6,8\}$? Conventional quantification is sufficient to disambiguate this type of expression:

$$\{(1,3)_n + (2,5)_n \mid 1 \le n \le 2\} = \{3,8\},$$
$$\{a + b \mid a \in \{1,3\} \wedge b \in \{2,5\}\} = \{3,5,6,8\}.$$

Alternatively we may use ***threads***. A thread links two or more classes in an expression. Between any two classes in the same thread, there must be a bijection, whether explicitly stated or implicitly obvious. When functions and relations are distributed over classes in the same thread, corresponding elements are distributed together; otherwise the functions and relations are distributed independently. Each thread in an expression is numbered, much like a subscript, to distinguish it from other threads. For example, we can write the class c appearing in two threads c either as $\underset{1}{c}$, $\underset{2}{c}$, or as $c_{:1}$, $c_{:2}$. The symbol $\underset{n}{c}$ is read "c thread n" and can be interpreted in a formula as a_n, where $a_n \in c$.

As an example, consider again $\{1,3\} + \{2,5\}$. If the two classes are in the same thread, then we use the following notation:

$$\underset{1}{\{1,3\}} + \underset{1}{\{2,5\}} = \{3,8\}.$$

The bijection in this case is easily understood since it is indicated by order of enumeration. If the two classes are in different threads, the we denote this as

$$\underset{1}{\{1,3\}} + \underset{2}{\{2,5\}} = \{3,5,6,8\}.$$

Classes in a thread may be multiple occurrences of same class, with the bijection being the identity map, as in

$$\underset{1}{\pm} 2 \underset{1}{\pm} 2 = \underset{1}{\pm} 4,$$

or the order of enumeration may indicate a nonidentity map, as in

$$\underset{1}{\pm} 2 \underset{1}{\mp} 2 = 0.$$

Threaded expressions may be alternatively represented with quantifiers or predicate class builders, as shown in the following examples:

$$\frac{\underset{1}{\mathbb{R}^+}}{\underset{1}{\mathbb{R}^+}} \quad \text{is equivalent to} \quad \left\{ \frac{a}{a} \mid a \in \mathbb{R}^+ \right\}$$

$$\frac{\underset{1}{\mathbb{R}^+}}{\underset{2}{\mathbb{R}^+}} \quad \text{is equivalent to} \quad \left\{ \frac{a}{b} \mid a, b \in \mathbb{R}^+ \right\}$$

$$\frac{\underset{1}{\mathbb{R}^+}}{\underset{1}{\mathbb{R}^+}} = 1 \quad \text{is equivalent to} \quad (\forall a \in \mathbb{R}^+) \frac{a}{a} = 1$$

$$\frac{\underset{1}{\mathbb{R}^+}}{\underset{2}{\mathbb{R}^+}} = \underset{3}{\mathbb{R}^+} \quad \text{is equivalent to} \quad (\forall a, b \in \mathbb{R}^+)(\exists c \in \mathbb{R}^+) \frac{a}{b} = c$$

$$\wedge\ (\forall c \in \mathbb{R}^+)(\exists a, b \in \mathbb{R}^+) \frac{a}{b} = c$$

Standard numeric classes

We can define some standard numeric classes as follows:

$$\begin{aligned}
\mathbb{B} &\equiv \{0, 1\} \\
\mathbb{N} &\equiv \bigcup_{k=0}^{\infty} k \\
\mathbb{Z} &\equiv \pm\mathbb{N} \\
\mathbb{Q} &\equiv \frac{\mathbb{Z}}{\mathbb{Z}^*} = \pm\frac{\underset{1}{\mathbb{N}^*}}{\underset{2}{\mathbb{N}^*}} \cup 0 \\
\mathbb{I} &\equiv \sum_{k=1}^{\infty} \underset{k}{\mathbb{B}}\, 2^{-k} \\
\mathbb{R} &\equiv \mathbb{Z} + \mathbb{I} = \mathbb{Z}\mathbb{I} \\
\mathbb{T} &\equiv e^{2\pi i\, \mathbb{I}} \\
\mathbb{C} &\equiv \underset{1}{\mathbb{R}} + \underset{2}{\mathbb{R}} i = |\mathbb{R}|\mathbb{T} = \mathbb{R}\, e^{\pi i\, \mathbb{I}}
\end{aligned}$$

In the definition of $\mathbb{I}$, the sum is not a limit, but a sum of an infinite number of terms, using a positive infinity from one of the **real infinite element**

extensions (p. 68). This approach to summation and real numbers is further explored in the numeristic theories of divergent series (see **Divergent Series** (p. 301–409)) and repeating decimals (see **Repeating Decimals** (p. 409–457)).

In this book:

- $\mathbb{N}$ includes zero.
- * means the omission of zero: $\mathbb{Q}^*$ means $\mathbb{Q} \setminus 0$, for instance.

Some set theoretic notation conflicts with class distribution. In this book, we will use the following alternative notation:

- Since $\mathbb{N} \times \mathbb{R}$ by class distribution means $\{nr \mid n \in \mathbb{N} \wedge r \in \mathbb{R}\}$ rather than $\{(n, r) \mid \mathbb{N} \wedge r \in \mathbb{R}\}$, we denote the latter $(\mathbb{N}, \mathbb{R})$.
- Since $\mathbb{R}^2$ by class distribution means $\{r^2 \mid r \in \mathbb{R}\}$ rather than $\{(r_1, r_2) \mid r_1, r_2 \in \mathbb{R}\}$, we denote the latter $\mathbb{R}^{\times 2}$.
- Since $\mathbb{R}^{\mathbb{C}}$ by class distribution means $\{r^c \mid r \in \mathbb{R} \wedge c \in \mathbb{C}\}$ rather than $\{f \mid f : \mathbb{C} \to \mathbb{R}\}$, we denote the latter $\mathbb{R}^{\swarrow \mathbb{C}}$.

Universe

In any given discussion, the ***universe*** is the largest class under consideration. Every class referred to in the discusion has an implied intersection with the universe.

A universe may be implicit or explicit. A universe may be explicitly indicated with the notation $U = u$ followed by one or more expressions, where u is the universe. Examples:

$$(U = \mathbb{R}) \qquad \sqrt{-1} = \sqrt{-1} \cap \mathbb{R} = \{\}$$
$$(U = \mathbb{C}) \qquad \sqrt{-1} = \sqrt{-1} \cap \mathbb{C} = \pm i$$
$$(U = \mathbb{H}) \qquad \sqrt{-1} = \sqrt{-1} \cap \mathbb{H} = \{ai + bj + ck \mid a, b, c \in \mathbb{R} \wedge a^2 + b^2 + c^2 = 1\}$$

The unary use of the operator $\setminus$, e.g. $\setminus c$, means $U \setminus c$, where U is the current universe.

Since classes may include functions, relations, and statments as well as numbers, the universe may also include such objects.

The empty class and the full class

The ***empty class*** or ***null class***, denoted $\emptyset$ or $\{\ \}$, is a class with no values. For example, $1 \cap 2 = \{\}$.

When a sentence distributes over the empty class, the result is the empty statement, no statement at all. Since no statement is both true and false, the empty statement is both true and false.

If a function f is undefined at a, we can say $f(a) = \emptyset$. The result of any arithmetic operation on $\emptyset$ is $\emptyset$, e.g. $1 + \emptyset = \emptyset$. This last statement is a nonempty statement because it uses class equality.

Similarly, we define the ***full class***, denoted $\not\infty$, as the complement of the empty class. Usually $\not\infty$ simply denotes the universe, such as $\mathbb{R}$ or $\mathbb{C}$.

We consider the place of the full class the real and complex arithmetic below, in **Indeterminate expressions and the full class** (p. 81).

Inverses

For any function f, its inverse is defined as a class:

$$f^{-1}(x) \equiv \{a \mid f(a) \supseteq x\},$$

for the general case of a multivalued f. For a single-valued f, this becomes:

$$f^{-1}(x) \equiv \{a \mid f(a) = x\}.$$

It follows that $f^{-1}(x) = \{\}$ is equivalent to $\neg\ (\exists a) f(a) = x$. It also follows that if $f(x) = y$, then $f^{-1}(y) \supseteq x$, and $f^{-1}(y) = x$ for single-valued injective f.

We then have:

$$\begin{aligned} f^0(x) &= x \\ f^{-1}(y) &= \{x \mid f(x) \supseteq y\} \\ f\left(f^{-1}(y)\right) &= f\left(\{x \mid f(x) \supseteq y\}\right) = y = f^0(y) \\ f^{-1}\left(f(x)\right) &= \{u \mid f(u) \supseteq f(x)\} \supseteq x = f^0(x). \end{aligned}$$

For a single-valued f:

$$
\begin{aligned}
f^0(x) &= x \\
f^{-1}(y) &= \{x \mid f(x) = y\} \\
f\left(f^{-1}(y)\right) &= f\left(\{x \mid f(x) = y\}\right) = y = f^0(y) \\
f^{-1}\left(f(x)\right) &= \{u \mid f(u) = f(x)\} \supseteq x = f^0(x) \\
&= x = f^0(x) \text{ for injective } f.
\end{aligned}
$$

One important consequence of this definition is that the inverse of a function is always itself a function. For instance, the inverse of $f(x) = x^2$ is $f^{-1}(x) = \pm\sqrt{x}$. The radical symbol is sometimes used as a multivalued inverse, i.e. $\sqrt[n]{b} \equiv \{a \in C \mid a^n = b\}$.

That inverse functions give us the same type of value as the original function is an important feature of numeristics. It means we can always retrace our steps and return to the starting point with a minimum of formulaic overhead.

Function mapping types are defined as follows:

- An ***injection*** or ***one-to-one-function*** is a single valued function whose inverse is single or empty valued everywhere.
- A ***surjection*** or ***onto function*** is a single valued function whose inverse is single or multiple valued everywhere.
- A ***bijection*** or ***one-to-one and onto function*** is a single valued function whose inverse is single valued everywhere.

INFINITY AND INFINITE ELEMENT EXTENSIONS

Infinity and division by zero

In conventional mathematics, given a function $f(c) \equiv c \cdot 0$, its inverse g, if it existed, would have to have the form $g(b) = \frac{b}{0}$. For $b = 0$, every c satisfies $c \cdot 0 = b$, while for $b \neq 0$, no c satisfies $c \cdot 0 = b$. Thus there is no function g which is the inverse of f for any element of the domain of f. This is why such division is undefined in conventional mathematics.

Numeristics deals with this situation by adding two features to conventional mathematics: (1) **classes** (p. 53) and (2) infinite elements. Every numeric expression evaluates to a class, unless explicitly restricted to an element. The class $\dfrac{0}{0}$ is the full class, and the class $\dfrac{b}{0}$ for $b \neq 0$ is a class of one or more infinite elements.

Numeristics adds infinite elements and classes so that they fit common observations, such as an infinitely removed point for $\tan \dfrac{\pi}{2}$. We then find that identities such as $\tan \theta = \dfrac{\sin \theta}{\cos \theta}$ and $\tan^2 \theta + 1 = \sec^2 \theta$ hold for all θ without exception or are minimally modified.

For any standard universe, there are several ways we can add infinite elements. For a discrete universe such as $\mathbb{N}$ or $\mathbb{Z}$, we can use the ***class count***. The count of a class c is the number of elements in c and is denoted $\#c$. We can then define infinity as the class count of the universe: $\infty \equiv \#\mathbb{N}$ or $\infty \equiv \#\mathbb{Z}$.

For a dense universe such as $\mathbb{Q}$, $\mathbb{R}$, or $\mathbb{C}$, we can use the class count, but we will find it more convenient to define infinity in terms of division by zero: $\infty \equiv \left|\dfrac{1}{0}\right|$.

We add at least this infinite value and possibly others to comprise the class of infinite elements in an extended universe. Below we discuss infinite element extensions of $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$.

If a number is zero or infinite, we say it is *abfinite*; otherwise we say it is *perfinite*. If two abfinite numbers are both zero or both infinite, we say they are *coabfinite*; if one is zero and the other is infinite, we say they are *contraabfinite*. The term *finitesimal* or *nonzero* refers to a number which is either perfinite or infinite.

Outside of numeristics, the symbol ∞ usually does not denote an actual quantity but is only used as part of a symbol to denote certain types of limit, sum, or integral. In contrast, numeristics regards infinity as a number or class of numbers.

Theorems for division by zero

We now investigate the properties that infinite elements should have in an extended number system. We want the properties of infinite elements to be as close as possible to those of finite numbers, and to be in accord with observation. We must be careful to show when and why a property of an infinite number must be different from that of a finite element. We will use these principles in the following proofs.

We initially assume that the usual commutative, associative, and distributive laws for multiplication hold in an extended system. Since division is the inverse of multiplication, $\frac{1}{b} \cdot b \supseteq 1$ for every element in an extended system, with equality holding for perfinite b.

To extend a system, we assume that there exists a class $\infty \equiv \frac{1}{0}$. At this point, we do not specify the elements of ∞, only that there is at least one.

INDETERMINACY OF 0/0: If the universe is an extended form of $F = \mathbb{Q}$ or $\mathbb{R}$ or $\mathbb{C}$ (where F is unextended and consists only of finite elements), then $\frac{0}{0} \supseteq F$.

PROOF. Let a be any element of the universe and $f(x) \equiv x \cdot 0$. Then

$$f(a) = a \cdot 0 = 0$$

$$\frac{0}{0} = f^{-1}(0) = \{a \mid a \cdot 0 = 0\} \supseteq F. \blacksquare$$

EXISTENCE OF ONE OR MORE INFINITE VALUES: $\boldsymbol{\infty \equiv \frac{1}{0}}$ is infinite.

PROOF. Let a be any nonzero element of $F = \mathbb{Q}$ or $\mathbb{R}$ or $\mathbb{C}$. Since F consists only of finite elements, a is perfinite. Then

$$\begin{aligned} \frac{1}{a} \cdot a &= 1 \\ \frac{1}{0} \cdot 0 &= \frac{0}{0} \supseteq F \overset{\{\}}{\neq} 1 \\ &\overset{\{\}}{\neq} 0. \end{aligned}$$

If $\frac{1}{0}$ were zero, $\frac{1}{0} \cdot 0$ would be 0. If $\frac{1}{0}$ were perfinite, $\frac{1}{0} \cdot 0$ would be 1. Since neither holds, $\frac{1}{0}$ is infinite. ■

ALTERNATE PROOF. For perfinite a and b, if $|a| < |b|$, then

$$\left|\frac{1}{a}\right| > \left|\frac{1}{b}\right|.$$

Here we make an additional assumption that the same property would hold for $a = 0$. In this case, since $0 < |b|$ for all b,

$$\left|\frac{1}{0}\right| > \left|\frac{1}{b}\right|$$

for all b. Hence $\left|\frac{1}{0}\right|$ is larger than any perfinite number, i.e. it is infinite. ■

ELEMENTARY ARITHMETIC OF $\boldsymbol{\infty}$:

(a) $\boldsymbol{\infty + a = \infty}$ **for any finite $\boldsymbol{a}$.**

(b) $\boldsymbol{a \cdot \infty = \infty}$ **for any perfinite $\boldsymbol{a}$.**

(c) $\boldsymbol{\infty \cdot \infty = \infty}$**.**

(d) $\boldsymbol{0 \cdot \infty = \frac{0}{0}}$**.**

(d) $\boldsymbol{\frac{\infty}{\infty} = \frac{0}{0}}$**.**

(e) $\boldsymbol{\infty - \infty = \frac{0}{0}}$**.**

(f) $\boldsymbol{\infty \subset \frac{0}{0}}$**.**

PROOF.

(a) Since $a \in \frac{0}{0}$,

$$\infty + a = \frac{1}{0} + a \subseteq \frac{1}{0} + \frac{0}{0} = \frac{1+0}{0} = \frac{1}{0} = \infty.$$

(b)

$$a \cdot \infty = a \cdot \frac{1}{0} = \frac{1}{\frac{1}{a}} \cdot \frac{1}{0} = \frac{1 \cdot 1}{\frac{1}{a} \cdot 0} = \frac{1}{0} = \infty.$$

(c)

$$\infty \cdot \infty = \frac{1}{0} \cdot \frac{1}{0} = \frac{1 \cdot 1}{0 \cdot 0} = \frac{1}{0} = \infty.$$

(d)

$$0 \cdot \infty = 0 \cdot \frac{1}{0} = \frac{0 \cdot 1}{0} = \frac{0}{0}.$$

(e)

$$\infty - \infty = \frac{1}{0} - \frac{1}{0} = \frac{1-1}{0} = \frac{0}{0}.$$

(f) Since $1 \in \frac{0}{0}$,

$$\infty = \frac{1}{0} \cdot 1 \subset \frac{1}{0} \cdot \frac{0}{0} = \frac{1 \cdot 0}{0} = \frac{0}{0}. \blacksquare$$

Thus $\frac{a}{a} = 1$ only for perfinite a, and $a - a = 0$ only for finite a, while for general a in an extended system, $\frac{a}{a} \supseteq 1$ and $a - a \supseteq 0$. We have also seen that $\frac{0}{0} = 0 \cdot \infty = \infty - \infty$, and that these indeterminate expressions include all infinite values and well as all finite values.

The above theorems are **class distributed equalities** (p. 55). They assume that $\frac{1}{0}$ is a nonempty class but do not address the internal structure of the class, only its overall behavior. The following five systems give a variety of specific structures to this class.

The first two systems are extensions of $\mathbb{R}$ and are discussed in detail in **Real infinite element extensions** (p. 68).

- The ***projectively extended real numbers*** adds a single infinite element. The conventional projectively extended system, described in [WPE], does not allow indeterminate expressions, since it does not have classes.

- The ***affinely extended real numbers*** adds two infinite elements, $+\infty$ and $-\infty$. The conventional affinely extended system, described in [WAE], lacks classes and does not allow indeterminate expressions.

The affinely extended real numbers furnish an example of how the above theorems do not hold if they are interpreted as **conjunctively distributed equalities** (p. 55). In the affinely extended system, $a(+\infty) = +\infty$ only for positive a; for negative a, $a(+\infty) = -\infty \neq +\infty$.

The next three systems are extensions of $\mathbb{C}$ and are discussed in detail in **Complex infinite element extensions** (p. 77).

- The ***single projectively extended complex numbers*** or ***Riemann sphere*** adds a single infinite element. The conventional Riemann sphere, discussed in [WRS], lacks classes and does not allow indeterminate expressions.

- The ***double projectively extended complex numbers*** adds an infinite number of infinite elements $\infty e^{i\mathbb{R}}$, where $\infty e^{is} = \infty e^{it}$ only if $\tan s = \tan t$ (i.e. $s \cong t \bmod \pi$). This system regards the complex plane as a two-dimensional real projective plane with an associated complex arithmetic.

- The ***affinely extended complex numbers*** adds an infinite number of infinite elements $\infty e^{i\mathbb{R}}$, where $\infty e^{is} = \infty e^{it}$ only if $e^{is} = e^{it}$ (i.e. $s \cong t \bmod 2\pi$).

Real infinite element extensions

Here we examine two methods of including infinite elements in the real numbers, the ***projectively extended*** real numbers and the ***affinely extended*** real numbers. These methods are known to conventional mathematics, although terminology and notation vary. The difference in numeristics is in the handling of multivalued expressions such as $\frac{0}{0}$ and $\infty - \infty$, which are undefined in the conventional approach but are multivalued classes in numeristics.

Projectively extended real numbers

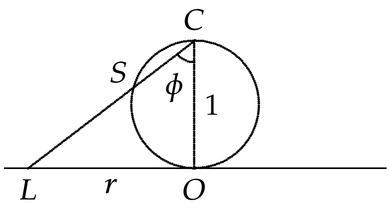

FIG. 14:
Projectively extended
real numbers,
one-point method
$r = \tan\phi$

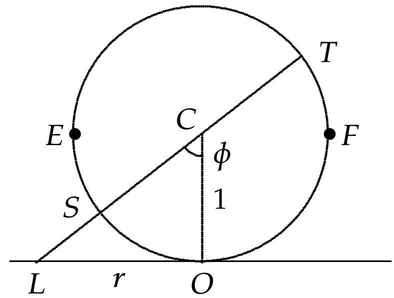

FIG. 15:
Projectively extended
real numbers,
two-point method
$r = \tan\phi$

The first method of adding infinite elements that we examine adds one infinite element, denoted ∞ or $\overline{\infty}$, to the real numbers. Figures 14 and 15 show two different methods of mapping real numbers to a circle. In Figure 14, every real number r at position L uniquely maps to some point S on the circle, and the infinite element ∞ maps to the point C at the top of the circle. The angle ϕ is called the ***colatitude*** of the point S. In Figure 15, r is mapped to a pair of points S and T, and ∞ is mapped to E and F.

Alternatively, instead of mapping to points, we can map to lines. In Figure 14, L can be mapped to the line SL, and ∞ is mapped to the line through P that is parallel to OL. In Figure 15, L can be mapped to the line ST, and ∞ is mapped to the line EF.

This extension of the real numbers is called the ***projectively extended real numbers*** and is denoted $\widehat{\mathbb{R}}$, $P^1(\mathbb{R})$, or $\mathbb{RP}^1$. See [WPE]. We also use the following symbols for important subsets.

$\mathbb{R}$		finite real numbers
$\mathbb{R}^*$	$\equiv \mathbb{R} \setminus 0$	perfinite real numbers
$\widehat{\mathbb{R}}$	$\equiv \mathbb{R} \cup \infty$	projectively extended real numbers
$\widehat{\mathbb{R}}^*$	$\equiv \mathbb{R} \cup \infty \setminus 0$	finitesimal projectively extended real numbers
$\widehat{\mathbb{R}}^\ddagger$	$\equiv \infty \cup 0$	abfinite projectively extended real numbers

In $\widehat{\mathbb{R}}$:

- $+\infty = -\infty$.
- For $c > 0$, we have $e^\infty = c^\infty = \{0, \infty\}$.
- For finite c, we may have one of two conventions: (1) both $c < \infty$ and $c > \infty$, or (2) neither $c < \infty$ nor $c > \infty$. We will normally use the first.
- Trichotomy fails for ∞, since for any finite real r, $r < \infty$ and $r > \infty$.
- Distributivity also fails for ∞:

$$\begin{aligned} 2\infty = \infty \quad &\subset \quad \infty + \infty = \not\infty \\ (2+0)\infty = 2 \cdot \infty = \infty \quad &\subset \quad 2 \cdot \infty + 0 \cdot \infty = \infty + \not\infty = \not\infty \\ (3 \cdot 1)^\infty = 3^\infty = \infty \quad &\subset \quad 3^\infty \cdot 1^\infty = \infty \cdot \not\infty = \not\infty \end{aligned}$$

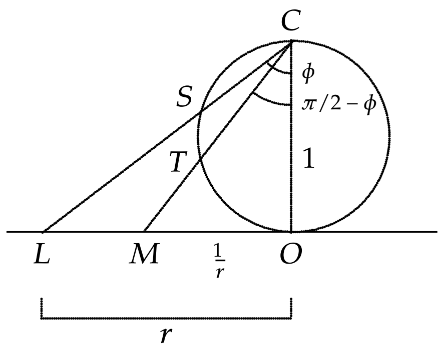

FIG. 16:
Geometric relation
of reciprocals in
projectively extended
real numbers,
one-point method

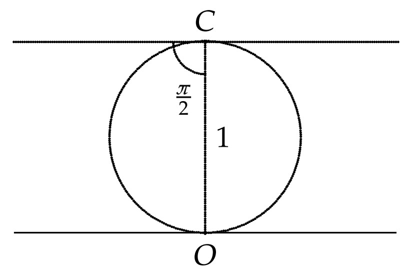

FIG. 17:
Geometric relation
of 0 and ∞ in
projectively extended
real numbers,
one-point method

Figures 16 and 17 show how pairs of reciprocals of projectively extended real numbers are mapped to pairs of points on the circle. Figure 16 uses the method of Figure 14. Colatitudes of reciprocals are supplemental, since $\tan\frac{\phi}{2} = \frac{1}{\cot\frac{\phi}{2}} = \frac{1}{\tan\frac{\pi}{2}-\phi}$.

Figure 17 uses the relation shown in Figure 16 to map ∞ as the reciprocal of 0 to the point P, whose colatitude is π.

Affinely extended real numbers

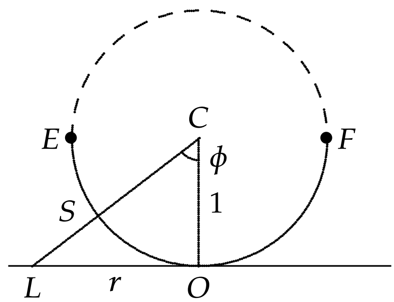

FIG. 18:
Affinely extended real numbers
$r = \tan\phi$

Figure 18 shows the second method of adding infinite elements to the real numbers, in which the single element $\overline{\infty}$ has been ***uncoiled*** into two infinite elements, $+\infty$ and $-\infty$, or ∞^+ and $-\infty^+$. In this figure also, every real number r at position L uniquely maps to some point S on the solid semicircle, but $-\infty$ maps to E and $+\infty$ to F.

This extended version of the real numbers is called the ***affinely extended real numbers*** and is denoted $\overline{\mathbb{R}}$. See [WAE].

In $\overline{\mathbb{R}}$:

- $+\infty \neq -\infty$.
- Trichotomy holds for $\pm\infty$. We have $-\infty < \infty$, $\infty \nless -\infty$, and for any finite real r, $r < \infty$ and $r > -\infty$, but $r \ngtr \infty$ and $r \nless -\infty$. Hence for any $r, s \in \overline{\mathbb{R}}$, exactly one of the alternatives $r < s$, $r = s$, and $r > s$ holds.
- Distibutivity holds for $\pm\infty$. For example, $+\infty + \infty = 2(+\infty) = +\infty$.

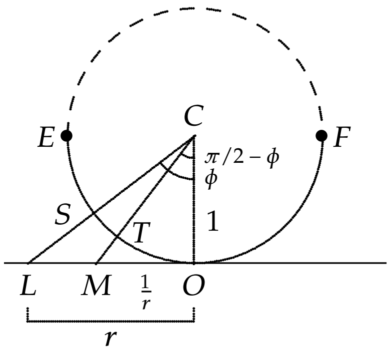

FIG. 19:
Geometric relation
of reciprocals in
affinely extended
real numbers

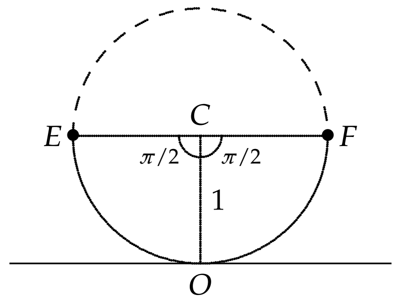

FIG. 20:
Geometric relation
of 0 and $\pm\infty$ in
affinely extended
real numbers

Figures 19 and 20 show how pairs of reciprocals of affinely extended real numbers are mapped to pairs of points on the semicircle. Figure 19 uses the method of Figure 18. Colatitudes of reciprocals are complementary, since $\tan\phi = \frac{1}{\cot\phi} = \frac{1}{\tan\left(\frac{\pi}{2}-\phi\right)}$.

Figure 20 uses the relation shown in Figure 19 to map $\pm\infty$ as the reciprocal of 0 to the points E and F, whose colatitudes are $\frac{\pi}{2}$.

ADDITION AND EXPONENTIATION OF $\pm\infty$:

(a) $\underset{1}{\pm\infty} + \underset{1}{\pm\infty} = \underset{1}{\pm\infty}$.

(b) $\underset{1}{\pm\infty} + \underset{2}{\pm\infty} = \not\phi$.

(c) $e^{+\infty} = +\infty$.

(d) $e^{-\infty} = 0$.

PROOF.

(a) $\underset{1}{\infty} + \underset{1}{\infty} = \{+\infty+\infty, -\infty-\infty\} = \{+\infty, -\infty\} = 2\infty = \infty$.

(b) $\underset{1}{\infty} + \underset{2}{\infty} = \{+\infty+\infty, +\infty-\infty, -\infty+\infty, -\infty-\infty\} = \{+\infty, \not\phi, -\infty\} = \not\phi$.

(c) $e^{+\infty} = \displaystyle\sum_{k=0}^{\infty} \frac{+\infty^k}{k!} = \sum_{k=0}^{\infty} +\infty = (+\infty)\cdot(+\infty) = +\infty.$

(d) $e^{-\infty} = \dfrac{1}{e^{+\infty}} = 0.$ ■

Table 21 compares selected arithmetic operations in the projectively and affinely extended real numbers.

TABLE 21: **Arithmetic operations in projectively and affinely extended real numbers**

In this table, a is finite, b and c are perfinite, p is finite positive.

$\widehat{\mathbb{R}}$	$\overline{\mathbb{R}}$
$\mathbb{R} \cup \overline{\infty}$	$\mathbb{R} \cup \{\infty^+, -\infty^+\}$
$\overline{\infty} \equiv \dfrac{1}{0}$	$\infty^* \equiv \left\lvert \dfrac{1}{0} \right\rvert$
$\dfrac{1}{0} = \overline{\infty}$	$\dfrac{1}{0} = \pm\infty^+$
$+\overline{\infty} = -\overline{\infty}$	$+\infty^+ \neq -\infty^*$
$a \pm \overline{\infty} = \overline{\infty}$	$a \pm \infty^* = \pm\infty^*$
$\overline{\infty} + \overline{\infty} = \not\infty$	$\infty^+ + \infty^+ = \infty^+$
$\overline{\infty} - \overline{\infty} = \not\infty$	$\infty^+ - \infty^+ = \not\infty$
$b\overline{\infty} = \overline{\infty}$	$p(\pm\infty^*) = \pm\infty^+$
$0\overline{\infty} = \not\infty$	$0(\pm\infty^+) = \not\infty$
$\overline{\infty} \cdot \overline{\infty} = \overline{\infty}$	$+\infty^*(\pm\infty^+) = \pm\infty^+, -\infty(\pm\infty^+) = \mp\infty^+$
$\dfrac{\overline{\infty}}{b} = \overline{\infty}$	$\dfrac{\pm\infty^+}{p} = \pm\infty^+$
$\dfrac{b}{\overline{\infty}} = 0$	$\dfrac{b}{\pm\infty^+} = 0$
$\dfrac{b}{0} = \overline{\infty}$	$\dfrac{b}{0} = \pm\infty^+$
$\dfrac{\overline{\infty}}{0} = \overline{\infty}$	$\dfrac{\infty^+}{0} = \pm\infty^+$
$\dfrac{0}{\overline{\infty}} = 0$	$\dfrac{0}{\pm\infty^+} = 0$

$\frac{0}{0} = \not\phi$	$\frac{0}{0} = \not\phi$
$\frac{\overline{\infty}}{\overline{\infty}} = \not\phi$	$\frac{\pm\infty^+}{\pm\infty^+} = \lvert\not\phi\rvert, \frac{\pm\infty^+}{\mp\infty^+} = -\lvert\not\phi\rvert$
$\frac{1}{x}$, $\tan x$ are continuous at $x = 0$	$\frac{1}{x}$, $\tan x$ are discontinuous at $x = 0$
$(b + c)\overline{\infty} \subset b\overline{\infty} + c\infty$	$(b + c)(\pm\infty^+) = b(\pm\infty^+) + c(\pm\infty^+)$
$(b + 0)\overline{\infty} \subset b\overline{\infty} + 0 \cdot \overline{\infty}$	$(b + 0)(\pm\infty^+) \subset b(\pm\infty^+) + 0(\pm\infty^+)$
$(b + \overline{\infty})\overline{\infty} = b\overline{\infty} + \overline{\infty} \cdot \overline{\infty}$	$(b \underset{1}{\pm} \infty^+)(\underset{2}{\pm}\infty^+) = b(\underset{1}{\pm}\infty^+) \underset{2}{\pm} 0(\underset{1}{\pm}\infty^+)$
$(0 + \overline{\infty})\overline{\infty} \subset 0 \cdot \overline{\infty} + \overline{\infty} \cdot \overline{\infty}$	$(0 \underset{1}{\pm} \infty^+)(\underset{2}{\pm}\infty^+) \subset 0(\underset{1}{\pm}\infty^+) \underset{2}{\pm} \infty^+(\underset{1}{\pm}\infty^+)$
$a < \overline{\infty} \wedge \overline{\infty} < a$	$a < \infty^+ \wedge -\infty^+ < a$
$b \subset \frac{b}{0} \cdot 0$	$b \subset \frac{b}{0} \cdot 0$
$b \subset \frac{b}{\overline{\infty}} \cdot \overline{\infty}$	$b \subset \frac{b}{\pm\infty^+} \cdot \pm\infty^+$
$a \subset (a + \overline{\infty}) - \overline{\infty}$	$a \subset (a \pm \infty^+) \mp \infty^+$
$e^{\overline{\infty}} = \{0, \overline{\infty}\}$	$e^{\infty^+} = \infty^+$
$e^{-\overline{\infty}} = \{0, \overline{\infty}\}$	$e^{-\infty^+} = 0$

Exponentiation is strictly monotonically increasing: for finite positive real q and r, if $q > r$, then $e^q > e^r$. Maintaining this property is desirable for positive infinite q, or if e^q is multivalued, then at least one value of e^q should be greater than e^r, i.e. $e^q \stackrel{\vee}{=} e^r$.

Since $|\infty|$ is greater than $|r|$ for any finite r, the exponential of any positive infinite value must include at least one positive infinite value. Similarly, the exponential of any negative infinite value must include zero.

$0^0 = \not\phi$	$0^0 = \not\phi$
$0^{\overline{\infty}} = \{0, \overline{\infty}\}$	$0^{\pm\infty^+} = \mp\infty^+$
$1^{\overline{\infty}} = \not\phi$	$1^{\pm\infty^+} = \not\phi$
$(-1)^{\overline{\infty}} = \{0, \overline{\infty}\}$	$(-1)^{\infty^+} = \{0, \pm\infty^+\}$
	$(-1)^{-\infty^+} = \{0, \pm\infty^+\}$
$\overline{\infty}^0 = \not\phi$	$(\pm\infty^+)^0 = \not\phi$
$\overline{\infty}^{\overline{\infty}} = \{0, \overline{\infty}\}$	$(\infty^+)^{\infty^+} = \infty^+$
	$(-\infty^+)^{+\infty^+} = \pm\infty^+$
	$(\pm\infty^+)^{-\infty^+} = 0$
$\ln \overline{\infty} = \overline{\infty}$	$\ln \infty^+ = \infty^+$
$\ln 0 = \overline{\infty}$	$\ln 0 = -\infty^+$
$\ln(-\overline{\infty}) = \overline{\infty}$	$\ln(-\infty^+) = \{\ \}$

$\lvert\overline{\infty}\rvert = \overline{\infty}$	$\lvert \pm \infty^+ \rvert = \infty^+$

Extended integers and rational numbers

We may classify ∞ as a natural number since it is the sum of other natural numbers: $\infty = 1 + 1 + 1 + \ldots$. In this case, we can add one or two infinite numbers to the natural numbers, integers, and rational numbers.

$$\widehat{\mathbb{N}} \equiv \mathbb{N} \cup \overline{\infty}$$
$$\overline{\mathbb{N}} \equiv \mathbb{N} \cup \infty^+$$
$$\widehat{\mathbb{Z}} \equiv \mathbb{Z} \cup \overline{\infty}$$
$$\overline{\mathbb{Z}} \equiv \mathbb{Z} \cup \pm\infty^+$$
$$\widehat{\mathbb{Q}} \equiv \mathbb{Q} \cup \overline{\infty}$$
$$\overline{\mathbb{Q}} \equiv \mathbb{Q} \cup \pm\infty^+$$

Some properties of these numbers change when they are extended this way. For instance, in the extended integers, the sum of an integer and a noninteger may be an integer.

The question may arise whether ∞ is even or odd. This depends on definitions. For example:

- If we define an even number as one which can be expressed in the form $2n$ where n is an integer, then ∞ is even, since $\infty = 2\infty$ and ∞ is an integer.

- If we define an odd number as an integer which is not even, the obviously ∞ is not odd.

- If we define an odd number as one which can be expressed in the form $2n + 1$ where n is an integer, then ∞ is odd, since $\infty = 2\infty + 1$ and ∞ is an integer.

Complex infinite element extensions

We now examine three methods of adding infinite elements to the complex numbers: the ***single projectively extended*** complex numbers, the ***double projectively extended*** complex numbers, and the ***affinely extended complex numbers***. Only the first of these methods is customarily defined in conventional mathematics.

The ***single projectively extended complex numbers***, commonly called the ***Riemann sphere***, adds a single infinite element, denoted ∞ ∞ or $\widehat{\infty}$, to the complex numbers. This system and is denoted $\widehat{\mathbb{C}}$, $P^1(\mathbb{C})$, or $\mathbb{CP}^1$. See [WRS].

Figure 14, which shows how each projectively extended real number is mapped to a point on a circle, also shows how each single projectively extended complex number is mapped to a point on a sphere, if we regard the line as any cross section of the complex plane through the origin and the circle as a cross section of the sphere. In the complex case, r is the radius from the origin, and the polar angle is perpendicular to the paper. This system is called single because it regards the complex numbers as a single complex dimension, rather than two real dimensions.

The ***double projectively extended complex numbers*** uncoils $\widehat{\infty}$ by adding a distinct infinite element for each pair of supplemental polar angles, i.e. a unique infinite element for each r in $[0, \pi)$. Each infinite element is called a ***directed infinity***. Since each infinite element is equal to its negative, it can be denoted $e^{ir}\overline{\infty}$. This system is denoted $\widetilde{\mathbb{C}}$.

Figure 15, which shows how each projectively extended real number is mapped to a pair of antipodal points on a circle, also shows how each double projectively extended complex number is mapped to a pair of points on a sphere. This system is called double because it regards the complex plane as a two-dimensional real projective plane with an associated complex arithmetic. See [WPP].

The ***affinely extended complex numbers*** further uncoils $\widehat{\infty}$ and $\overline{\infty}$ by adding a distinct infinite element for each polar angle, i.e. a unique directed infinite element for each r in $[0, 2\pi)$. An infinite element can be denoted $e^{ir}\infty$ or $e^{ir}\infty^+$, where $\infty \equiv \infty^+ \equiv \left|\frac{1}{0}\right|$. This system is denoted $\overline{\mathbb{C}}$. Figure 18, which shows how each affinely extended real number is mapped to a point on a semicircle, also shows how each affinely extended complex number is mapped to a point on a hemisphere.

In $\overline{\mathbb{C}}$, distribution holds:

$$\infty e^{ir} + \infty e^{ir} = 2\infty e^{ir} = \infty e^{ir}.$$

But in $\widetilde{\mathbb{C}}$ and $\widehat{\mathbb{C}}$, distribution does not hold:

$$\overline{\infty}e^{ir} + \overline{\infty}e^{ir} = \overline{\infty}e^{ir} - \overline{\infty}e^{ir} = \not\infty \neq 2\overline{\infty}e^{ir} = \overline{\infty}e^{ir}$$

$$\widehat{\infty} + \widehat{\infty} = \widehat{\infty} - \widehat{\infty} = \not\infty \neq 2\widehat{\infty} = \widehat{\infty}$$

Table 22 shows compares selected arithmetic operations in these three complex infinite element extensions.

TABLE 22: **Arithmetic operations in the single projectively extended complex numbers, the double projectively extended complex numbers, and the affinely extended complex numbers**

In this table:
a and d are finite complex, b and c are perfinite complex,
p is finite positive real, q is perfinite positive real,
r and s are real.
Complex numbers are given in polar form re^{ir} since rectangular form $a + bi$ does not properly distinguish infinite elements.

$\widehat{\mathbb{C}}$	$\widetilde{\mathbb{C}}$	$\overline{\mathbb{C}}$
$\mathbb{C} \cup \widehat{\infty}$	$\mathbb{C} \cup \overline{\infty}e^{i\mathbb{R}}$	$\mathbb{C} \cup \infty^+ e^{i\mathbb{R}}$
$e^{ir}\widehat{\infty} = \widehat{\infty}$	$e^{ir}\overline{\infty}$ is unique for $r \in [0, \pi)$	$e^{ir}\infty^+$ is unique for $r \in [0, 2\pi)$
$\widehat{\infty} \equiv \dfrac{1}{0}$	$\overline{\infty} \equiv \pm\left\lvert\dfrac{1}{0}\right\rvert$	$\infty^+ \equiv \left\lvert\dfrac{1}{0}\right\rvert$
$\dfrac{1}{0} = \widehat{\infty}$	$\dfrac{1}{0} = \overline{\infty}e^{i\mathbb{R}}$	$\dfrac{1}{0} = \infty^+ e^{i\mathbb{R}}$
$+\widehat{\infty} = -\widehat{\infty} = i\widehat{\infty} = -i\widehat{\infty}$	$+\overline{\infty} = -\overline{\infty} \neq i\overline{\infty} = -i\overline{\infty}$	$+\infty^+ \neq -\infty^+ \neq i\infty^+ \neq -i\infty^+$
$e^{ir}\widehat{\infty} = \widehat{\infty}$	$e^{ir}\overline{\infty} = e^{i(r+\pi)}\overline{\infty}$	$e^{ir}\infty^+ = e^{i(r+2\pi)}\infty^+$
$a + \widehat{\infty} = \widehat{\infty}$	$a + e^{ir}\overline{\infty} = \pm e^{ir}\overline{\infty}$	$a + e^{ir}\infty^+ = e^{ir}\infty^+$

$\widehat{\infty}+\widehat{\infty}=\widehat{\infty}-\widehat{\infty}=\not\infty$	$e^{ir}\overline{\infty}+e^{is}\overline{\infty}=\left(\pm e^{ir}+\pm e^{is}\right)\overline{\infty}$	$e^{ir}\infty^{+}+e^{is}\infty^{+}=\left(e^{ir}+e^{is}\right)\infty$
$b\widehat{\infty}=\widehat{\infty}$	$qe^{ir}\left(e^{is}\overline{\infty}\right)=\pm e^{i(r+s)}\overline{\infty}$	$qe^{ir}\left(e^{is}\infty^{+}\right)=e^{i(r+s)}\infty^{+}$
$0\widehat{\infty}=\not\infty$	$0e^{ir}\overline{\infty}=\not\infty$	$0e^{ir}\infty^{+}=\not\infty$
$\widehat{\infty}\cdot\widehat{\infty}=\widehat{\infty}$	$\left(e^{ir}\overline{\infty}\right)\cdot\left(e^{is}\overline{\infty}\right)=\pm e^{i(r+s)}\overline{\infty}$	$\left(e^{ir}\infty^{+}\right)\cdot\left(e^{is}\infty^{+}\right)=e^{i(r+s)}\infty$
$\dfrac{\widehat{\infty}}{b}=\widehat{\infty}$	$\dfrac{e^{ir}\overline{\infty}}{e^{is}q}=\pm e^{i(r-s)}\overline{\infty}$	$\dfrac{e^{ir}\infty^{+}}{e^{is}q}=e^{i(r-s)}\infty^{+}$
$\dfrac{b}{\widehat{\infty}}=0$	$\dfrac{b}{e^{ir}\overline{\infty}}=0$	$\dfrac{b}{e^{ir}\infty^{+}}=0$
$\dfrac{b}{0}=\widehat{\infty}$	$\dfrac{b}{0}=U\overline{\infty}=e^{\pi iI}\overline{\infty}$	$\dfrac{b}{0}=U\infty^{+}=e^{2\pi iI}\infty^{+}$
$\dfrac{\widehat{\infty}}{0}=\widehat{\infty}$	$\dfrac{e^{ir}\overline{\infty}}{0}=e^{\pi iI}\overline{\infty}$	$\dfrac{e^{ir}\infty^{+}}{0}=e^{2\pi iI}\infty^{+}$
$\dfrac{0}{\widehat{\infty}}=0$	$\dfrac{0}{e^{ir}\overline{\infty}}=0$	$\dfrac{0}{e^{ir}\infty^{+}}=0$
$\dfrac{0}{0}=\not\infty$	$\dfrac{0}{0}=\not\infty$	$\dfrac{0}{0}=\not\infty$
$\dfrac{\widehat{\infty}}{\widehat{\infty}}=\not\infty$	$\dfrac{e^{ir}\overline{\infty}}{e^{is}\overline{\infty}}=e^{i(r-s)}\widehat{\mathbb{R}}$	$\dfrac{e^{ir}\infty^{+}}{e^{is}\infty^{+}}=e^{i(r-s)}\lvert\mathbb{R}\rvert$
$\dfrac{1}{x}$, $\tan x$ are continuous at $x=0$	$\dfrac{1}{x}$, $\tan x$ are continuous at $x=0$ in the real direction, discontinuous in other directions	$\dfrac{1}{x}$, $\tan x$ are discontinuous at $x=0$
$(b+c)\widehat{\infty}\subset b\widehat{\infty}+c\widehat{\infty}$ for $b+c\neq 0$	$(b+c)\overline{\infty}e^{ir}\subset b\overline{\infty}e^{ir}+c\overline{\infty}e^{ir}$ for $b+c\neq 0$ and $\dfrac{b}{c}\notin\mathbb{R}$	$(b+c)\infty^{+}e^{ir}\subset b\infty^{+}e^{ir}+c\infty e^{ir}$ for $b+c\neq 0$ and $\dfrac{b}{c}\notin\lvert\mathbb{R}\rvert$
$(b+0)\widehat{\infty}\subset b\widehat{\infty}+0\cdot\widehat{\infty}$	$(b+0)\overline{\infty}e^{ir}\subset b\overline{\infty}e^{ir}+0\cdot\overline{\infty}e^{ir}$	$(b+0)\infty^{+}e^{ir}\subset b\infty^{+}e^{ir}+0\cdot\infty^{+}e^{ir}$
$(b+\widehat{\infty})\widehat{\infty}=b\widehat{\infty}+\widehat{\infty}\cdot\widehat{\infty}$	$(b+\overline{\infty}e^{is})\overline{\infty}e^{ir}\subset b\overline{\infty}e^{ir}+\overline{\infty}e^{is}\overline{\infty}e^{ir}$ for $\dfrac{b}{e^{is}}\notin\mathbb{R}$	$(b+\infty^{+}e^{is})\infty^{+}e^{ir}\subset b\infty^{+}e^{ir}+\infty^{+}e^{is}\infty^{+}e^{ir}$ for $\dfrac{b}{e^{is}}\notin\lvert\mathbb{R}\rvert$

$(0+\widehat{\infty})\widehat{\infty} \subset$ $\quad 0\cdot\widehat{\infty}+\widehat{\infty}\cdot\widehat{\infty}$	$(0+\overline{\infty}e^{is})\overline{\infty}e^{ir} \subset$ $\quad 0\cdot\overline{\infty}e^{ir}+\overline{\infty}e^{is}\overline{\infty}e^{ir}$	$(0+\infty^+e^{is})\infty^+e^{ir} \subset$ $\quad 0\cdot\infty e^{ir}+\infty e^{is}\infty e^{ir}$
$b \subset \frac{b}{0}\cdot 0$	$b \subset \frac{b}{0}\cdot 0$	$b \subset \frac{b}{0}\cdot 0$
$b \subset \frac{b}{\widehat{\infty}}\cdot\widehat{\infty}$	$b \subset \frac{b}{\overline{\infty}e^{ir}}\cdot\overline{\infty}e^{ir}$	$b \subset \frac{b}{\infty^+e^{ir}}\cdot\infty^+e^{ir}$
$a \subset (a+\widehat{\infty})-\widehat{\infty}$	$a \subset \left(a+\overline{\infty}e^{ir}\right)-\overline{\infty}e^{ir}$	$a \subset \left(a+\infty^+e^{ir}\right)-\infty^+e^{ir}$
$e^{\widehat{\infty}} = \{0,\widehat{\infty}\}$	$e^{\overline{\infty}} = \{0,\overline{\infty}e^{i\mathbb{R}}\}$	$e^{\infty^+} = \infty^+e^{i\mathbb{R}}$
$e^{-\widehat{\infty}} = \{0,\widehat{\infty}\}$	$e^{-\overline{\infty}} = \{0,\overline{\infty}e^{i\mathbb{R}}\}$	$e^{-\infty^+} = 0$
$e^{i\widehat{\infty}} = \{0,\widehat{\infty}\}$	$e^{i\overline{\infty}} = \{0,\overline{\infty}e^{i\mathbb{R}}\}$	$e^{i\infty^+} = \{0,\infty^+e^{i\mathbb{R}}\}$
$e^{e^{ir}\widehat{\infty}} = \{0,\widehat{\infty}\}$	$e^{e^{ir}\overline{\infty}} = \{0,\overline{\infty}e^{i\mathbb{R}}\}$	$e^{e^{ir}\infty^+} =$ $\quad\infty^+e^{i\mathbb{R}}$ for $\operatorname{Re} e^{ir} > 0$, $\quad 0$ for $\operatorname{Re} e^{ir} < 0$ $\quad\{0,\infty^+e^{i\mathbb{R}}\}$ for $\operatorname{Re} e^{ir} = 0$

In $\overline{\mathbb{R}}$, we have $e^{\infty^+} = \infty^+$ and $e^{-\infty^+} = 0$. In $\overline{\mathbb{C}}$,

$$\infty^+ = r\infty^+ = \infty^+ + s + it$$

for real r, s, t and $r > 0$, and

$$e^{\infty^+} = e^{\infty^+ + s + it} = e^{\infty^+ + s}e^{it} = e^{\infty^+}e^{it}.$$

The value $e^{\infty^+} = \infty^+e^{i\mathbb{R}}$ satisfies these conditions. Similarly $e^{-\infty^+} = e^{-\infty^+}e^{it}$ is satisfied by $e^{-\infty^+} = 0$, and $e^{i\infty^+} = e^{i\infty^+}e^{s}$ is satisfied by $e^{i\infty^+} = \mathbb{C}^*$.

In $\widehat{\mathbb{R}}$, we have $e^{\overline{\infty}} = \{0,\overline{\infty}\}$. In $\widetilde{\mathbb{C}}$,

$$\overline{\infty} = r\overline{\infty} = \overline{\infty} + s + it$$

for real r, s, t, and

$$e^{\overline{\infty}} = e^{\overline{\infty} + s + it} = e^{\overline{\infty} + s}e^{it} = e^{\overline{\infty}}e^{it}.$$

The value $e^{\overline{\infty}} = \left\{0,\overline{\infty}e^{ir}\right\}$ satisfies these conditions. Since $e^{-\overline{\infty}} = e^{\overline{\infty}}$, we have $e^{-\overline{\infty}} = \left\{0,\overline{\infty}e^{ir}\right\}$. Similarly, $e^{i\overline{\infty}} = e^{i\overline{\infty}}e^{s}$ is satisfied by $e^{i\overline{\infty}} = \mathbb{C}^*$.

In $\widehat{\mathbb{C}}$,

$$\widehat{\infty} = z\widehat{\infty} = \widehat{\infty} + s + it$$

for real s, t and nonzero complex z, and

$$e^{\widehat{\infty}} = e^{\widehat{\infty} + s + it} = e^{\widehat{\infty}}e^{it} = e^{-\widehat{\infty}}e^{it} = e^{i\widehat{\infty}}e^{it}.$$

The value $e^{\widehat{\infty}} = \widehat{\mathbb{C}}$ satisfies these conditions.

$\ln 1 = 2\mathbb{Z}\pi i$	$\ln 1 = 2\mathbb{Z}\pi i$	$\ln 1 = 2\mathbb{Z}\pi i$
$\ln(-1) = (2\mathbb{Z}+1)\pi i$	$\ln(-1) = (2\mathbb{Z}+1)\pi i$	$\ln(-1) = (2\mathbb{Z}+1)\pi i$
$\ln i = (2\mathbb{Z}+\frac{1}{2})\pi i$	$\ln i = (2\mathbb{Z}+\frac{1}{2})\pi i$	$\ln i = (2\mathbb{Z}+\frac{1}{2})\pi i$
$\ln\widehat{\infty} = \widehat{\infty}$	$\ln\overline{\infty} = \ln(-\overline{\infty}) =$ $\quad\overline{\infty} + \mathbb{Z}\pi i = \overline{\infty}$	$\ln\infty^+ =$ $\quad\infty^+ + 2\mathbb{Z}\pi i = \infty$
$\ln(-\widehat{\infty}) = \ln\widehat{\infty} = \widehat{\infty}$	$\ln(-\overline{\infty}) = \ln\overline{\infty} = \overline{\infty}$	$\ln(-\infty^+) =$

$$\infty^{+} + (2\mathbb{Z}+1)\pi i = \infty^{+}$$

$\ln(i\widehat{\infty}) = \ln\widehat{\infty} = \widehat{\infty}$	$\ln(i\overline{\infty}) = \ln(-i\overline{\infty}) = \overline{\infty} + (\mathbb{Z}+\frac{1}{2})\pi i = \overline{\infty}$	$\ln(i\infty^{+}) = \infty^{+} + (2\mathbb{Z}+\frac{1}{2})\pi i = \infty^{+}$
$\ln 0 = \widehat{\infty}$	$\ln 0 = \overline{\infty}$	$\ln 0 = -\infty^{+}$

In nonextended arithmetic:

$$\ln 1 = 2\mathbb{Z}\pi i \qquad \ln(\pm 1) = 2\mathbb{Z}\pi i + (2\mathbb{Z}+1)\pi i = \mathbb{Z}\pi i$$

$$\ln(-1) = (2\mathbb{Z}+1)\pi i \qquad \ln i = (2\mathbb{Z}+\tfrac{1}{2})\pi i$$

In $\overline{\mathbb{C}}$:

$$\ln\infty^{+} = \ln(1\cdot\infty^{+}) = \infty^{+} + 2\mathbb{Z}\pi i = \infty^{+} \qquad \ln(i\infty^{+}) = \infty^{+} + (2\mathbb{Z}+\tfrac{1}{2})\pi i = \infty^{+}$$

$$\ln(-\infty^{+}) = \ln(-1\cdot\infty^{+}) = \infty^{+} + (2\mathbb{Z}+1)\pi i = \infty^{+} \qquad \ln 0 = -\infty^{+}$$

In $\widetilde{\mathbb{C}}$:

$$\ln\overline{\infty} = \ln(-\overline{\infty}) = \overline{\infty} + \mathbb{Z}\pi i = \overline{\infty} \qquad \ln 0 = -\overline{\infty} = \overline{\infty}$$

$$\ln(i\overline{\infty}) = \overline{\infty} + (2\mathbb{Z}+\tfrac{1}{2})\pi i = \overline{\infty}$$

In $\widehat{\mathbb{C}}$:

$$\ln\widehat{\infty} = \ln(\widehat{\infty}e^{i\mathbb{R}}) = \widehat{\infty}$$

$$\ln 0 = -\widehat{\infty} = \widehat{\infty}$$

$\sqrt{\widehat{\infty}} = \widehat{\infty}$	$\sqrt{\overline{\infty}} = \overline{\infty}$	$\sqrt{\infty^{+}} = \pm\infty^{+}$
$\lvert\widehat{\infty}\rvert = \widehat{\infty}$	$\lvert\overline{\infty}e^{ir}\rvert = \overline{\infty}$	$\lvert\infty^{+}e^{ir}\rvert = \infty^{+}$

Indeterminate expressions and the full class

Conventional infinite element extensions leave indeterminate expressions such as $\frac{0}{0}$ and $\infty - \infty$ undefined, since they cannot handle multivalued expressions. Any assignment of such expressions to single values creates inconsistencies. For example, if we define $0\cdot\infty$ as 1, then associativity of multiplication fails: $2\cdot(0\cdot\infty) = 2$, but $(2\cdot 0)\cdot\infty = 1$. If we define $(+\infty)+(-\infty)$ as 0, then associativity of addition fails: $1+[(+\infty)+(-\infty)] = 1$, but $[1+(+\infty)]+(-\infty) = 0$.

This means that infinite element extensions, whether numerisitic or conventional, under addition or multiplication or both, do not satisfy the axioms of conventional algebraic structures such as group, ring, or field, since there is no single valued binary operation which satisfies the respective axioms and is defined for all elements. For example, the affinely extended real numbers are not even a semigroup under addition, since $(+\infty)+(-\infty)$ is either undefined (conventional) or $\not\infty$ (numeristic).

In numeristics, indeterminate expressions play an important role of connecting classes. For example, even though 0 is a natural number, $\frac{0}{0}$ includes nonintegral, irrational, and imaginary elements.

The above tables state that indeterminate expressions such as $\frac{0}{0}$ and $\infty - \infty$ are equal to $\not\infty$, but we must be aware that, while they include all values in the elementary classes $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ that we have considered so far, they may not include absolutely all numbers. For instance, there are classes in which there exist a such that $0a \neq 0$, so $a \notin \frac{0}{0}$ but $a \in \not\infty$.

To clarify this situation, we can use **universe notation** (p. 61) or intersection: for example, we can say

$$(U = \mathbb{R}) \qquad \frac{0}{0} = \not\infty = \mathbb{R}$$

or

$$\frac{0}{0} \cap \mathbb{R} = \not\infty \cap \mathbb{R} = \mathbb{R}.$$

A similar situation may occur with determinate expressions, such as the different interpretations of $\sqrt{-1}$ in the complex numbers $\mathbb{C}$ and the quaternions $\mathbb{H}$:

$$\begin{aligned} (U = \mathbb{C}) &\qquad \sqrt{-1} = \sqrt{-1} \cap \mathbb{C} = \pm i \\ (U = \mathbb{H}) &\qquad \sqrt{-1} = \sqrt{-1} \cap \mathbb{H} = \{ai + bj + ck \mid a, b, c \in \mathbb{R} \wedge a^2 + b^2 + c^2 = 1\} \\ &\qquad \phantom{\sqrt{-1}} = ie_{1}^{k\mathbb{R}e_{2}^{i\mathbb{R}}} \end{aligned}$$

Tangent scale plots

A ***tangent scale*** plot is a way of visualizing infinite values in both domain and range of a function. Tangent scale is analogous to logarithmic scale. Tangent scale uses the arctangent function to map the interval $[-\infty, +\infty]$ to $\left[-\frac{\pi}{2}, +\frac{\pi}{2}\right]$.

A ***rectangular tangent scale*** maps the infinite plane to a finite square. Given a function plotted in rectangular coordinates, the abscissa and ordinate are condensed to a finite size with an arctangent transformation. If x and

$y = f(x)$ are the variables of the function, the transformed coordinates in the tangent scale plot are $X \equiv \tan^{-1} x$ and $Y \equiv \tan^{-1} y$ respectively.

Figures 23–26 show several examples of rectangular tangent scale plots. The thin lines show the axes and limits of the graph region.

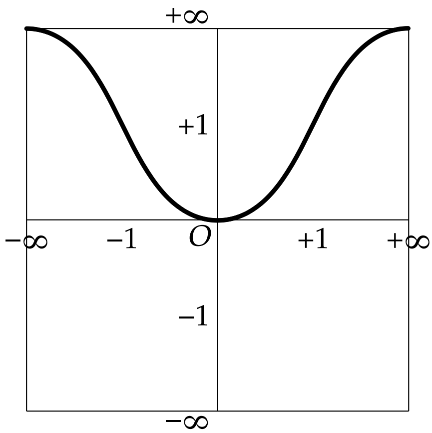

FIG. 23:
Tangent scale
plot of $y = x^2$

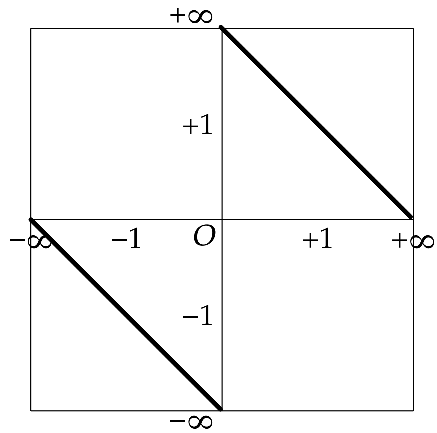

FIG. 24:
Tangent scale
plot of $y = \frac{1}{x}$

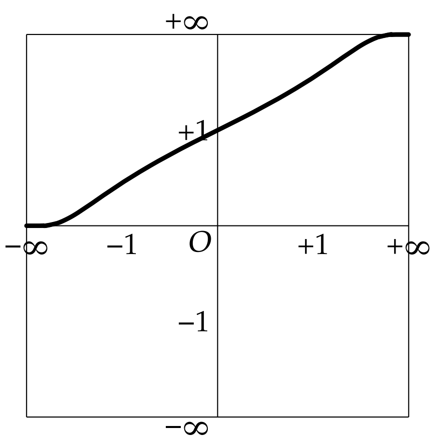

FIG. 25:
Tangent scale
plot of $y = e^x$

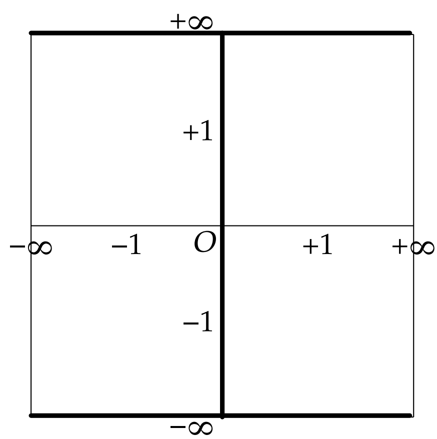

FIG. 26:
Tangent scale
plot of $y = \dfrac{x}{0}$

A ***polar tangent scale*** maps the infinite plane to a finite circle. Given a function plotted in polar coordinates, the radial coordinate is condensed to a finite size with an arctangent transformation. The angular coordinate is not changed. If θ and $r = f(\theta)$ are the variables of the function, the transformed coordinates in the tangent scale plot are θ and $R \equiv \tan^{-1} r$ respectively.

Figures 27–30 show several examples of polar tangent scale plots. The thin lines show the axes, and the thin circle shows the limit of the graph region.

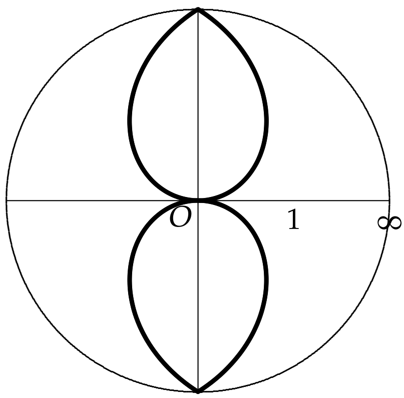

FIG. 27:
Tangent scale
plot of $r = \tan\theta$

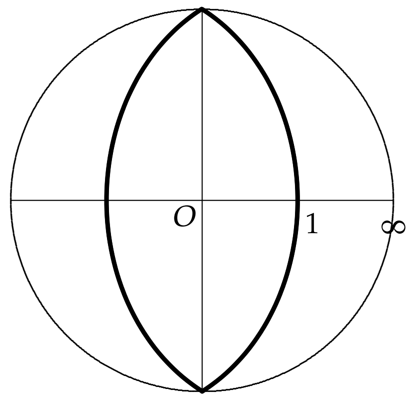

FIG. 28:
Tangent scale
plot of $r = \sec\theta$

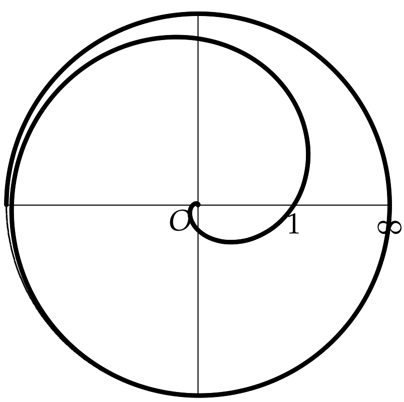

FIG. 29:
Tangent scale
plot of $r = e^{\theta}$

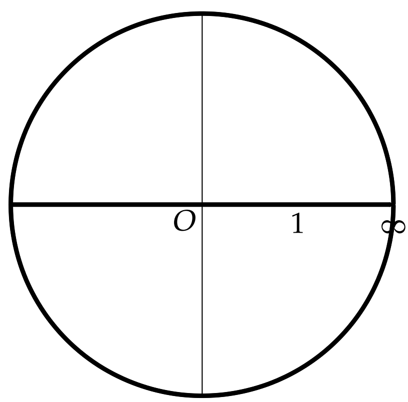

FIG. 30:
Tangent scale
plot of $r = \dfrac{\theta}{0}$

FURTHER NUMERISTIC CALCULATIONS

Signum function

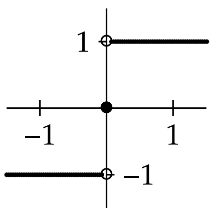

FIG. 31:
Conventional signum function $f(x) = \operatorname{sgn} x$

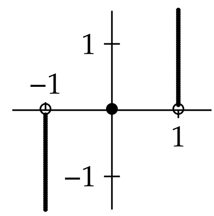

FIG. 32:
Conventional signum function inverse $f^{-1}(x) = \operatorname{sgn}^{-1} x$

Figure 31 shows the usual form of the signum (or sign) function $\operatorname{sgn} x$, which can be defined by either

$$f(x) = \operatorname{sgn} x = \begin{cases} \dfrac{x}{|x|} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

or

$$f(x) = \operatorname{sgn} x = \begin{cases} -1 & \text{for } x < 0 \\ 0 & \text{for } x = 0 \\ +1 & \text{for } x > 0 \end{cases} .$$

Figure 32 shows the inverse $\operatorname{sgn}^{-1} x$, which is not single valued, and is therefore not a function in the conventional sense, but is a function in the numeristic sense. It can also be expressed as

$$f^{-1}(x) = \operatorname{sgn}^{-1} x = \begin{cases} \mathbb{R}^- & \text{for } x = -1 \\ 0 & \text{for } x = 0 \\ \mathbb{R}^+ & \text{for } x = +1 \\ \{\} & \text{otherwise} \end{cases} .$$

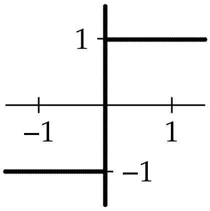

FIG. 33:
Alternate signum function $f(x) = \mathrm{sgn}_2\, x$

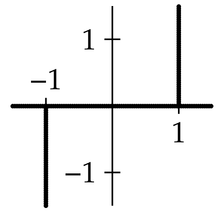

FIG. 34:
Alternate signum function inverse $f^{-1}(x) = \mathrm{sgn}_2^{-1}\, x$

Figure 33 shows a revised form of the signum function, $\mathrm{sgn}_2\, x$, defined as

$$\mathrm{sgn}_2\, x = \frac{x}{|x|}$$

for all x, which can also be expressed as

$$g(x) = \mathrm{sgn}_2\, x = \begin{cases} -1 & \text{for } x < 0 \\ \mathbb{R} & \text{for } x = 0 \\ +1 & \text{for } x > 0 \end{cases}.$$

The vertical line at $x = 0$ shows that the value at this point is the indeterminate class $\mathrm{sgn}_2\, 0 = \frac{0}{0}$.

Figure 34 shows the inverse, $\mathrm{sgn}_2^{-1}\, x$, which can be expressed as

$$\mathrm{sgn}_2^{-1}(x) = \begin{cases} -|\mathbb{R}| & \text{for } x = -1 \\ |\mathbb{R}| & \text{for } x = +1 \\ 0 & \text{otherwise} \end{cases}.$$

Neither $\mathrm{sgn}_2\, x$ nor $\mathrm{sgn}_2^{-1}\, x$ are single valued and therefore cannot be conventional functions, but both are numeristic functions.

Argument function

For converting rectangular coordinates to polar coordinates, it is common in the conventional literature to see the formula $\theta = \tan^{-1} \frac{y}{x}$. The intention of this formula is to find the angle θ subtended by the x axis and by the line from the origin to the point (x, y) which is given in rectangular coordinates.

The problem is that this formula only works partially. While $\tan\theta = \frac{y}{x}$ is valid, $\tan^{-1}\frac{y}{x}$ returns too many values—it returns the correct values $\theta + 2\pi\mathbb{Z}$, but it also returns the incorrect values $\theta + 2\pi\mathbb{Z} + \pi$.

Many programming languages include a function, often called atan2, which calculates the principal value of this angle correctly. This is also single valued. The following numeristic definition of this function uses class intersection to return all the correct values without incorrect values:

$$\operatorname{atan2}(x, y) \equiv \cos^{-1}\frac{y}{\sqrt{x^2+y^2}} \cap \sin^{-1}\frac{x}{\sqrt{x^2+y^2}}.$$

This uses the fact that $\sin^{-1}$ and $\cos^{-1}$ each return angles in two quadrants, and the intersection of them returns the original angle.

The argument of a complex number can thus be defined:

$$\arg z \equiv \operatorname{atan2}(\operatorname{Re} z, \operatorname{Im} z) = \cos^{-1}\frac{\operatorname{Im} z}{|z|} \cap \sin^{-1}\frac{\operatorname{Re} z}{|z|}.$$

At the origin, the argument is indeterminate:

$$\arg 0 = \cos^{-1}\frac{0}{0} \cap \sin^{-1}\frac{0}{0} = \not\infty.$$

Solution of $x = rx$

As a demonstration of numerisitic techniques, we consider the equation $x = rx$. A conventional solution could run as follows:

$$x - rx = 0$$
$$x(1 - r) = 0,$$

from which we conclude that $x = 0$, except for $r = 1$, where x is indeterminate.

This is not a complete numeristic solution, since it assumes that for any a and b, $a - a = 0$, and that $ab = 0$ implies either $a = 0$ or $b = 0$ or both. In numeristics, both of these assumptions are valid only for finite a and b.

We now examine a numeristic solution, which adds all the neglected cases.

1. $r = 1$: $x = \not\infty$.

2. $r = 0$:

    a. x finite: $x = 0$.

    b. x infinite: $x = \infty$. Here we allow "=" to also mean "$\supseteq$" in the original equation.

3. Other finite r:

    a. x finite:

$$\begin{aligned} x - rx &= 0 \\ x(1-r) &= 0 \\ x &= 0 \end{aligned}$$

    b. x infinite: $x = \infty$.

4. Infinite r: Invert the equation and follow the case $r = 0$:

$$\begin{aligned} \frac{1}{x} &= 0\frac{1}{x} \\ \frac{1}{x} &= 0, \infty \\ x &= 0, \infty \end{aligned}$$

Singular matrices

The inverse of a 2×2 matrix is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

We can use infinite elements to apply this to a singular matrix:

$$\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} = \frac{1}{0} \begin{pmatrix} 2 & -2 \\ -1 & 1 \end{pmatrix}.$$

In the projectively extended real numbers, this yields

$$\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} \infty & \infty \\ \infty & \infty \end{pmatrix} = \infty \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

while in the affinely extended real numbers, this is

$$\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} \pm\infty & \mp\infty \\ \mp\infty & \pm\infty \end{pmatrix} = \pm\infty \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

The product of the original matrix and its inverse are

$$\infty\begin{pmatrix}1 & 2\\ 1 & 2\end{pmatrix}\begin{pmatrix}2 & -2\\ -1 & 1\end{pmatrix} = \infty\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix} = \begin{pmatrix}\not\infty_{11} & \not\infty_{21}\\ \not\infty_{12} & \not\infty_{22}\end{pmatrix} = \mathscr{M}_{2,2}(\mathbb{R}).$$

We can also use determinants of singular matrices to solve degenerate cases of simultaneous equations.

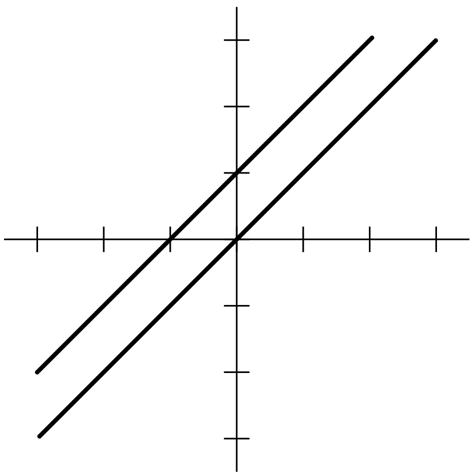

FIG. 35: Parallel simultaneous equations $x - y = -1$ and $x - y = 0$

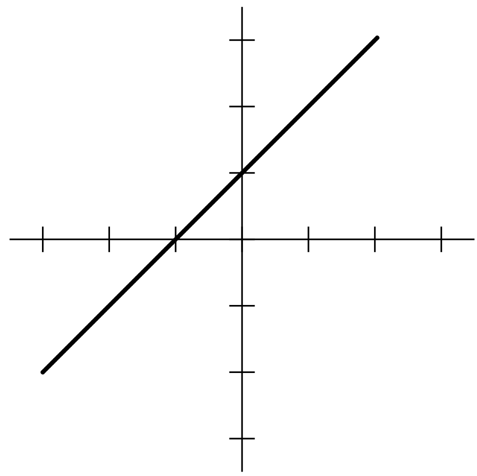

FIG. 36: Coincident simultaneous equations $x - y = -1$ and $2x - 2y = -2$

Figure 35 shows a system of two equations whose graphs are parallel. The solution by Cramer's rule is

$$x = \frac{\begin{vmatrix}-1 & -1\\ 0 & -1\end{vmatrix}}{\begin{vmatrix}1 & -1\\ 1 & -1\end{vmatrix}} = \frac{2}{0} = \pm\infty$$

$$y = \frac{\begin{vmatrix}-1 & -1\\ 1 & 0\end{vmatrix}}{\begin{vmatrix}1 & -1\\ 1 & -1\end{vmatrix}} = \frac{2}{0} = \pm\infty$$

The algebraic solution is one (projective) or two (affine) points at infinity, which agrees with the geometric solution as the meeting point of parallel lines.

Figure 36 shows a system of two equations whose graphs coincide. Again by Cramer's rule, the solution is

$$x = \frac{\begin{vmatrix} -1 & -1 \\ -2 & -2 \end{vmatrix}}{\begin{vmatrix} 1 & -1 \\ 2 & -2 \end{vmatrix}} = \frac{0}{0} = \mathbb{R}$$

$$y = \frac{\begin{vmatrix} -1 & -1 \\ 1 & -2 \end{vmatrix}}{\begin{vmatrix} 1 & -1 \\ 2 & -2 \end{vmatrix}} = \frac{0}{0} = \mathbb{R}$$

The algebratic solution shows that there is a solution for every value of x and y, which agrees with the geometric solution of the whole graph.

HOW NUMERISTICS HANDLES RUSSELL'S PARADOX

Russell's paradox (or antinomy) demonstrates a weakness of naive set theory, the predecessor to axiomatic set theories. In naive set theory, a set is allowed to be an element of itself. Russell's paradox considers the set S which contains all sets that do not contain themselves. The existence of such a set leads to a contradiction: if S contains itself, then by definition it does not contain itself, and if it does not contain itself, then again by definition it contains itself.

Axiomatic set theories avoid this problem in various ways. Zermelo-Fraenkel set theory restricts the elements that sets can contain. Bernays-Gödel set theory makes a distinction between a *class*, which can contain elements, and a *set*, which can be an element; all sets are classes, but not all classes are sets.

Mereology (p. 118) has a similar paradox, in which we consider an object T such that every object that is not a proper part of itself is a proper part of T. Assuming T exists, if T is a part of itself, by definition it is not, and if it is not, then by definition it is. Mereological theories avoid this through an involved set of axioms restricting the use of the proper part relation.

Numeristic classes differ fundamentally from sets and set-theoretic classes, primarily in their flat structure, by which a class containing a single element is identical to the element itself. Every numeristic class contains itself, and, if it is an element, it is an element of itself. There is no numeristic class that does not contain itself. The class of all classes that do not contain themselves is therefore the empty class, the class of all elements satisfying contradictory conditions. The empty class is unmanifest, whereas sets and mereological objects are always manifest.

ABSTRACTION

General considerations

The numeristic view of the **ultraprimitives** (p. 48) of infinity, unity, and zero establishes that there is nothing more abstract than these ultraprimitives. In this light, the attribute "abstract" in the term *abstract structures* is something of a misnomer.

We can consider an abstract structure to be a single valued operation or a pair of single valued operations which satisfies certain conditions. For example, the class of groups is the class of single valued operations satisfying the group axioms, each operation being restricted to an appropiate class of elements.

The operations in this type of abstract structure are always single valued and thus cannot include indeterminate forms, which as we have seen naturally arise in the arithmetic of even very simple classes.

Moreover, the class of axioms in any abstract structure is always finite, whereas Gödel's incompleteness theorem establishes that no finite class of axioms can give us complete knowledge of any system that includes the natural numbers, addition, multiplication, and quantifiers. This is one of the reasons that numeristics does not use axioms, instead relying on subjective and objective observation to establish rigor.

While there is often useful information in an abstract structure, the vision of the whole is lost. Numeristic ultraprimitives restore this vision, when they are experienced on the level of pure subjectivity.

We identify two kinds of abstraction:

- ***Abstraction of rules***: Categorizing or classifying a structure based on a portion of its properties; the type of abstraction in abstract algebra.

- ***Abstraction of reference***: Realization of the full extent of a structure, by transcending from object referral to subject referral; the type of abstraction that evolves in numeristics with the subjective experience of ultraprimitives, coupled with the objective experience of applications of numeristics.

In the first kind of abstraction, there is much freedom of classification and definition. This freedom should not lead us to believe that mathematical truth is relative or arbitrary, or that mathematics is a game or an activity in which we don't know what we are doing. We are free to choose definitions and axioms, but we cannot choose what system in nature this may describe. Conversely, we can choose which system in nature to investigate, and definitions to describe it, but we cannot arbitrarily choose the axioms which the system satisfies.

For example, in non-Euclidean geometry, changing the parallel postulate was really a discovery that the altered systems are consistent and apply to non-Euclidean surfaces. This involved alteration of *definitions* as well as axioms, such as redefining a line as a geodesic in elliptic geometry. The application of a system of geometry with no parallel lines to the surface of an ellipsoid is not arbitrary but a discovery about nature.

Numeristicity

A mathematical expression is ***numeristic*** if it expressed in terms of simple numbers, functions, and relations, using classes and infinite arithmetic when appropriate. The ultimate in numeristicity is the completely unified nature of the **ultraprimitives** (p. 48).

Numeristicity is a subjective quality, like elegance and rigor. It is comparable: one expression may be more numeristic than another.

As an example we consider three different models of the dihedral group D_n. This group consists of: r_k, rotations by k places; and s_k, reflections over a vertex k, of which only $\frac{n}{2}$ are unique if n is even.

- The first model of D_n is a semidirect product of cyclic groups $\mathbb{Z}_2$ and $\mathbb{Z}_n$. The two groups are composed of elements and operations that are synthetically defined, and the operation connecting them is also synthetically defined. This is the least numeristic model in this list.

- A more numeristic model uses matrix multiplication:

$$r_k = \begin{pmatrix} \cos\frac{2\pi k}{n} & -\sin\frac{2\pi k}{n} \\ \sin\frac{2\pi k}{n} & \cos\frac{2\pi k}{n} \end{pmatrix}$$

$$s_k = \begin{pmatrix} \cos\frac{2\pi k}{n} & \sin\frac{2\pi k}{n} \\ \sin\frac{2\pi k}{n} & -\cos\frac{2\pi k}{n} \end{pmatrix}$$

  This model uses trigonometric functions, which naturally occur in real and complex arithmetics, but they are held together by a synthetic matrix structure. This is more numeristic than the previous model.

- An even more numeristic model uses complex multiplcation:

$$r_k = e^{\frac{2\pi i k}{n}}$$

$$s_k = \left(e^{\frac{-2\pi i k}{n}}\right)^* e^{\frac{2\pi i k}{n}}$$

  where * denotes the complex conjugate

  This model uses only one complex expression and avoids synthetic structures. This is the most numeristic of the three models.

Further observations:

- Quaternions are highly numeristic, because they are an extension of the complex numbers.

- Structures such as groups, rings, and fields are less numeristic, because they include only some properties of numbers. These structures have little of the self-referral quality of numbers.

- Sets are the least numeristic structures in modern mathematics, because they completely lack self-referral. A set can map to a natural number, for instance, but it cannot *be* a natural number.

Multivalued abstract structures

It is quite straightforward to extend abstract structures numeristically, i.e. using infinite elements and classes.

Consider the example of the group of positive real numbers with multiplication. If zero is included, then the result is only a monoid (a group without inverses, or a semigroup with identity), since 0 has no inverse.

But if we allow multivalued multiplication, include an infinite element ∞, and modify the inverse axiom slightly, then 0 has an inverse: $0 \cdot \infty = \varnothing \supseteq 1$. This structure is then a multivalued group.

Another example is a pair of finite multivalued groups: $E \equiv \{0, 1, \infty\}$ with affinely extended multiplication, and $L \equiv \{-\infty, 0, +\infty\}$ with affinely extended addition. E and L are isomorphic, with the relation $L = \ln E$.

Some theorems of single valued abstract structures fail in their multivalued equivalents. For instance, the order of the multivalued group $A \equiv \{0, 1, -1, i, -i, \infty\}$ with Riemann multiplication is 6, but the order of the subgroup $B \equiv \{1, -1, i, -i\}$ of A is 4, which does not divide the order of A. The order of the coset (or coclass) $0B = \{0\}$ is 1, which is not the same as the order of B.

APPENDIX: OTHER FOUNDATIONAL THEORIES

Maharishi Vedic Mathematics

"Vedic" means referring to Veda, an ancient body of knowledge preserved in India. Vedic literature contains much that is scientific and mathematical. Maharishi refers to Maharishi Mahesh Yogi, who revived the knowlege and experience of Veda.

Maharishi Vedic Mathematics can be described as the mathematics of nature or the mathematics of pure consciousness. Pure consciousness is a state of pure subjectivity, independent of any objects of experience. Mathematically, it is focused on the **ultraprimitives** (p. 48) described above, especially zero.

The experience and understanding of pure consciousness is the inspiration for the subjective side of numeristics. The objective side is provided by modern mathematics. Numeristics is an attempt to bring these two together into a single compatible field of knowledge.

> Vedic Mathematics is the structuring dynamics of Natural Law; it spontaneously designs the source, course, and goal of Natural Law—the orderly theme of evolution.
>
> Vedic Mathematics, the system of maintaining absolute order, is the reality of self-referral consciousness, which, fully awake within itself, forms the structures of the Veda and Vedic Literature, and further proceeds to structure the fundamentals of creation in the most perfect, eternal, symmetrical order, and eternally glorifies creation on the ground of evolution.
>
> Vedic Mathematics is the quality of infinte organizing power inherent in the structure of self-referral consciousness—pure knowledge—the Veda.
>
> As Veda is structured in consciousness, Vedic Mathematics is the mathematics of consciousness; coexistence of simultaneity and sequence characterize Vedic Mathematics.

As self-Referral consciousness is the Unity (Saṁhitā) of observer (Ṛishi), process of observation (Devatā), and observed (Chhandas), Vedic Mathematics, being the mathematics of self-referral consciousness, is the mathematics of the relationship between these four values—Saṁhitā, Ṛishi, Devatā, Chhandas.

Vedic Mathematics is the mathematics of relationship; it is the science of relationship. Vedic Mathematics is the system of maintaining perfect order in all relationships.

Vedic Mathematics, being the mathematics of the order-generating principle of pure consicousness, it itself the mathematician, the process of deriving results, and the conclusion; whatever consciousness is and wherever consciousness is, there is the structure of Vedic Mathematics, the source of perfect order.

—Maharishi Mahesh Yogi, [M96, p. 338–340]

The mechanics of ordering have to be mathematically derived in order for the knowledge to be really complete, and also for the infinite organizing power of knowledge to be precisely, properly, and thoroughly applied so that life can be naturally lived on the ground of orderly evolution, so that nothing shadows life—nothing shadows the immortal, eternal continuum of bliss, which is the nature of the self-sufficient, self-referral quality of the Absolute Number, from where everything emerges, through which everything is sustained, and to which everything evolves.

Unless the Absolute Number is enlivened in conscious awareness, unless the all-dimensional value of the Absolute Number is lively on the level of *SmṚiti*—the lively level of memory that maintains order and steers the evolutionary process—the process of computation, the process of ordering, cannot be explained, and cannot be practically lived in life.

It is a joy to mention here that Transcendental Meditation is the process of maintaining connectedness with the Absolute Number—the source of the creative process—and through this programme, the precision of evolution and order in the process of creation is enlivened in human awareness, and is expressed in all thought, speech, and action.

—Maharishi Mahesh Yogi, [M96, p. 616–617]

> We admire the achievement of scientists in every field of modern science—Physics, Chemistry, etc.—who have presented in one symbol the entire knowledge of the ever-expanding universe. What remains to be achieved is that every mathematical symbol is able to whisper *I am Totality—Ahaṁ Brahmāsmi.*
>
> What remains to be acheived is that every physical expression of total knowledge (mathematical symbol), is awakened to feel and say and behave with the total competence of the WHOLENESS of knowledge; what remains to be achieved is the enlivenment of the structure of knowledge in which one single symbol of Mathematics is a self-sufficiently lively field of intelligence WHICH CAN OPERATE FROM WITHIN ITSELF and self-sufficiently perform with precision and order from the level of the entire creative potential of intelligence of Cosmic Life; what remains to be achieved is the realization of the reality *"Anoranīyān* is *mahato mahīyān"*—smaller than the smallest is bigger than the biggest; what remains to be realized is the enlivenment of the silent objectivity of the mathematical symbol into the lively dynamism of the intelligence within it; what remains to be achieved is just one step from the object to the subject—from the objectivity of the mathematical expression to the field of subjectivity within it, so that the mathematician can identify his self-referral intelligence with the structure of intelligence within the physical structure of the mathematical formula.
>
> This last step of knowledge, evolving from the objective quality of its structure to its lively subjective basis, is provided by my Vedic Mathematics; therefore my Vedic Approach (subjective approach), my approach of knowledge, my science of knowledge, through its subjective approach, has competence to enliven the spark of knowledge contained in any mathematical symbol (formula) of total knowledge from every field of modern science.
>
> —Maharishi Mahesh Yogi, [M95, p. 296–298]

Skolem's primitive recursive arithmetic

In the early 20th century, Thoralf Skolem developed a formal foundational system based on natural numbers, standard logic, and primitive recursion. Skolem's system later became known as primitive recursive arithmetic and was used by Kurt Gödel in the proofs of his famous incompleteness theorems. Skolem's primitive recursive arithmetic is developed in detail in [S23].

Skolem's primitives include the following:

- classical logic (first order logic with quantifiers)
- the natural number 1
- successor operation of a natural number
- equality of natural numbers
- primitive recursion

His definitions include:

- order relations (using the successor operation)
- multiplication of natural numbers (using recursion)
- divisibility of natural numbers
- subtraction and division of natural numbers (in those cases where the result is a natural number)
- greatest common divisor and least common multiple
- prime numbers

The theorems include:

- associative and commutative laws of addition
- trichotomy of order relations
- distributive law of multiplication over addition
- associative and commutative laws of multiplication
- prime factorization

Skolem constructs a formal foundational theory with numbers and without sets. Like **Weyl** (p. 104), using only a very few primitives, he develops a substantial numeric theory. One of his primitives is primitive recursion, which is a partial value of the basic self referral property of consciousness.

Skolem's primary goal in this paper is to develop a theory of natural numbers. He regards this theory as finitistic, in the sense that it contains no infinite elements and thus avoids the transfinite numbers of set theory. As he remarks in the concluding section of the paper: "[O]ne can doubt that there is any justification for the actual infinite or the transfinite." [S23, p. 332].

However, his system does generate an infinite number of finite numbers, and the number of referents of his primitive natural number 1 is infinite. His system therefore cannot count the number of numbers or the number of referents to any of the numbers.

The two quotes below, from Skolem's other works of this period, describe some aspects of the thinking which went into his creation of primitive recursive arithmetic.

> 7. ...[T]he notion that really matters in these logical investigations, namely "proposition following from certain assumptions", also is an inductive (recursive) one: the propositions we consider are those that are derivable by means of an *arbitrary finite number* of applications of the axioms. Thus the idea of the *arbitrary finite* is essential, and it would necessarily lead to a vicious circle if the notion "finite" were itself based, as in set theory, on certain axioms whose consistency would then in turn have to be investigated.
>
> Set theoreticians are usually of the opinion that the notion of integer should be defined and that the principle of mathematical induction should be proved. But it is clear that we cannot define or prove ad infinitum; sooner or later we come to something that is not further definable or provable. Our only concern, then, should be that the initial foundations be something immediately clear, natural, and not open to question. This condition is satisfied by the notion of integer and by inductive inferences, but it is decidedly not satisfied by set-theoretic axioms of the type of Zermelo's or anything else of that kind; if we were to accept the reduction of the former notions to the latter, the set-theoretic notions would have to be simpler than mathematical induction, and reasoning with them less open to question, but this runs entirely counter to the actual state of affairs.
>
> In a paper [Hi22] Hilbert makes the following remark about Poincaré's assertion that the principle of mathematical induction is not provable: "His objection that this principle could not be proved in any way other than by mathematical induction itself is unjustified and is refuted by my theory." But then the big question is whether we can prove this principle by means of simpler principles and *without using any property of finite expressions or formulas that in turn rests upon*

mathematical induction or is equivalent to it. It seems to me that this latter point was not sufficiently taken into consideration by Hilbert. For example, there is in his paper (bottom of page 170), for a lemma, a proof in which he makes use of the fact that in any arithmetic proof in which a certain sign occurs that sign must necessarily occur for a first time. Evident though this property may be on the basis of our perceptual intuition of finite expressions, a formal proof of it can surely be given only by means of mathematical induction. In set theory, at any rate, we go to the trouble of proving that every ordered finite set is well-ordered, that is, that every subset has a first element. Now why should we carefully prove this last proposition, but not the one above, which asserts that the corresponding property holds of finite arithmetic expressions occurring in proofs? Or is the use of this property not equivalent to an induction inference?

I do not go into Hilbert's paper in more detail, especially since I have seen only his first communication. I just want to add the following remark: It is odd to see that, since the attempt to find a foundation for arithmetic in set theory has not been very successful because of the logical difficulties inherent in the latter, attempts, and indeed very contrived ones, are now being made to find a different foundation for it—as if arithmetic had not already an adequate foundation in inductive inferences and recursive definitions.

8. So long as we are on purely axiomatic ground there is, of course, nothing special to be remarked concerning the principle of choice (though, as a matter of fact, new sets are *not* generated *univocally* by applications of this axiom); but if many mathematicians—indeed, I believe, most of them—do not want to accept the principle of choice, it is because they do not have an axiomatic conception of set theory at all. They think of sets as given by specification of arbitrary collections; but then they also demand that every set be definable. We can, after all, ask: What does it mean for a set to exist if it can perhaps never be defined? It seems clear that this existence can only be a manner of speaking, which can lead only to purely formal propositions—perhaps made up of very beautiful *words*— about objects *called* sets. But most mathematicians want mathematics to deal, ultimately with performable computing operations and not to consist of formal propositions about objects called this or that.

Concluding remark

The most important result above is that set-theoretic notions are relative. I had already communicated it orally to F. Bernstein in Göttingen in the winter of 1915–16. There are two reasons why I have not published anything about it until now: first, I have in the meantime been occupied with other problems; second, I believed that it was so clear that axiomatization in terms of sets was not a satisfactory ultimate foundation of mathematics that mathematicians would, for the most part, not be very much concerned with it. But in recent times I have seen to my surprise that so many mathematicians think that these axioms of set theory provide the ideal foundation for mathematics; therefore it seemed to me that the time had come to publish a critique.

—Thoralf Skolem, [S22, p. 299–301] (emphasis in the original)

I here permit myself a remark about the relation between the fundamental notions of logic and those of arithmetic. No matter whether we introduce the notion of propositional function in the first or the second way, we are confronted with the idea of the integer. For, even when the notion of propositional function is introduced axiomatically, we shall have to consider (for instance, in investigations concerning consistency) what we can derive by using the axioms an arbitrary finite number of times. On the other hand, it is not possible to characterize the number sequence logically without the notion of propositional function. For such a characterization must be equivalent to the principle of mathematical induction, and this reads as follows: If a propositional function $A(x)$ holds for $x = 1$ and if $A(x+1)$ is true whenever $A(x)$ is true, then $A(x)$ is true for every x. In signs, it takes the form

$$\prod_U \left(\overline{U(1)} + \sum_x U(x)\overline{U(x+1)} + \prod_y U(y) \right)$$

[in modern notation

$$(\forall U)\left[\neg\, U(1) \vee (\exists x)(U(x) \wedge U(x+1)) \vee (\forall y)U(y)\right].$$

This proposition clearly involves the totality of propositional functions. Therefore, the attempt to base the notions of logic upon those of arithmetic, or vice versa, seems to me to be mistaken. The foundations for both must be laid simultaneously and in an interrelated way.

—Thoralf Skolem, [S28, p. 517]

Weyl's foundational system of the continuum

Also in the early 20th century, Hermann Weyl developed a theory of the real numbers, which he intended as an alternative to set theory as a foundation of analysis (calculus). Weyl bases his theory of the real continuum on natural numbers, basic logical operations, and primitive recursion, without transfinite set theory or proof by contradiction. This system of the real continuum is developed in [W32].

Weyl's primitives include the following:

- classical logic (first order logic with quantifiers)
- sets which have only numbers, or ordered multiples of numbers, as elements
- the natural numbers
- successor operation of a natural number
- identity (equality)
- iteration (primitive recursion)

His definitions include:

- relations
- order relations (using the successor operation)
- multiplication of natural numbers (using recursion)
- cardinality of sets
- fractions and rational numbers
- zero and negative rational numbers
- addition, subtraction, and multiplication of rational numbers
- real numbers (as cuts of rational numbers)

- addition, subtraction, multiplication, and division of real numbers, excluding division by zero
- exponentiation of real numbers by natural numbers (by recursion)
- algebraic numbers
- complex numbers (as real number pairs)
- sequences, limits, and convergece
- infinite series and power series
- continuity
- function inverses

Weyl indicates that it is possible to define the exponential function, logarithms, differentiation, and integration in his system, but he does not actually define them.

The theorems include:

- associative and commutative laws of addition
- trichotomy of order relations
- distributive law of multiplication over addition
- associative and commutative laws of multiplication
- Cauchy convergence principle
- Heine-Borel theorem in the one dimensional case of real intervals

Like **Skolem** (p. 99), Weyl develops a substantial numeric theory using only a few primitives, which include natural numbers and primitive recursion. His theory has sets, but these sets include only numbers and ordered multiples of numbers, so there are no transfinite numbers. He defines division of real numbers but excludes division by zero. His system is less formal than most other foundational theories, which seems to be the result of his stated aim of providing a firm foundation for analysis. He anticipates numeristics by using the natural numbers as a primitive rather than defining them as sets.

The quotes below describe some aspects of the thinking which went into Weyl's creation of his theory of the continuum.

> It is not the purpose of this work to cover the "firm rock" on which the house of analysis is founded with a fake wooden structure of formalism—a structure which can fool the reader and, ultimately, the author into believing that it is the true foundation. Rather, I shall show that this house is to a large degree built on sand. I believe that I can replace this shifting foundation with pillars of enduring strength.
>
> —Hermann Weyl, [W87, p. 1]

> It is characteristic of every mathematical discipline that 1) it is based on a sphere of operation such as we have presupposed here from the beginning; that 2) the natural numbers along with the relation *S* [successor relation] which connects them are always associated with this sphere; and that 3) over and above this composite sphere of operations, a realm of new ideal objects, of sets and functional connections is erected by means of the mathematical process which may, if necessary, be repated arbitrarily often. The old explantion of mathematics as the doctrine of number and space has, in view of the more recent development of our science, been judged to be too narrow. But, clearly, even in such disciplines as pure geometry, analysis situs [topology], group theory, and so on, the natural numbers are, from the start, related to the objects under consideration. So from now on we shall assume that at least one category of object underlies our investigation and that at least one of these underlying categories is that of the natural numbers. If there is more than one such category, we should recall the observation in §1 that each blank of a judgment scheme of a primitive or derived relation is affiliated with its own definite category of object. If the underlying sphere of operation described at the beginning of ths paragraph is that of the natural numbers, without anything further being added, then we arrive at *pure number theory*, which forms the centerpiece of mathematics; its concepts and results are clearly of significance for *every* mathematical discipline.
>
> If the natural numbers belong to the sphere of operations, then a new, important, and specifically mathematical principle of definition joins those enumerated in §2; namely, the principle of it iteration (definition by complete induction) by virtue of which the natural numbers first come into contact with the objects of the remaining categories of the underlying sphere of operations (if there are any).
>
> —Hermann Weyl, [W87, p. 25–26] (emphasis in the original)

> [W]e are less certain than ever about the ultimate foundations of (logic and) mathematics; like everybody and everything in the world today, we have our "crisis". We have had it for nearly fifty years. Outwardly it does not seem to hamper our daily work, and yet I for one confess that it has had a considerable practical influence on my mathematical life: it directed my interests to fields I considered relatively "safe", and it has been a constant drain on my enthusiasm and determination with which I pursued my research work. The experience is probably shared by other mathematicians who are not indifferent to what their scientific endeavours mean in the contexts of man's whole caring and knowing, suffering and creative existence in the world.
>
> —Hermann Weyl, [W46, p. 13]

> The *circulus vitiosis* [vicious circle, of circular reasoning in the foundations of mathematics], which is cloaked by the hazy nature of the usual concept of set and function, but which we reveal here, is surely not an easily dispatched formal defect in the construction of analysis. Knowledge of its foundamental significance is something which, at this particular moment, cannot be conveyed to the reader by a lot of words. But the more distinctly the logical fabric of analysis is brought to givenness and the more deeply and completely the glance of consciousness penetrates it, the clearer it becomes that, given the current approach to foundational matters, every cell (so to speak) of this mighty organism is permeated by the poison of contradiction and that a thorough revision is necessary to remedy the situation.
>
> A "hierarchical" version of analysis is artificial and useless. It loses sight of its proper object, i.e. number (cf. note 24). Clearly we must take the other path—that is, we must restrict the existence concept to the basic categories (here, the natural and rational numbers) and must not apply it in connection with the system of properties and relations (or the sets, real numbers, and so on, corresponding to them). In other words, the only natural strategy is *to abide by the narrower iteration procedure.* Further, only this procedure guarantees too that all concepts and results, quantities and operations of such a "precision analysis" are to be grasped as idealizations of analogues in a mathematics of approximation operating with "round numbers." This is of crucial significance with regard to *applications.*
>
> —Hermann Weyl, [W87, p. 32] (emphasis in the original)

The concept of function has two historical roots. *First*, this concept was suggested by the "natural dependencies" which prevail in the material world—the dependencies which consist, on the one hand, in the fact that conditions and states of real things are variable over *time*, the paradigmatic independent variable, on the other hand, in the *causal* connections between action and consequence. The arithmetical-algebraic operations form a *second*, and entirely independent, source of the concept "function." For, in bygone days, analysis regarded a *function* as an expression formed from the independent variables by finitely many applications of four primary rules of arithmetic and a few elementary transcendental ones. Of course, these elementary operations have never been clearly and fully defined. And the historical development of mathematics has again and again pushed beyond boundaries which were drawn much too narrowly (even though those responsible for this development were not always entirely aware of what they were doing).

These two independent sources of the concept of function join together in the concept "law of nature." For in a law of nature, a natural dependence is represented as a function constructed in a purely conceptual-arithmetical way. Galileo's laws of falling bodies are the first great example. The modern development of mathematics has revealed that the algebraic principles of construction of earlier versions of analysis are much too narrow either for a general and logically natural construction of analysis or for the role which the concept "function" has to play in the formulation of the laws which govern material events. General logical principles of construction must replace the earlier algebraic ones. Renouncing such a construction altogether, as modern analysis (judging by the wording of its definitions) seems to have done, would mean losing oneself entirely in the fog; and, at the same time, the general notion of natural law would evaporate into emptiness. (But, happily, here too what one says and what one does are two different things.)

I may or may not have managed to fully uncover the requisite general logical principles of construction—which are based, on the one hand, on the concepts "and," "or," "not," and "there is," on the other, on the specifically mathematical concepts of set, function, and natural number (of iteration). (In any case, assembling these principles is not a matter of convention, but of logical discernment.) The one entirely certain thing is that the negative part of my remarks, i.e., the critique of the previous foundations of analysis and, in particular, the indicication of the circularity in them, are all sound. And one must follow my path in order to discover a way out.

With the help of a tradition bound up with that complex of notions which even today enjoys absolute primacy in mathematics and which is connected above all with the names Dedekind and Cantor, I have discovered, traversed, and here set forth my own way out of this circle. Only after having done so did I become acquainted with the ideas of Frege and Russell which point in exactly the same direction. Both in his pioneering little treatise (1884) and in the detailed work (1893), Frege stresses emphatically that by a "set" he means merely the scope (i.e., extension) of a concept and by a "correspondence" merely the scope or, as he says, the "value-range" of a relation. Russell's theory of logical types corresponds to the formation of levels mentioned in §6 and is motivated by his "vicious-circle principle": "No totality can contain members defined in terms of itself." Of course, Poincaré's very uncertain remarks about impredicative definitions should also be noted here. But Frege, Russell, and Poincarè all neglect to mention what I regard as the crucial point, namely, that the principles of definition must be used to give a precise account of the sphere of the properties and relations to which the sets and mappings correspond. Russell's definition of the natural numbers as equivalence classes (a technique which he borrows from Frege) and his "Axiom of Reducibility" indicate clearly that, in spite of our agreement on certain matters, Russell and I are separated by a veritable abyss. So it is only to be expected that he discusses neither the "narrower procedure" nor the concept of function introduced at the end of §6.

My investigations began with an examination of Zermelo's axioms for set theory, which constitute an exact and complete formulation of the foundations of the Dedekind-Cantor theory. Zermelo's explanation of the concept "definite set-theoretic predicate," which he employs in the crucial "Subset"-Axiom III, appeared unsatisfactory to me. And in my effort to fix this concept more precisely, I was led to the principles of definition of §2. My attempt to formulate these principles as axioms of set formation and to express the requirement that sets be formed only by finitely many applications of the principles of construction embodied in the axioms—and, indeed, to do this *without presupposing the concept of the natural numbers*—drove me to a vast and ever more complicated formulation but, unfortunately, not to any satisfactory result. Only when I had achieved certain general philosophical insights (which, incidentally, required that I renounce conventionalism), did I realize that I was wrestling with a scholastic pseudo-problem. And I became firmly convinced (in agreement with Poincaré, whose philosophical position I share in so few other respects) that *the idea of iteration, i.e., of the sequence of the natural numbers, is an ultimate foundation of mathematical thought*—in spite of Dedekind's

> "theory of chains" which seeks to give a logical foundation for definition and inference by complete induction without employing our intuition of the natural numbers. For if it is true that the basic concepts of set theory can be grasped only through this "pure" intuition, it is unnecessary and deceptive to turn around then and offer a set-theoretic foundation for the concept "natural number." Moreover, I must find the theory of chains guilty of a *circulus vitiosus*. If we are to use our principles to erect a mathematical theory, we need a foundation—i.e., a basic category and a fundamental relation. As I see it, mathematics owes its greatness precisely to the fact that in nearly all its theorems what is essentially infinite is given a finite resolution. But this "infinitude" of the mathematical problems springs from the very foundation of mathematics—namely, the infinite sequence of the natural numbers and the concept of existence relevant to it. "Fermat's last theorem," for example, is intrinsically meaningful and either true or false. But I cannot rule on its truth or falsity by employing a systematic procedure for sequentially inserting all numbers in both sides of Fermat's equation. Even though, viewed in this light, this task is infinite, it will be reduced to a finite one by the mathematical proof (which, of course, in this notorious case, still eludes us).
>
> —Hermann Weyl, [W87, p. 45–49] (emphasis in the original)

Fuller's synergetics

Buckminster Fuller is widely recognized as one of the most outstanding thinkers of the 20th century. He is especially recognized for his early promotion of ecological principles.

Fuller coined the term ***synergetics*** to denote a mathematical study that he showed has a wide variety of applications in many disciplines. Synergetics popularized the concept of synergy, which was obscure only a few decades ago but is now common knowledge.

Synergetics, while it uses certain mathematical principles heavily, is intended as a comprehensive systems theory rather than a mathematical theory. It does however address several issues in the foundations of mathematics.

Below we examine several quotations from his comprehesive books on synergetics and compare them to numeristics. The main work is [Fu75], which is supplemented by [Fu79], and the two are combined in an online version, [Fu97]. These books are subtitled *Explorations in the Geometry of Thinking*. The paragraphs in these books are all numbered with a unified numbering scheme. References below are to paragraph numbers rather than pages.

The following two passages define *synergy* and *synergetics*.

> 101.01 Synergy means behavior of whole systems unpredicted by the behavior of their parts taken separately.

> 200.01 Synergetics promulgates a system mensuration employing 60-degree vectorial coordination comprehensive to both physics and chemistry, and to both arithmetic and geometry, in rational whole numbers.

The following two passages show that synergetics is firmly based on experience. A similar theme for numeristics is discussed in **Objective considerations** (p. 38) and **Subjective considerations** (p. 38).

> 502.31 The difference between synergetics and conventional mathematics is that it is derived from experience and is always considerate of experience, whereas conventional mathematics is based upon "axioms" that were imaginatively conceived and that were inconsiderate of information progressively harvested through microscopes, telescopes, and electronic probings into the nonsensorially tunable ranges of the electromagnetic spectrum. Whereas solids, straight lines, continuous surfaces, and infinity seemed imaginatively obvious, i.e., axiomatic; physics has discovered none of the foregoing to be experimentally demonstrable. The imaginary "abstraction" was so logical, valid, and obviously nonsolid, nonsubstantial in the preinstrumentally-informed history of the musings of man that the mathematician assumed abstraction to be systemic conceptuality, i.e., metaphysical absolutely devoid of experience: He began with oversight.

> 220.03 Pure principles are usable. They are reducible from theory to practice.

In the following two passages, Fuller refers to *physical* and *metaphysical* principles, which roughly correspond to the *objective* and *subjective* concerns of numeristics.

> 163.00 No generalized principles have ever been discovered that contradict other generalized principles. All the generalized principles are interaccomdative. Some of them are synchronously interaccomodative; that is, some of them accomodate the other by synchronized nonsimultaneity. Many of them are interaccomodative simultaneously. Some interact at mathematically exponential rates of interaugmentation. Because the physical is time, the relative endurances of all special-case physical experiences are proportional to the synchronous periodicity of asssociability of the complex principles involved. Metaphysical generalizations are timeless, i.e. eternal. Because the metaphysical is abstract, weightless, sizeless, and eternal, metaphysical experiences have no endurance limits and are eternally compatible with all other metaphysical experiences. What is a *metaphysical experience?* It is comprehending the relationships of eternal principles. The means of communication is physical. That which is communicated, i.e. understood, is metaphysical. The symbols with which mathematics is communicatingly described are physical. A mathematical principle is metaphysical and independent of whether X, Y or A, B are symbolically employed.

> 164.00 The discovery by human mind, i.e. intellect, of eternally generalized principles that are only intellectually comprehendable and only intuitively apprehended—and only intellectually comprehended principles being further discovered to be interaccomodative—altogether discloses what can only be complexedly defined as a *design*, design being a complex of interaccomodation whose omniintegrity of interaccomodation order can only be itself described as intellectually immaculate. Human mind (intellect) has experimentally demonstrated at least limited access to the eternal design intellectually governing eternally regenerative Universe.

The following two quotes show that, unlike numeristics, synergetics does not admit the possibility of consciousness without an object, i.e. pure consciousness, the basis of **ultraprimitives** (p. 48).

> 302.00 . . . *Consciousness* means an awareness of others. . . .

> 502.24 Consciousness is experience. Experience is complex consciousness of being, of self coexisting with all the nonself. Experience is plural and nonsimultaneous. Experience is recurrent consciousness of sequences of self reexperiencing similar events. Reexperienced consciousness is recognition. Re-cognitions generate identifications. Re-cognition of within-self rhythms of heartbeat or other identities generates a matrix continuum of time consciousness upon

> which, as on blank music lines, are superimposed all the observances by self of the nonself occurrences.

For Fuller, experience is always limited by objects and cannot experience infinity. Therefore, even though synergetics includes many general statements, it cannot generalize reliably.

This leads to inconsistency. For example, he does not admit the use of infinity or straight lines in 502.31 (above) and 502.41 (below), but he uses infinity and straight lines in the caption of Fig. 923.10 (below).

> 502.41 In speaking of his "purely imaginary straight line," the mathematician uses four words, all of which were invented by man to accommodate his need to communicate his experience to self or others:
>
> (a) Purely: This word comes from the relativity of man's experiences in relation to impurities or "undesirable presences."
>
> (b) Imaginary: "Image-inary" means man's communication of what he thinks it is that he thinks his brain is doing with the objects of his experience. His discovery of general conceptual principles characterizing all of his several experiences—as the rock having insideness and outsideness, the many pebbles having their corners knocked off and developing roundness—means that there could be pure "roundness" and thus he imagined a perfect sphere.
>
> (c) Straight: Man's experiences with curvilinear paths suggested that the waviness could be reduced to straightness, but there was naught in his experience to validate that nonexperienced assumption. Physics finds only waves. Some are of exquisitely high frequency, but inherently discontinuous because consisting of separate event packages. They are oscillating to and from negative Universe, that is to say, in pulsation.
>
> (d) Line: Line is a leading, the description of man's continual discovery of the angularly observable directional sequences of events. Lines are trajectories or traceries of event happenings in respect to the environmental events of the event happening.

[caption] Fig. 923.10 *Constant Volume of A and B Quanta Modules:*

> A. A comparison of the end views of the A and B Quanta Modules shows that they have equal volumes by virtue of the fact that they have equal base areas and identical altitudes.
>
> B. It follows from this that if a line, originating at the center of area of the triangular base of a regular tetrahedron, is projected through the apex of the tetrahedron to infinity, is subdivided into equal Increments, it will give rise to additional Modules to infinity. Each additional Module will have the same volume as the original A or B Module, and as the incremental line approaches infinity the Modules will tend to become lines, but lines still having the same volume as the original A or B Module.
>
> [A and B quanta modules are certain irregular tetrahedra which are subdivisions of a regular tetrahedron]

Another inconsistency, in 502.31 and 502.41 (above), is the non-admittance of straight lines, continuous surfaces, solids, etc. on the grounds that physics has never found them. But physics has never found a perfect triangle or tetrahedron either, yet most of the book is devoted to a discussion of such figures, as in the following passages.

> 610.01 By structure, we mean a self-stabilizing pattern. The triangle is the only self-stabilizing polygon.
>
> 610.02 By structure, we mean omnitriangulated. The triangle is the only structure. Unless it is self-regeneratively stabilized, it is not a structure.
>
> 614.04 Each of the angles of a triangle is interstabilized. Each of the angles was originally amorphous—i.e., unstable—but they became stable because each edge of a triangle is a lever. With minumum effort, the ends of the levers control the opposite angles with a push-pull, opposite-edge vector. A triangle is the means by which each side stabilizes the opposite angle with minimum effort.
>
> 614.05 The stable structural behavior of a whole triangle, which consists of three edges and three individually and independetly unstable angles (or a total of six components), is not predicted by any one or two of its angles of edges taken by themselves. A triangle (a

> structure) is synergetic: it is a behavior of a whole unpredicted by the behavior of any of its six parts considered only separately.

Fortunately, these drawbacks are minor and do not seriously affect the overall scheme of synergetics, which contains a large number of highly original and useful results.

The subjective side of numeristics allows us to find that the mathematical structures such as straight lines, surfaces, and solids exist in pure form in the metaphysical realm, and are found in impure, ever changing form in the physical realm. This connection is why mathematical application of these concepts is useful.

Set theories

Usually "set theory" is referred to only in the singular, but in fact there are several varieties. See [Ho12]. By far the most commonly used and de facto standard is ***Zermelo-Fraenkel*** set theory (ZF), or its extension by the Axiom of Choice (ZFC). The set-theoretic notion of class comes from an equivalent axiomatization called ***Von Neumann–Bernays–Gödel*** set theory (NBG).

In these two theories, all mathematical structures are made from sets. The structure of sets is hierarchical and must be carefully restricted to avoid issues such as **Russell's paradox** (p. 92).

Numeristic classes, in contrast, are secondary, being the simultaneous presence of numbers, functions, relations, or statements, the latter being taken as primitives. The flat structure of classes and the existence of the empty class avoids the need for a set of axioms restricting their formation. The empty class is completely unmanifest, whereas sets are always manifest.

Below we briefly describe some significant alternative set theories. All of these alternative theories essentially suffer from the same problems as described above in **Inadequacies of set theory** (p. 44).

Internal set theory was develped by Edward Nelson as an alternative axiomatic basis for nonstandard analysis [N77]. It enriches ZFC by adding nonstandard sets to the standard sets of ZFC.

New Foundations was developed by Quine in 1937. It is a typed theory, and it has ***universal set*** (a set which includes all other sets), but it has no foundational elements and thereby allows infinite descent. It avoids Russel's antinomy by allowing only ***stratified*** formulae, e.g. $a \in b$ is stratified if a and b are of different types but not if they are of the same type. The axiom of choice can be shown to be false in this system. The axiom of infinity is a theorem, since the negation of the axiom of choice implies that there exists an infinite set.

Structural set theories contrast with ***material set theories***, which include ZF. Instead of being constructed from one or more atoms, a set in a structural theory is defined only through functions and relations that involve it [NSS]. The canonical example of a structural set theory is the Elementary Theory of the Category of Sets (ETCS), an axiomatization of set theory designed to be congruent with **category theory** (p. 116).

Reverse mathematics attempts to find axioms which are necessary to prove ordinary mathematical theorems. Proponents of this approach often reject set theory as too expressive, thus generating too much hierarchy and leaving the door open to poorly resolved issues which have little or no bearing on the rest of mathematics. Reverse mathematics often uses subsystems of second order arithmetic, in which quantifiers can range over sets of numbers in addition to individual numbers. A recent book of Simpson is often regarded as important [Si09].

Category theory

A ***category*** may be defined in set-theoretic terms as a collection of objects (elements) and arrows (functions), for example the category of groups and group homomorphisms. See [Ma14]. Similarly, most of the "abstract" structures investigated by modern mathematics are categories: rings, fields, vector spaces, topological spaces, etc.

Since a collection of objects and functions depends on the definition of set, this definition depends on set theory. Alternatively it is possible to define categories independently of set theory, by defining the category of all categories, in which sets are one category. This makes category theory an alternative foundational theory.

A certain type of category knows as a ***topos*** forms the basis of another foundational theory.

Category theory and topos theory suffer from many of the same problems as set theory, as discussed above in **Inadequacies of set theory** (p. 44)

and **A numeristic view of abstraction** (p. 93). From the numeristic perspective, the main problem with such theories is that they do not fully account for the structures they include, and as such they really only classify rather than define.

Type theory

Type theory has many variations; see [ShT]. In most of them, every ***term*** (syntactic element) has a ***type***, for example the number 5 has the type of integer. This assignment is called a ***judgment*** ([ShJ]). A function uses judgments to restrict its domain and range to specific types. ***Rules*** govern the transformation of terms through their types. Propositions can have their own type, so type theory can encode logic. Type theory can encode sets or conversely.

Type theory is naturally connected to typed programming in computer science. From a numeristic perspective, it is significant that the structure underlying computer data (machine code) is numeric, and thus typed programming is actually dependent on numbers rather than the reverse. Likewise, type theory, along with set theory and category theory, are dependent on number, since counting logically precedes all distinctions such as type, judgment, and rule. See **Inadequacies of set theory** (p. 44).

Principia Mathematica

This rather famous but little-used work [WR27, WR62] by Alfred North Whitehead and Bertrand Russell lays out a foundational theory starting from logic.

Principia Mathematica uses sets but in a different way from the now-standard Zermelo-Fraenkel and Von Neumann–Bernays–Gödel theories described above. For instance, it defines 1 as the set of all sets with a single element, i.e., in modern notation:

$$1 \equiv \{x \mid (\exists y \mid x = \{y\})\},$$

apparently heedless of the circularity of this definition, since each object in the definition is an instance of 1.

Principia Mathematica has the same basic problems as set theories:

- lack of self reference

- heavily dualism with no fundamental unity
- circular definition of number, as just noted
- definition of numbers suitable only for natural numbers

  Other problems:

- To avoid **Russell's paradox** (p. 92), it allows sets to be members of sets through types, which restrict the membership relation. This approach is now generally regarded as arbitrary and inelegant.
- It only attempts to address the arithmetic of counting numbers.

Mereology

Mereology is the philosophical and mathematical study of the relationship between wholes and parts. See [V16]. The study is both ancient and modern.

There are many varieties of mereology and much debate within the field. Mereological collections bear some resemblance to numeristic classes, but numistics for the most part uses set theoretical terms and notation, with adjustments, and avoids mereological terms not used in set theory.

The following approximate equivalences hold in some varieties of mereology:

fusion, sum, or object	class
atom	element of a class
universal object W	full class $\not\propto$
null object N	empty class $\emptyset$
parthood relation Pxy	class inclusion $x \subseteq y$
proper part relation $PPxy$	proper class inclusion $x \subset y$

Mereological theories often aim to clarify the relation of parts and wholes, and they often include axiomatizations of the parthood relation, such as the one in [CV99]. In some mereological theories, parthood has a wide sense and may refer to nonmathematical objects, e.g. "the handle is part of the cup", "elections are a part of democracy".

A wide sense of parthood is sometimes poorly founded and can result in "mereological loops". For example, it may be said that the statement $2 + 3 = 3 + 2$ is a proper part of the commutativity of addition because it is one instance out of infinitely many of that property, but commutativity of addition is a proper part of $2+3 = 3+2$ because addition is a proper part of this statement and has other properties as well.

Leśniewski [Le16, Le27] developed a mereological system which he intended to be an alternative foundational theory of mathematics, while a later refinement of this system by Goodman and Quine [Go47] remained chiefly in the philosophy of mathematics. These theories are now collectively known as "classical mereology" [Hv09]. These theories are formulated from the philosophical viewpoint of nominalism, which rejects "abstract" objects such as sets (while accepting abstractions such as quantifiers), and finitism, which rejects infinite numbers (while accepting quantified statements with an infinite number of cases).

In [HK16], it has been shown that mereological axioms alone (those focused on parthood) cannot yield set membership and thus cannot form a theory equivalent to set theory, but instead must be supplemented by axioms postulating the existence of singletons and the empty set.

Numeristics rejects sets and similar constructions for a foundation of numbers, not because they are abstract, but because they are not abstract enough, since they lack the property of self referral which numbers possess.

In classical mereology, the notion of fusion or sum is close to that of a numeristic class: In classical mereological theories, everything is part of itself, which is also a property of numeristic classes, $a = a$ for any number, function, or relation a.

Classical mereology does not have a null object, so that the intersection of two disjoint objects is undefined, whereas in numeristics this is well defined as the empty class. Classical mereology also does not seem to have distribution of statements over classes.

Classical mereology is a largely axiomatic approach, while numeristics uses primitives and ultraprimitives as its foundation. Ultraprimitive unity encompasses not only numbers, functions, relations, etc. but also the intelligence with governs numbers, including properties such as the commutativity of addition.

Wheel theory

Wheel theory is an abstract theory which extends a ring or integral domain to allow unrestricted division by zero, including the normally indeterminate expression $\frac{0}{0}$. See [Se97] and [Ca04].

Wheel axioms include addition and multiplication and also a unary operation $/$ or $\frac{1}{\cdot}$ called *reciprocal*. The notation x/y or $\frac{x}{y}$ means $x \cdot /y$ or $x\frac{1}{y}$, not $x \cdot y^{-1}$, where y^{-1} is the multiplicative inverse.

A common application of wheel theory is to the projectively extended real numbers $\widehat{\mathbb{R}}$ and to the single projectively extended complex numbers $\widehat{\mathbb{C}}$. These result in further extensions denoted $\mathbb{R}_{\perp}^{\infty}$ and $\mathbb{C}_{\perp}^{\infty}$. In these systems, $/0 = \infty$, and there is an additional element $\perp \equiv 0/0$. (It is not clear how the symbol $\perp$ is pronounced–we could adopt the TEX name, "perp".) $\perp$ functions in some ways like the numeristic $\varnothing$, e.g. $x + 0/0 = 0/0$ for any element x in the wheel. $\mathbb{R}_{\perp}^{\infty}$ and $\mathbb{C}_{\perp}^{\infty}$ can thus be regarded as $\mathbb{R}$ and $\mathbb{C}$ with ∞ and $\perp$ added, and with arithmetics in some respects as described above in Real infinite element extensions and Complex infinite element extensions.

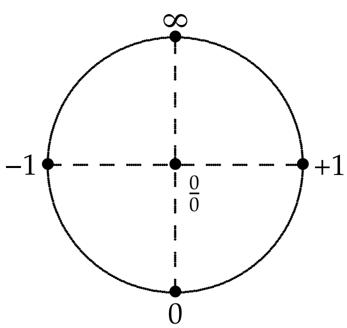

FIG. 37:
Wheel of projective extension of real numbers, mapped to circle and hub

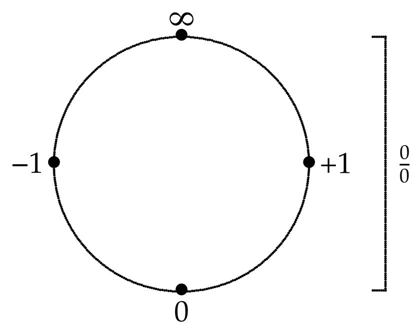

FIG. 38:
Numeristic view of projective extension of real numbers, mapped to circle

Figure 37 shows $\mathbb{R}_{\perp}^{\infty}$ mapped to a circle and a hub, with $\frac{0}{0} =\perp$ at the hub. The dotted lines show a link from four of the points on the circle, representing the link from every point on the circle to the hub through properties such as $x + \frac{0}{0} = \frac{0}{0}$. Figure 38 shows $\widehat{\mathbb{R}}$ mapped to a circle, with $\frac{0}{0} = \not\infty$ as the whole circle.

Wheel theory does not use classes, so $\perp$ is an element rather than a multivalued class. $\perp$ and its basic behavior are postulated via a reciprocal operator, which is defined separately from multiplication. From the projective extension, we have $\frac{1}{0} = \infty$, but in the wheel, 0^{-1} does not exist, since there is no x such that $x \cdot 0 = 1$. Instead, we have $0 \cdot \infty =\perp$, which is completely distinct from 1, so ∞ cannot be the multiplicative inverse of 0.

In numeristics, the equivalence of $\frac{0}{0}$ and $\not\infty$ is derived from division being the inverse of multiplication and all operations being allowed to take on multiple values through classes. So $0 \cdot \infty$ includes 1 as one of its values, which means ∞ is the multiplicative inverse of 0, and $\frac{1}{0} = 0^{-1} = \infty$.

The wheel axioms use only the equality relation, which leaves order relations and the role of $\perp$ in order relations unaddressed. In numeristics, $\not\infty$ can be used along with all other multivalued numeric expressions in order relations, using class distribution. For example:

$$\not\infty \overset{\vee}{=} 0$$

$$\neg\,(\not\infty \overset{\wedge}{=} 0)$$

Edmund Berkeley's Guide to Mathematics

Edmund Berkeley in [B66] has not attempted to formulate a foundational theory, but only to give a gentle introduction to some simple mathematical ideas. Yet some of the ideas he espouses are remarkably similar to those of **Maharishi Vedic Mathematics** (p. 97).

In Chapter 1, in a section called "What Kind of Thing is Mathematics?" Berkeley shows how pervasive mathematics is.

> [M]athematics deals or can deal with any objects or observations whatever, whether real or imaginary. Numbers can count atoms or stars, motor cars or people, angels or devils. A sequence of numbers can specify the sequence of stations on a railroad, or the sequence of days in a year, or the sequence of points in an argument. Almost no other science is as general as mathematics.
>
> —Edmund Berkeley, [B66, p. 23]

> Some people have found out what mathematics is about, and what kind of thing it is, and have themselves taken hold in one way or another of a significant part of the great power that mathematics contains. To these people there is no finer or more exciting instrument of the human mind than mathematics.
>
> But we cannot say that most people have understood this. Education in mathematics up to now has not spread widely this degree of understanding "What is mathematics?"
>
> —Edmund Berkeley, [B66, p. 24]

In Chapter 2, in a section called "The Unconscious Use of Common Mathematical Ideas", Berkeley shows how often we use mathematical ideas subconsciously.

> In almost all the sentences you say, in almost all the thoughts you think, you make use of ideas that are really and essentially mathematical. In fact, you have had a lot of experience with many important mathematical ideas, and you can use them correctly nearly all the time—although it is quite likely that you are not conscious of this fact in the least.
>
> —Edmund Berkeley, [B66, p. 33]

He then observes that this paragraph alone contains the following mathematical concepts: "almost all", "nearly all", "least", "many", "-s" (for plural), "not", "and", and "in", to which we can add "likely".

Further, he observes that everyday language often contains terms denoting shapes, sizes, comparisons, indefinite and definite numbers, order, and approximation, all of which are mathematical in character. Since we normally use these terms without thinking of them as mathematics, we are actually mathematicians without realizing it, and rather successful ones at that.

Wildberger's math foundations

Norman Wildberger's foundational theory is a contemporary theory, mostly presented in video form. Here we examine several quotes from one of his written documents.

> Mathematics does not require 'Axioms'. The job of a pure mathematician is not to build some elaborate castle in the sky, and to proclaim that it stands upon the strength of some arbitrarily chosen assumptions. The job is to investigate the mathematical reality of the world in which we live. For this, no assumptions are necessary. Careful observation is necessary, clear definitions are necessary, and correct use of language and logic are necessary.
>
> —Norman Wildberger, [Wi06, p. 8]

This view is largely shared by numeristics, as well as by **Skolem** (p. 99) and **Weyl** (p. 104).

> But at no point does one need to start invoking the existence of objects or procedures that we cannot see, specify, or implement.—[Wi06, p. 8]

The agreement with numeristics ends at this point. No number can be seen or directly observed with the senses alone. A number is a purely mental structure. It is not a sense-perceptible object. It can be *applied* to sense-perceptible objects, but it can be applied to other objects as well, including other numbers or to itself.

> And where is the infinite set $\mathbb{N}$? The answer is—nowhere. It doesn't exist. It is a convenient metaphysical fiction that allows mathematicians to be sloppy in formulating various questions and arguments. It allows us to avoid issues of specification and replace concrete understandings with woolly abstractions.—[Wi06, p. 14]

Denying the existence of infinity is also a metaphysical claim, since denial takes place in an abstract realm beyond the senses, beyond the physical. By Wildberger's logic, every number is a "convenient metaphysical fiction," since numbers are also in an abstract realm beyond the senses.

Moreover, infinity is used in physics. A quantum wave function, for example, extends infinitely far in space and time, and the center of a black hole is a point where the space-time curvature and density are both infinite.

Even if mathematics does not allow infinity as a number, infinity still occurs in more covert form. The statement that $a + b = b + a$ for all integers, for instance, applies to an infinite number of cases, which obviously cannot all be examined individually. Such reasoning with the infinite is metaphysical, but it is the only practical way to do mathematics.

At one point, Wildberger defines an integer w which is "a number so vast that it requires all the particles of the universe" to write its decimal expansion.

> Perhaps you believe that even though you cannot write down numbers bigger than w, you can still abstractly contemplate them! This is a metaphysical claim.
>
> What does a number bigger than w mean, if there is nothing that it counts, and it can't even be written down?—[Wi06, p. 15]

One thing a number bigger than w might count is the number of possible permutations of particles used for the writing of the decimal expansion of w, or the number of points that those particles occupy.

Elsewhere Wildberger denies that irrational numbers exist and has devised a system of "rational trigonometry" to avoid them. However, as others have pointed out, it seems that Wildberger rejects irrational numbers because he confuses the definition of a number with its digital representation; since an irrational number can never be completely written out in decimal form, he seems to think that it can never be properly specified.

But of course, irrational numbers such as $\sqrt{2}$ and π have exact definitions, which is the very thing that enables us to write them in decimal form to any desired level of accuracy and have that decimal form agree with observation, such as computing the length of a diagonal or the area of a circle. Metaphysical mathematical concepts thereby show that they apply to physical reality.