NUMERISTICS

A NUMBER-BASED FOUNDATIONAL THEORY OF MATHEMATICS

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SUMMARY

Numeristics is a number-based alternative foundational theory of mathematics. Numeristics is inspired in part by the recent revival of the Vedic tradition of India, as expressed by Maharishi Mahesh Yogi in his Vedic Mathematics and has antecedents in the work of Skolem and Weyl. This monograph does not assume any familiarity with this material.

Several reasons are given for set theory being inadequate as a foundational theory:

- The axioms of set theory have always been controversial.
- Russells paradox means that, unlike numbers, sets cannot be self-referral
- With few exceptions, a definition of a mathematical structure as a set or sets does not allow us to obtain the properties of that structure from its definition.
- since given number has different definitions if it used as, say, a natural number, rational number, real number, etc., sets do not locate the abstract level of number underlying these different uses
- Set theory and logic are dependent on numbers and therefore cannot be used to define numbers.

The fundamental ideas of numeristics are then given:

- ***Ultraprimitves***, a deep level of number which show the essential self-referral property of number.
- ***Primitives***, properties of numbers which are roughly equivalent to axioms.
- ***Numeristic classes***, groupings of numbers which are somewhat similar to sets but have a flat structure.
- ***Infinite element extensions***, infinite numbers which are added to the standard number systems.

Together these structures allow unrestricted arithmetic and provide an elegant computational framework.

Numeristics includes an alternative approach to analysis called ***equipoint analysis***, described in a separate monograph.

All things that are known have number; for without this nothing whatever could possibly be thought of or known.—Philolaus, quoted in [Cu11].

The infinite! No other question has ever moved so profoundly the spirit of man; no other idea has so fruitfully stimulated his intellect; yet no other concept stands in greater need of clarification than that of the infinite.—David Hilbert

[T]he more distinctly the logical fabric of analysis is brought to givenness and the more deeply and completely the glance of consciousness penetrates it, the clearer it becomes that, given the current approach to foundational matters, every cell (so to speak) of this mighty organism is permeated by the poison of contradiction and that a thorough revision is necessary to remedy the situation.—Hermann Weyl, [W87, p. 32]

Set theoreticians are usually of the opinion that the notion of integer should be defined and that the principle of mathematical induction should be proved. But it is clear that we cannot define or prove ad infinitum; sooner or later we come to something that is not further definable or provable. Our only concern, then, should be that the initial foundations be something immediately clear, natural, and not open to question. This condition is satisfied by the notion of integer and by inductive inferences, but it is decidedly not satisfied by set-theoretic axioms of the type of Zermelo's or anything else of that kind; if we were to accept the reduction of the former notions to the latter, the set-theoretic notions would have to be simpler than mathematical induction, and reasoning with them less open to question, but this runs entirely counter to the actual state of affairs.—Thoralf Skolem, [S22, p. 299]

ऋ॒चो अ॒क्षरे॑ प॒रमे व्यो॑म॒न्
Ṛicho akṣhare parame vyoman
The eternal expressions of knowledge are located in the collapse of infinity to its point, in the transcendental field of pure consciousness.
—Rig Veda 1.164.39, Atharva Veda 9.10.18, Shvetashvatara Upanishad 4.8

DEFINITION AND SCOPE

Numeristics is a number-based alternative foundational theory of mathematics. It postulates that the real basis of mathematics is number, rather than sets.

Numeristics includes many ideas that are inspired by the recent revival of knowledge and experience in the Vedic tradition of India, as expressed by Maharishi Mahesh Yogi in his Vedic Mathematics. Source material in Maharishi Vedic Mathematics is summarized in [CS] and includes especially [M96]. A overview of the field is given by this author in [CI]. The present monograph does not assume any familiarity with Maharishi Vedic Mathematics.

This monograph gives the fundamental ideas of numeristics as they apply to arithmetic and elementary algebra. Numeristics includes an alternative approach to analysis called *equipoint analysis*, described in [CE], and to divergent series in [CD] and to repeating decimals in [CR].

WHY NUMERISTICS

Philosophy of numeristics

The numeristic approach to mathematics holds that mathematics has two purposes:

1. Objective: To successfully improve the outer environment through practical applications; and

2. Subjective: To successfully develop the inner environment of the practitioner of mathematics mentally, emotionally, and spritually.

These goals can and should complement each other. Subjective development helps us to solve problems more easily, with fewer mistakes and more balance. Balanced focus on the objective brings benefits to the world at large.

Excessive concern with axioms does not contribute to the fulfillment of either of these goals. Such axiomatics can leave us suspended between the objective and subjective goals without fulfilling either one, and give us a set of conditional statments, without informing us about the context of any of the premises of the axioms.

Axiomatics typically takes smaller logical steps which can bring out mistakes in reasoning. However, the risk is great that it becomes merely a display of intellect without going outside the bounds of intellect, whether into the area of objective applications in the world of the senses, or into the subjective realm of spirituality beyond the intellect.

Therefore numeristics, at least in its early stages, does not use an axiomatic approach. Instead, it uses ***primitives***, as explained below. It also offers a more abstract approach with ***ultraprimitives***, also explained below.

Numeristics aims to develop a theory which fits closely with experience, both objective and subjective. It further aims to integrate these two by expressing the connection between them.

Objective considerations

Mathematics is a science. Like other sciences, its conclusions can be regarded as valid only if they have been empirically validated through objective means. Ancient branches of mathematics have long been validated through physical application, but not all modern branches have been.

What is called mathematical proof is really derivation, a chain of logic connecting axioms and previously proven theorems to a new theorem. It cannot be considered complete proof because it assumes axioms without proof. Since the Renaissance, the presentation of mathematics as a whole has increasingly emphasized formality and neglected objective verfication. This has led to an increasingly prevalent belief among mathematicians that mathematics is a game that derives its authority from social consensus, rather than from objective validation.

Numeristics attempts to improve this situation by using only thoroughly verified principles of number and space as the foundation of mathematics.

Subjective considerations

Numeristics is also based on sound principles of consciousness. We can define consciousness as self referral. Complete self referrral is pure consciousness.

As we will examine more closely below, numbers have a simple self referral nature. Numeristics thus takes number as the most basic of mathematical structures.

Inadequacies of set theory

Set theory is currently held by the vast majority of mathematicians to be a universal basis of mathematics, at least on a formal level. The modern neglect of objective verification in mathematics as a whole has had an important effect on the development of set theory. This development, starting in the late 19th century, has been purely subjective, focused on paper proofs only, and devoid of concern with objective verification.

As far as this author has been able to determine, the assertions of set theory about the infinite have never been proved by physical experiment. Robin Ticciati, author of a well known reference work of the mathematics of quantum field theory [T99], when asked if he knew of any use of mathematics in quantum theory that depended on a set theoretical result, responded in the negative [T03].

Set theory met with much controversy in its early days. See the Appendix for source material that shows that Thoralf Skolem and Hermann Weyl substantially disagreed with the supposition that set theory could form a proper foundation for mathematics.

The axioms of set theory have been notoriously controversial. For instance, the axiom of infinity, which asserts the existence of an infinite set, encountered considerable controversy when it was introduced, which to this day has never been completely settled. Other axioms are even more controversial, such as the axiom of choice and the generalized continuum hypothesis.

Russell's paradox means that set theory must be constructed so that a set cannot be a member of itself. Since the only thing that sets can really do is include sets and be members of sets, this strikes a fatal blow to any aspiration of making set theory self-referent.

Even the existence of infinite sets must be regarded as uncertain if infinite sets cannot be located in nature. Numbers are obviously found in nature, in the sense that numerical laws of mathematics have long been known to govern objects of nature.

With few exceptions, the set theoretical definitions of mathematical structures, including numbers, do not allow us to obtain the properties of those structures from their supposed definitions. The properties must instead be supplied from non-set-theoretical considerations. For this reason alone, set theory should not be considered a true foundational theory, but at best only a modeling theory.

The number 1 has different set theoretical definitions depending on whether it is considered a natural number, integer, rational number, real number, complex number, or other role. Set theory therefore does not access the abstract level which underlies all of the different uses of such numbers.

Although it is claimed that set theory defines numbers, this reasoning is circular. Set theory and the system of logic it is built upon are implicitly dependent on numbers. Both set theory and logic assume fundamental dualities and multiplicities, such as true and false, axioms and sets, inside and oustide of sets, the multiplicity of axioms. Dualities are implicit uses of the number 2, and multiplicities are implicit uses of higher numbers.

Even this consideration pales besides the implicit use of the number 1, which occurs each time we express or even think of any object of attention, and the number 0, which logically precedes all expressions and objects of attention.

The set theoretical model of 1, $\{0\} = \{\{\ \}\}$, has no inverse, meaning that there is no "negative set" which when applied to $\{0\}$ yields $0 = \{\ \}$. Set theory covers up emptiness instead of exploring it.

From the foregoing it should be clear that any system that explains number must account for the whole range of manifestation, from the subtlest thinking level to the most obvious, and it must account for both subjective and objective phenomena. It must also be clear that any such system cannot be based on intellectual values alone, since intellectual conception and expression necessarily take place in a field of multiplicity. The intellect, by itself, cannot properly account for unity and thus, by itself, is not an appropriate tool for exploring numbers.

Set theory is thus a form of hardened positivism that is utterly incapable of dealing with the subtleties of consciousness.

Antecedents to numeristics

In the early 20th century, Skolem and Weyl independently anticipated some of the features of numeristics by attempting to construct foundational systems that did not use set theory.

In [S23], Skolem proves a variety of elementary number theoretical results using a system of natural numbers, standard logic, and first order recursion. This is known as primitive recursive arithmetic and was used by Gödel in the proofs of his famous incompleteness theorems in 1931. See Skolem's recursive foundational system in the Appendix.

In [W32], Weyl develops a theory of the real numbers, which he intended as an alternative to set theory as a foundation of analysis (calculus). Weyl bases his theory of the real continuum on natural numbers, basic logical operations, and primitive recursion, without transfinite set theory or proof by contradiction. See Weyl's foundational system of the continuum in the Appendix.

Some foundational theories based on mereology have developed the concept of ***fusion*** or ***sum*** which is similar to the important numeristic concept of class. Numeristic classes have a flat structure which contrasts with the hierarchical structure of sets.

BASIC MECHANICS OF NUMERISTICS

Ultraprimitives

Numeristics bases mathmatics purely on number. Numeristics starts with three numeric ***ultraprimitives*** or alternate ways of expressing the ultimate value of mathematics. These three are ***infinity***, ***unity***, and ***zero***.

Infinity. Number, as with everything else, ultimately starts from the infinite. The infinite is inexhaustible and therefore only partly conceptualizable.

The infinite in its totality is beyond human conception but within the range of human experience. Vedic Mathematics shows how the infinite can be directly experienced as unbounded, pure consciousness, a fourth state of consciousness distinct from waking, dreaming, and sleeping. The Vedic tradition of India is very familiar with this state of consciousness and gives it many names, among them *samādhi* and *turīya*. It can be experienced in innumerable ways, but a systematic way of experiencing this fourth state of consciousness is through the TM (Transcendental Meditation) program. See [M96 p. 434–445]

The infinite may be visualized in ordinary space, since, even within a finite extent of space, the number of points and possible curves is infinite.

The infinite may be partially conceptualized in terms of finite numbers, by finding infinity within a number, or by finding a number within infinity.

Unity. Unity, the number 1, is the first mathematical manifestation. It expresses the unified nature of infinity.

Whenever a number is used to measure an object of experience, we can consider the number to be an attribute of the object, or we can consider the object to be an instance of the number. Since any conceivable single object is one object, everything conceivable is within unity. Unity is obviously within infinity, but the infinite is also within unity.

The number one, since it is an identifiable object of attention, is an instance of itself. To put it another way, one is one number. This is the *principle of self-referral*. If we define consciousness as self-referral, then consciousness is essentially the state of unity.

Zero. The number 0 represents the unmanifest quality of pure consciousness. It is silence and balance. Whenever any mathematical object manifests, its opposite also manifests. Each positive number has a negative; each function has an inverse; every statement has a negation.

In Vedic Mathematics, zero is called the absolute number, because it is the unmanifest state from which all manifestation begins. See [M96 p. 611–634], [M05a], and [M05b].

Multiplicity. Zero and one observing each other give rise to the number two, and from there multiplicity comes out. Unity is found within two, since two is composed of two units, and two is found with unity, since two is one number, an instance of unity. The number two gives rise to the two values in classical logic of true and false and to all distinctions generally.

With multiple numbers, the potential of transformation comes about, and this manifests as functions. The values of true and false give rise to relations, especially equality.

The self referral nature of number gives rise to multiplication and division. Infinity can result from division by zero, which numeristics allows in a careful way, as described below.

Zero steps to multiple steps. Maharishi Vedic Mathematics is focused on the above ultraprimitives. It is a system of ***mathematics without steps***, a spontaneous knowing or cognition ([M96 p. 558–559]), as contrasted with the system of ***mathematics with steps*** in modern mathematics ([M96 p. 626–627]). Numeristics is an attempt to develop a system of stepwise mathematics that is in harmony with the ideals and practices of zero-step Maharishi Vedic Mathematics.

Numeristics emphasizes what may be called ***observational rigor***, by which we mean keeping mathematical expressions in tune with observations of both inner and outer nature. The inner observation of ultraprimitive unity gives a stable platform for properly assessing the outer observations of multiplicity. This stability is more useful than axioms because it operates from a deeper level of awareness.

Knower, known, and knowing. Vedic Science identifies three divisions of knowledge: the knower, the known, and the process of knowing. It also identifies a level of unification, pure knowledge or pure awareness, that underlies and unites these three. In numeristics, we associate the three divisions with numbers/quantities (known), functions/operators (process of knowing), and relations (knower). The knowledge value itself is associated with a complete mathematical statement, which unites the three divided values of numbers, functions, and relations.

A function is an abstraction of a constant by allowing the transformation of one number into another. Functions thus a more abstract level of numbers, but they are associated with the process of knowing since they emphasize transformation.

A relation is a function that has a logical value and is thus an abstraction of a function. Logical values are yet more subtle level of numbers since they connect the measurement level of number to the knowledge level of number, and truth values identify true and false statements. Relations are associated with the knower since logical values are much more associated with the knower than the known.

Figure 1 shows these identifications in a sample equation.

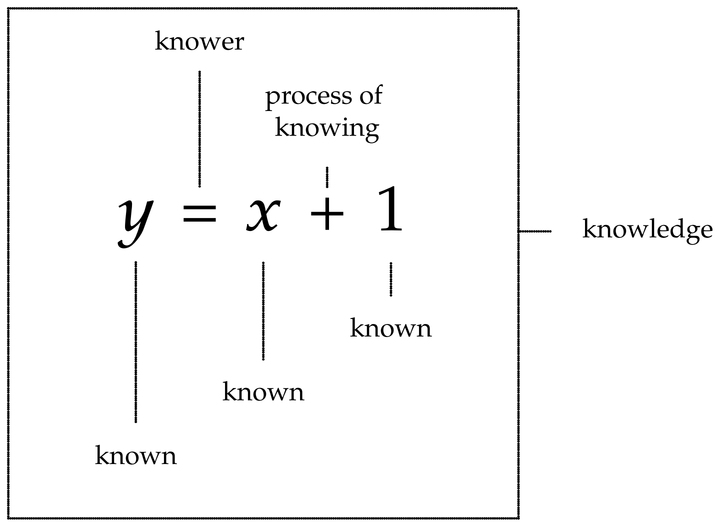

FIG. 1:
Knower, knowing, and known
in a mathematical statement

Primitives

In its multiple step phase, numeristics starts with the following ***primitives***, which function somewhat as axioms. These primitives are generally those which existed in "classical mathematics," by which we mean mathematics as it historically existed before the development of set theory, and which is currently taught at the primary, secondary, and upper division undergraduate levels.

- Natural, integral, rational, real, and complex numbers
- Addition, multiplication, exponentiation of these numbers and their inverses
- Usual commutative, associative, and distributive properties of these numbers
- Elementary equality and order relations

- Euclidan geometry
- Ordinary classical logic with quantifiers (first order logic)

We do *not* include the following.

- We do not include sets or categories.
- We do not include abstract structures, non-Euclidean geometries, or numbers beyond the complex numbers at this point. We will extend to them at a later point.
- Calculus and analysis are handled in a separate document [CE].
- Infinite series are also handled in a separate document [CD].

To the above primitives, we will soon add the following.

- Classes, which handle multiple unordered values
- Infinite element extensions, i.e. one or more infinite numeric values

We also accept the following principles.

The ***principle of freedom:*** We are free to perform any arithmetic operation, as long as we put it in correct context. This means that every numeric operation has a numeric result. No numeric operation is undefined. A numeric operation may be multivalued or empty valued, as described below.

The ***principle of reversal:*** Every operation can be reversed. This is because zero is the balance point, both a static balance point of positive and negative, and a dynamic balance point of an operation and its inverse.

The ***principle of extension:*** Number systems can be naturally extended, sometimes through reversal, and operations can be naturally extended to extended number systems.

Classes

A numeristic ***class*** is a potentially multivalued number or other numeric or number-like construction. Classes have a flat structure: Every number is a single valued class; a class containing a single number is identical to the number. A class containing multiple numbers distributes operations on it and statements about it over each constituent number.

Since numeristics and set theory do not mix but have some similar concepts, for numeristic classes we will use set theoretical notation with numeristic meanings. We denote classes with the same braces that we use to denote sets, e.g. $\{+1, -1\}$. We may also put multiple functions and relations into classes, e.g. $\pm = \{+, -\}$.

Flat class structure means that for any class c, $c = \{c\}$. Flat structure allows arithmetic operations on classes in a straightforward way. For example, if $x^2 = 1$, then x is a class with the two elements $+1$ and -1, and we say $x = \pm 1 = \{+1, -1\}$, and $x + 1 = \pm 1 + 1 = \{0, 2\}$. For a list that is otherwise clearly delimited, we may drop the braces and write an expression such as $\pm 1 + 1 = 0, 2$.

We use several other notations from set theory in numeristics. It must be emphasized that these notations have somewhat different meanings in numeristics from those in set theory. Below are samples of notation we can use to describe the numeristic class ± 1:

$$\begin{aligned}
\pm 1 &= +1, -1 \\
&= \{1, -1\} \\
&= \{a \mid a^2 = 1\} \\
&= 1 \cup -1 \\
&= \bigcup_{k=1}^{2} (-1)^k.
\end{aligned}$$

A *subclass* is a class that is completely included in another class. We use the subset symbol for subclasses, e.g. $1 \subset \pm 1$. We may also indicate inclusion through equality with a condition, for example:

$$\pm 1 = +1 \text{ when } \pm 1 \text{ is positive}$$

We use the notation $\cap$ to denote the class of elements common to two classes, e.g. $\{1, 2, 3\} \cap \{3, 4, 5\} = 3$, and the notation $\setminus$ to denote removal of elements: $c \setminus d := \{a \in c \mid a \notin d\}$. The unary use of $\setminus$, e.g. $\setminus c$, means $S \setminus c$, where S is the current base class in which we are working.

There may also be classes of functions, relations, and statements. An indefinite integral is an example of a class of functions, as explained in [CE].

Elements

An *element* is a class which does not contain any smaller subclasses. If a is an element of b, then we use the notation $a \in b$ or $b \ni a$, e.g. $1 \in \pm 1$. Since an element is also a class, $a \in b$ implies $a \subseteq b$.

A class that is not an element is said to be ***multivalued***. An element may also be called a ***single valued*** class.

Elements may be relative to levels of sensitivity. A class that is an element at one level of sensitivity may be multivalued at another level of sensitivity. We do not use sensitivity levels in this monograph, but we do use them in [CE].

In numeristics, we do not *define* any element, whether a number, function, or relation, *as* a class. The numeristic view is that this is neither necessary nor sufficient. Rather, we define classes as collections of elements, and take numbers, functions, and relations as primitives, since both the concepts and the knowledge of how to use them emerge in a natural, obvious way from the experience of the absolute number and from objective application.

Functions

It will generally be assumed that a function returns a class, unless it is explicitly indicated as being single valued. This means that functions are generally multivalued.

It will also be generally assumed that a function may accept classes as arguments. This requires distribution, as explained below.

A relation is a function that returns a logical value.

A ***compound*** is a special purpose function. Examples:

- An infinite sequence is a function from $\mathbb{N}$ to a class whose elements are terms of the sequence.
- A finite sequence is a function from $\{1 \ldots n\}$ to an element class.
- An ordered set is a finite sequence.
- An ordered pair is an ordered set with two elements, a function from $\{1, 2\}$ to an element class.

- A matrix is a function from a class of ordered pairs (the row and column indexes) to an element class.

Distribution

Functions and relations on classes may or may not distribute over their elements, depending on their type.

Arithmetic functions, such as $+$, $-$, and $\sqrt{}$, operate on elements. When applied to a single class, they distribute over the class elements to form another class of elements. For instance, $\pm 2 + 1$ means $\{-1, 3\}$, and in general, for any class c and function f, $f(c)$ means $\{f(a) \mid a \in c\}$.

Class functions, such as $\cap$, $\cup$, $\setminus$ (class subtraction), and ^{c} (complement), operate on classes as a whole and do not distribute over class elements.

Arithmetic relations, such as $<$, $\leq$, $>$, $\geq$, and $\equiv \ldots \bmod$ (congruence), relate elements. When one of the operands is a class, the relation distributes over the class elements, yielding a class of relation statements.

An arithmetic relation distributing over a class may denote a class of statements, but such a class should usually produce a single statement by being joined with some logical connective, which may be conjunction, inclusive disjunction, or exclusive disjunction. For example, $\pm 2 < 5$ may be interpreted as $(2 < 5) \wedge (-2 < 5)$, or as $(2 < 5) \vee (-2 < 5)$, or as $(2 < 5) \veebar (-2 < 5)$.

An arithmetic relation R distributing over a class c as $R(c)$ could thus have one of four interpretations: a class of statements $\{R(a) | a \in c\}$, a conjunctive interpretation $(\forall a \in c)R(a)$, an inclusive disjunctive interpretation $(\exists a \in c)R(a)$, or an exclusive disjunctive interpretation $(\exists! a \in c)R(a)$. In this monograph, we usually assume the conjunctive interpretation.

Class relations, such as $\subset$, $\subseteq$, $\supset$, $\supseteq$, $\cap$, $\cup$, relate classes as a whole and do not distribute over class elements.

The equality relation $=$ actually has two different types, class equality and distributed equality, and distributed equality has several subtypes. $(\pm 2)^2 = 4$, for instance, may mean:

- the class equality $\{2^2, (-2)^2\} = \{4\}$, meaning the two classes have the same elements; or
- a distributed equality such as $2^2 = 4 \wedge (-2)^2 = 4$, or $2^2 = 4 \vee (-2)^2 = 4$.

A distribted eqality is meaningful only when an ***implicit function*** mapping corresponding elements of the classes is clearly understood, and the logical connective is understood.

When necessary, we use the notation in Table 2 to distinguish these types.

TABLE 2: **Class and distributed equalities and inequalities**

In this table, a and b are elements, c and d are classes, f is a function from c and d, and g is a function from d to c.

Symbol	Meaning	Implicit function	Equivalent
$c \overset{\{\}}{=} d$	$(\forall a)(a \in c \;\Leftrightarrow\; a \in d)$		$\neg(c \overset{\{\}}{\neq} d)$
$c \overset{\wedge}{=} d$	$\bigwedge_{a \in c} a = f(a)$	$f : c \to d$ bijective	$\neg(c \overset{\vee}{\neq} d)$
$c \overset{\vee}{=} d$	$\bigvee_{a \in c} a = f(a)$	$f : c \to d$ surjective	$\neg(c \overset{\wedge}{\neq} d)$
	or		
	$\bigvee_{b \in d} b = g(b)$	$g : d \to c$ surjective	
$c \overset{\{\}}{\neq} d$	$\neg(\forall a)(a \in c \;\Leftrightarrow\; a \in d)$		$\neg(c \overset{\{\}}{=} d)$
$c \overset{\wedge}{\neq} d$	$\bigwedge_{a \in c} a \neq f(a)$	$f : c \to d$ bijective	$\neg(c \overset{\vee}{=} d)$
$c \overset{\vee}{\neq} d$	$\bigvee_{a \in c} a \neq f(a)$	$f : c \to d$ surjective	$\neg(c \overset{\wedge}{=} d)$
	or		
	$\bigvee_{b \in d} b \neq g(b)$	$g : d \to c$ surjective	

Examples:

$$\{2,3\}^2 \overset{\{\}}{=} \{4,9\} \overset{\{\}}{=} \{9,4\}$$

$$\{2,3\}^2 \overset{\wedge}{=} \{4,9\}$$

$$\{2,3\}^2 \overset{\vee}{\neq} \{9,4\}$$

$$\pm 2 \overset{\{\}}{=} 2(\pm 1) \overset{\{\}}{=} 2(\mp 1)$$

$$\pm 2 \overset{\wedge}{=} 2(\pm 1)$$

$$\pm 2 \overset{\vee}{\neq} 2(\mp 1)$$

$$\pm 2 \overset{\vee}{=} 2$$

$$\pm 2 \overset{\vee}{\neq} 1$$

$$\mathbb{Z}^* \overset{\{\}}{=} -\mathbb{Z}^*, \text{where } \mathbb{Z}^* \text{ denotes the nonzero integers}$$

$$\mathbb{Z}^* \overset{\wedge}{\neq} -\mathbb{Z}^*$$

Classes are unordered. Significance of order within class lists is only to define the implicit function. and does not indicate order within the classes. This principle governs the use of $\pm$ and $\mp$, as shown in the above examples.

This notation is only necessary to distinguish between class and distributed equalities. Often these are equivalent, especially $\overset{\{\}}{=}$ and $\overset{\wedge}{=}$, or the type is understood, in which case it is sufficient to use $=$.

Multiple distribution

Arithmetic expressions involving multiple classes require more explicit disambiguation. $\pm$, for instance, can be regarded as a class of operations $\{+,-\}$, whether unary or binary. Multiple occurences of $\pm$ represent positions in which an operation may be chosen, e.g. $\pm 3 \pm 1$. By itself, such a formula is ambiguous: Does it represent $\{4,2,-2,-4\}$ or $\{4,-4\}$?

Conventional quantification is sufficient to disambiguate this type of expression, but we may also use a more elegant notation in which we number the independent occurrences and indicate them with a special subscript, e.g. $c_{:1}$, $c_{:2}$, etc.

Using this notation, we have $\{4,2,-2,-4\} \overset{\{\}}{=} \pm_{:1} 3 \pm_{:2} 1$ and $\{4,-4\} \overset{\{\}}{=} \pm_{:1} 3 \pm_{:1} 1$. The double statement $-\pm 2 \overset{\wedge}{=} \mp 2 \overset{\wedge}{\neq} \pm 2$ we can also express as $-\pm_{:1} 2 \overset{\wedge}{=} \pm_{:2} 2 \overset{\wedge}{\neq} \pm_{:1} 2$, and so ± 2 is a solution of $x_{:1} = -x_{:2}$ but not of $x_{:1} = -x_{:1}$.

Standard numeric classes

Letting $b := \{0, 1\}$, we can define some standard numeric classes as follows:

$$\begin{aligned}
\mathbb{N} &:= \bigcup_{k=0}^{\infty} k \\
\mathbb{Z} &:= \pm\mathbb{N} \\
\mathbb{Q} &:= \frac{\mathbb{Z}_{:1}}{\mathbb{Z}_{:2}} \\
I &:= \sum_{k=1}^{\infty} b_{:k} 2^{-k} \\
\mathbb{R} &:= \bigcup_{k=-\infty}^{\infty} I_{:k} + k = \bigcup_{k=-\infty}^{\infty} b_{:k} 2^{k} \\
\mathbb{C} &:= \mathbb{R}_{:1} + \mathbb{R}_{:2}\boldsymbol{i}
\end{aligned}$$

Three dimensional real space $\mathbb{R}^3$, including points at infinity, can be expressed as $\{(a_1, a_2, a_3) \mid a_1, a_2, a_3 \in \mathbb{R}\}$ or more succinctly as $(\mathbb{R}_{:1}, \mathbb{R}_{:2}, \mathbb{R}_{:3})$. This is a slight inconsistency of notation, since numeristically $\mathbb{R}^3$ should mean $\{a^3 \mid a \in \mathbb{R}\}$, but since this is the same as $\mathbb{R}$, we use the former meaning for $\mathbb{R}^3$.

The empty class and the full class

The ***empty class*** or ***null class***, denoted $\emptyset$ or $\{\}$, is a class with no values. For example, $1 \cap 2 = \{\}$.

When a sentence distributes over the empty class, the result is the empty statement, no statement at all.

If a function f is undefined at a, we can say $f(a) = \emptyset$. The result of any arithmetic operation on $\emptyset$ is $\emptyset$, e.g. $1 + \emptyset = \emptyset$.

Similarly, we define the ***full class***, denoted $\not\!\emptyset$, as the complement of the empty class. At this point, we will use $\not\!\emptyset$ to denote the base class in which we are working, such as $\mathbb{R}$ or $\mathbb{C}$.

Properly, however, the full class is the class of all numbers. The full extent of numbers is infinite, as infinitely beyond the capacities of the human intellect as the numeric infinity is beyond numeric unity. Therefore we do not attempt to rigorously define number or gives a precise boundary to the full class.

We consider the place of the full class the real and complex arithmetic below, in Indeterminate expressions and the full class.

Inverses

One important consequence of the flat structure of numeristic classes is that the inverse of a function is always itself a function. For instance, the inverse of $f(x) = x^2$ is $f^{-1}(x) = \pm\sqrt{x}$.

For a general function f, $f^{-1}(x) = \{a \mid f(a) = x\}$. It follows that $f^{-1}(x) = \{\}$ is equivalent to $\neg(\exists a) f(a) = x$. It also follows that if $f(x) = y$, then $f^{-1}(y) \supseteq x$.

We may use the radical symbol as a multivalued inverse, i.e. $\sqrt[n]{b} := \{a \in C \mid a^n = b\}$.

That inverse functions give us the same type of value as the original function is an important feature of numeristics. It means we can always retrace our steps and return to the starting point with a minimum of formulaic overhead.

INFINITY AND INFINITE ELEMENT EXTENSIONS

Infinity and division by zero

The principle of freedom of numeric operation includes division by zero. As a numeric value, we define infinity as $\left|\frac{1}{0}\right|$ and denote it ∞. We add this and possibly other infinite values to comprise the class of infinite points in a larger numeric class. For any base space, such as $\mathbb{R}$ or $\mathbb{C}$, there are several ways we can add infinite elements, some of which are discussed below.

Outside of numeristics, the symbol ∞ usually does not denote an actual quantity but is only used as a modifier to denote certain types of limit, sum, or integral. By contrast, in numeristics infinity is a number or class of numbers.

Since $a \cdot 0 = 0$ for any finite a, $\frac{0}{0}$ includes all finite numbers. ∞ is unchanged by the addition of any finite number, i.e. $\infty + a = \infty$ for any finite a.

Similarly, we have $\infty - \infty = \frac{0}{0}$. Thus, $a - a = 0$ only for finite a, while for general a, $a - a \supseteq 0$.

Since $0 \cdot 0 = 0$ and $\infty \cdot \infty = \left|\frac{1}{0}\right|\left|\frac{1}{0}\right| = \left|\frac{1}{0}\right| = \infty$, we have $0, \infty \in \frac{0}{0} = 0 \cdot \infty = \frac{\infty}{\infty}$. Thus, in the base spaces we are investigating here, $\frac{0}{0} = \not\emptyset$. While we may use the word "indeterminate" to describe such expressions, we should remember that they are fully defined as the unrestricted or full class. We completely avoid describing division by zero as "undefined," since it yields one or more well defined infinite values.

When $a = 0$ or ∞, we say it is *afinite*; otherwise we say it is *perfinite*. $\frac{a}{a} = 1$ only for perfinite a, while for afinite a, $\frac{a}{a} \supset 1$. Thus for general a, $\frac{a}{a} \supseteq 1$.

Real infinite element extensions

Here we examine two methods of adding infinite elements to the real numbers, the ***projectively extended*** real numbers and the ***affinely extended*** real numbers. These methods are known to conventional mathematics, although terminology and notation vary. The difference in numeristics is in the handling of multivalued expressions such as $\frac{0}{0}$ and $\infty - \infty$, which are undefined in the conventional approach but are multivalued classes in numeristics.

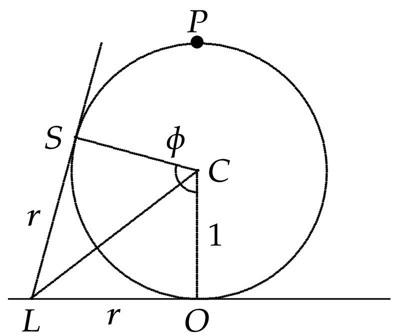

FIG. 3:
Projectively extended
real numbers,
one-point method
$r = \tan \frac{\phi}{2}$

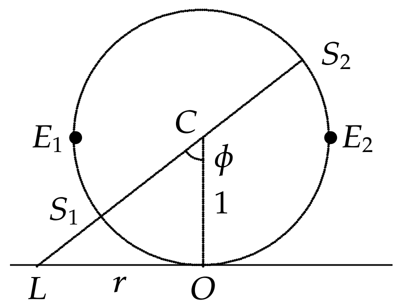

FIG. 4:
Projectively extended
real numbers,
two-point method
$r = \tan \phi$

The first method of adding infinite elements that we examine adds one infinite element to the real numbers. Figures 3 and 4 show two different methods of mapping real numbers to a circle. In Figure 3, every real number r at position L uniquely maps to some point S on the circle, and the infinite element ∞ maps to the point P at the top of the circle. The angle ϕ is called the ***colatitude*** of the point S. In Figure 4, r is mapped to a pair of points S_1 and S_2, and ∞ is mapped to E_1 and E_2.

This extended version of the real numbers is called the ***projectively extended real numbers*** and is denoted $\widehat{\mathbb{R}}$, $P^1(\mathbb{R})$, or $\mathbb{RP}^1$.

In $\widehat{\mathbb{R}}$, $+\infty = -\infty$, and trichotomy fails for ∞. For finite c, we may have one of two conventions: (1) both $c < \infty$ and $c > \infty$, or (2) neither $c < \infty$ nor $c > \infty$. We also have $e^\infty = c^\infty = 0, \infty$.

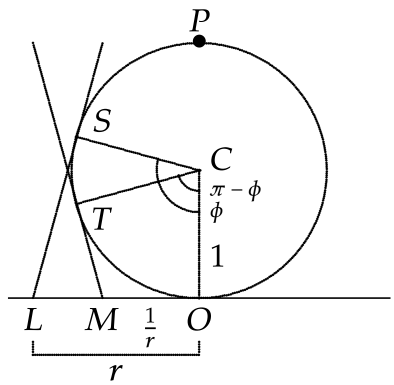

Fig. 5:
Geometric relation
of reciprocals in
projectively extended
real numbers,
one-point method

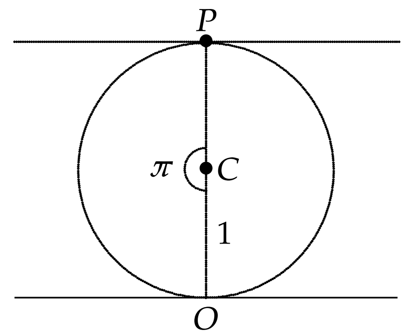

Fig. 6:
Geometric relation
of 0 and ∞ in
projectively extended
real numbers,
one-point method

Figures 5 and 6 show how pairs of reciprocals of projectively extended real numbers are mapped to pairs of points on the circle. Figure 5 uses the method of Figure 3. Colatitudes of reciprocals are supplemental, since $\tan \frac{\phi}{2} = \frac{1}{\cot \frac{\phi}{2}} = \frac{1}{\tan \frac{\pi-\phi}{2}}$.

Figure 6 uses the relation shown in Figure 5 to map ∞ as the reciprocal of 0 to the point P, whose colatitude is π.

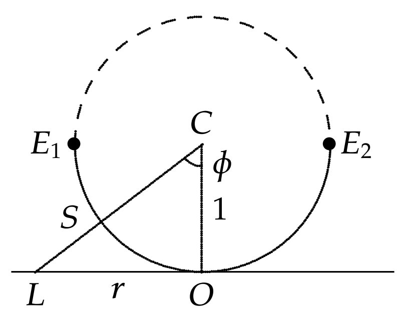

Fig. 7:
Affinely extended real numbers
$r = \tan \phi$

Figure 7 shows the second method of adding infinite elements, which adds two infinite elements to the real numbers, $+\infty$ and $-\infty$. In this figure also, every real number

r at position L uniquely maps to some point S on the solid semicircle, but $-\infty$ maps to E_1 and $+\infty$ to E_2.

This extended version of the real numbers is called the ***affinely extended real numbers*** and is denoted $\overline{\mathbb{R}}$.

In $\overline{\mathbb{R}}$, $+\infty \neq -\infty$, and trichotomy holds for infinite elements: For finite c, $c < +\infty$, $c > -\infty$, $e^{+\infty} = +\infty$, and $e^{-\infty} = 0$.

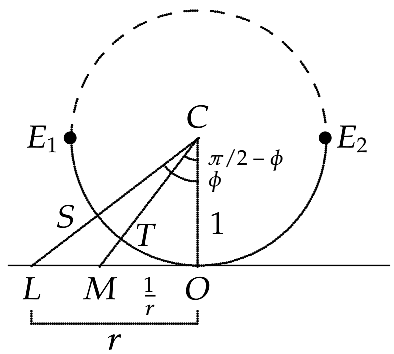

FIG. 8:
Geometric relation of reciprocals in affinely extended real numbers

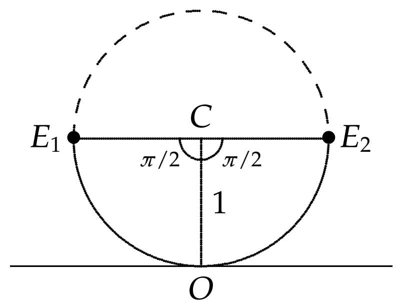

FIG. 9:
Geometric relation of 0 and $\pm\infty$ in affinely extended real numbers

Figures 8 and 9 show how pairs of reciprocals of affinely extended real numbers are mapped to pairs of points on the semicircle. Figure 8 uses the method of Figure 7. Colatitudes of reciprocals are complementary, since $\tan\phi = \frac{1}{\cot\phi} = \frac{1}{\tan\left(\frac{\pi}{2}-\phi\right)}$.

Figure 9 uses the relation shown in Figure 8 to map $\pm\infty$ as the reciprocal of 0 to the points E_1 and E_2, whose colatitudes are $\frac{\pi}{2}$.

Table 10 compares selected arithmetic operations in the projectively and affinely extended real numbers. Most properties of finite numbers hold in the extended systems, but not all; for example, in the projective system:

$$(2+0)\infty = 2\cdot\infty = \infty \quad \subset \quad 2\cdot\infty + 0\cdot\infty = \infty + \not\phi = \not\phi$$
$$(3\cdot 1)^{\infty} = 3^{\infty} = \infty \quad \subset \quad 3^{\infty}\cdot 1^{\infty} = \infty\cdot\not\phi = \not\phi$$

TABLE 10: **Arithmetic operations in projectively and affinely extended real numbers**

In this table, a is finite, b and c are perfinite, p is finite positive.

$\widehat{\mathbb{R}}$	$\overline{\mathbb{R}}$
$\mathbb{R} \cup \infty$	$\mathbb{R} \cup \{+\infty, -\infty\}$
$\infty := \frac{1}{0}$	$\infty := \left\lvert\frac{1}{0}\right\rvert$
$\frac{1}{0} = \infty$	$\frac{1}{0} = \pm\infty$
$+\infty = -\infty$	$+\infty \neq -\infty$
$a \pm \infty = \infty$	$a \pm \infty = \pm\infty$
$\infty + \infty = \not\infty$	$\infty + \infty = \infty$
$\infty - \infty = \not\infty$	$\infty - \infty = \not\infty$
$b\infty = \infty$	$p(\pm\infty) = \pm\infty$
$0\infty = \not\infty$	$0(\pm\infty) = \not\infty$
$\infty \cdot \infty = \infty$	$+\infty(\pm\infty) = \pm\infty, -\infty(\pm\infty) = \mp\infty$
$\frac{\infty}{b} = \infty$	$\frac{\pm\infty}{p} = \pm\infty$
$\frac{b}{\infty} = 0$	$\frac{b}{\pm\infty} = 0$
$\frac{b}{0} = \infty$	$\frac{b}{0} = \pm\infty$
$\frac{\infty}{0} = \infty$	$\frac{+\infty}{0} = \pm\infty$
$\frac{0}{\infty} = 0$	$\frac{0}{\pm\infty} = 0$
$\frac{0}{0} = \not\infty$	$\frac{0}{0} = \not\infty$
$\frac{\infty}{\infty} = \not\infty$	$\frac{\pm\infty}{\pm\infty} = \lvert\not\infty\rvert, \frac{\pm\infty}{\mp\infty} = -\lvert\not\infty\rvert$
$\frac{1}{x}$, $\tan x$ are continuous at $x = 0$	$\frac{1}{x}$, $\tan x$ are discontinuous at $x = 0$
$(b + c)\infty \subset b\infty + c\infty$	$(b + c)(\pm\infty) = b(\pm\infty) + c(\pm\infty)$
$(b + 0)\infty \subset b\infty + 0 \cdot \infty$	$(b + 0)(\pm\infty) \subset b(\pm\infty) + 0(\pm\infty)$
$(b + \infty)\infty = b\infty + \infty \cdot \infty$	$(b \pm_{:1} \infty)(\pm_{:2}\infty) = b(\pm_{:1}\infty) \pm_{:2} 0(\pm_{:1}\infty)$
$(0 + \infty)\infty \subset 0 \cdot \infty + \infty \cdot \infty$	$(0 \pm_{:1} \infty)(\pm_{:2}\infty) \subset 0(\pm_{:1}\infty) \pm_{:2} \infty(\pm_{:1}\infty)$

$a < \infty \wedge \infty < a$	$a < +\infty \wedge -\infty < a$
$b \subset \frac{b}{0} \cdot 0$	$b \subset \frac{b}{0} \cdot 0$
$b \subset \frac{b}{\infty} \cdot \infty$	$b \subset \frac{b}{\pm\infty} \cdot \pm\infty$
$a \subset (a + \infty) - \infty$	$a \subset (a \pm \infty) \mp \infty$
$e^{\infty} = \{0, \infty\}$	$e^{+\infty} = +\infty$
$e^{-\infty} = \{0, \infty\}$	$e^{-\infty} = 0$
$0^0 = \not\phi$	$0^0 = \not\phi$
$0^{\infty} = \{0, \infty\}$	$0^{\pm\infty} = \mp\infty$
$1^{\infty} = \not\phi$	$1^{\pm\infty} = \not\phi$
$(-1)^{\infty} = \{0, \infty\}$	$(-1)^{+\infty} = \{0, \pm\infty\}$
	$(-1)^{-\infty} = \{0, \pm\infty\}$
$\infty^0 = \not\phi$	$(\pm\infty)^0 = \not\phi$
$\infty^{\infty} = \{0, \infty\}$	$(+\infty)^{+\infty} = +\infty$
	$(-\infty)^{+\infty} = \pm\infty$
	$(\pm\infty)^{-\infty} = 0$
$\ln \infty = \infty$	$\ln \infty = +\infty$
$\ln 0 = \infty$	$\ln 0 = -\infty$
$\ln(-\infty) = \infty$	$\ln(-\infty) = \{\ \}$
$\lvert\infty\rvert = \infty$	$\lvert \pm \infty\rvert = +\infty$

Complex infinite element extensions

We now examine three methods of adding infinite elements to the complex numbers: the ***single projectively extended*** complex numbers, the ***double projectively extended*** complex numbers, and the ***affinely extended complex numbers***. Only the first of these methods is customarily defined in conventional mathematics.

The ***single projectively extended complex numbers***, commonly called the ***Riemann sphere***, adds a single infinite element to the complex numbers and is denoted $\widetilde{\mathbb{C}}$, $P^1(\mathbb{C})$, or $\mathbb{CP}^1$. Figure 3, which shows how each projectively extended real number is mapped to a point on a circle, also shows how each single projectively extended complex number is mapped to a point on a sphere, if we regard the line as any cross section of the complex plane through the origin and the circle as a cross section of the sphere. In the complex case, r is the radius from the origin, and the polar angle is perpendicular to the paper. This system is called single because it regards the complex numbers as a single complex dimension, rather than two real dimensions.

The ***double projectively extended complex numbers*** add a distinct infinite element for each pair of supplemental polar angles, i.e. a unique infinite element for each θ such that $0 \le \theta < \pi$. Each infinite element is called a ***directed infinity*** and can be denoted $e^{i\theta}\infty$, where $\infty = \left|\frac{1}{0}\right|$. This system is denoted $\widehat{\mathbb{C}}$. Figure 4, which shows how each projectively extended real number is mapped to a pair of antipodal points on a circle, also shows how each double projectively extended complex number is mapped to a pair of points on a sphere. This system is called double because it regards the complex plane as a two-dimensional real projective plane with an associated complex arithmetic.

The ***affinely extended complex numbers*** add a distinct infinite element for each polar angle, i.e. a unique directed infinite element for each θ such that $0 \le \theta < 2\pi$. This system is denoted $\overline{\mathbb{C}}$. Figure 7, which shows how each affinely extended real number is mapped to a point on a semicircle, also shows how each affinely extended complex number is mapped to a point on a hemisphere.

Table 11 shows compares selected arithmetic operations in these three complex infinite element extensions.

TABLE 11: **Arithmetic operations in the single projectively extended complex numbers, the double projectively extended complex numbers, and the affinely extended complex numbers**

In this table, a and d are finite complex, b and c are perfinite complex,
p is finite positive real, q is perfinite positive real,
r and s are real, w and z are perfinite complex,
$[0, 1)$.
Complex numbers are given in polar form $re^{i\theta}$ since rectangular form $a + bi$ does not properly distinguish infinite elements.

$\widetilde{\mathbb{C}}$	$\widehat{\mathbb{C}}$	$\overline{\mathbb{C}}$
$\mathbb{C} \cup \infty$	$\mathbb{C} \cup \infty e^{i\mathbb{R}}$	$\mathbb{C} \cup \infty e^{i\mathbb{R}}$
$e^{ir}\infty = \infty$	$e^{ir}\infty$ is unique for $r \in [0, \pi)$	$e^{ir}\infty$ is unique for $r \in [0, 2\pi)$
$\infty := \frac{1}{0}$	$\infty := \left\lvert\frac{1}{0}\right\rvert$	$\infty := \left\lvert\frac{1}{0}\right\rvert$
$\frac{1}{0} = \infty$	$\frac{1}{0} = \infty e^{i\mathbb{R}}$	$\frac{1}{0} = \infty e^{i\mathbb{R}}$
$+\infty = -\infty = i\infty = -i\infty$	$+\infty = -\infty \neq i\infty = -i\infty$	$+\infty \neq -\infty \neq i\infty \neq -i\infty$

$e^{ir}\infty = \infty$	$e^{ir}\infty = e^{i(r+\pi)}\infty$	$e^{ir}\infty = e^{i(r+2\pi)}\infty$
$a + \infty = \infty$	$a + e^{ir}\infty = \pm e^{ir}\infty$	$a + e^{ir}\infty = e^{ir}\infty$
$\infty + \infty = \infty - \infty = \not\infty$	$e^{ir}\infty + e^{is}\infty =$ $\left(\pm e^{ir} + \pm e^{is}\right)\infty$	$e^{ir}\infty + e^{is}\infty =$ $\left(e^{ir} + e^{is}\right)\infty$
$b\infty = \infty$	$qe^{ir}\left(e^{is}\infty\right) = \pm e^{i(r+s)}\infty$	$qe^{ir}\left(e^{is}\infty\right) = e^{i(r+s)}\infty$
$0\infty = \not\infty$	$0e^{ir}\infty = \not\infty$	$0e^{ir}\infty = \not\infty$
$\infty \cdot \infty = \infty$	$\left(e^{ir}\infty\right) \cdot \left(e^{is}\infty\right) =$ $\pm e^{i(r+s)}\infty$	$\left(e^{ir}\infty\right) \cdot \left(e^{is}\infty\right) =$ $e^{i(r+s)}\infty$
$\dfrac{\infty}{b} = \infty$	$\dfrac{e^{ir}\infty}{e^{is}q} = \pm e^{i(r-s)}\infty$	$\dfrac{e^{ir}\infty}{e^{is}q} = e^{i(r-s)}\infty$
$\dfrac{b}{\infty} = 0$	$\dfrac{b}{e^{ir}\infty} = 0$	$\dfrac{b}{e^{ir}\infty} = 0$
$\dfrac{b}{0} = \infty$	$\dfrac{b}{0} = U\infty = e^{\pi iI}\infty$	$\dfrac{b}{0} = U\infty = e^{2\pi iI}\infty$
$\dfrac{\infty}{0} = \infty$	$\dfrac{e^{ir}\infty}{0} = e^{\pi iI}\infty$	$\dfrac{e^{ir}\infty}{0} = e^{2\pi iI}\infty$
$\dfrac{0}{\infty} = 0$	$\dfrac{0}{e^{ir}\infty} = 0$	$\dfrac{0}{e^{ir}\infty} = 0$
$\dfrac{0}{0} = \not\infty$	$\dfrac{0}{0} = \not\infty$	$\dfrac{0}{0} = \not\infty$
$\dfrac{\infty}{\infty} = \not\infty$	$\dfrac{e^{ir}\infty}{e^{is}\infty} = e^{i(r-s)}\widehat{\mathbb{R}}$	$\dfrac{e^{ir}\infty}{e^{is}\infty} = e^{i(r-s)}\lvert\overline{\mathbb{R}}\rvert$
$\dfrac{1}{x}$, $\tan x$ are continuous at $x = 0$	$\dfrac{1}{x}$, $\tan x$ are continuous at $x = 0$ in the real direction, discontinuous in other directions	$\dfrac{1}{x}$, $\tan x$ are discontinuous at $x = 0$
$(b + c)\infty \subset$ $b\infty + c\infty$ for $b + c \neq 0$	$(b + c)\infty e^{ir} \subset$ $b\infty e^{ir} + c\infty e^{ir}$ for $b + c \neq 0$ and $\dfrac{b}{c} \notin \mathbb{R}$	$(b + c)\infty e^{ir} \subset$ $b\infty e^{ir} + c\infty e^{ir}$ for $b + c \neq 0$ and $\dfrac{b}{c} \notin \lvert\mathbb{R}\rvert$
$(b + 0)\infty \subset$ $b\infty + 0 \cdot \infty$	$(b + 0)\infty e^{ir} \subset$ $b\infty e^{ir} + 0 \cdot \infty e^{ir}$	$(b + 0)\infty e^{ir} \subset$ $b\infty e^{ir} + 0 \cdot \infty e^{ir}$
$(b + \infty)\infty =$ $b\infty + \infty \cdot \infty$	$(b + \infty e^{is})\infty e^{ir} \subset$ $b\infty e^{ir} + \infty e^{is}\infty e^{ir}$ for $\dfrac{b}{e^{is}} \notin \mathbb{R}$	$(b + \infty e^{is})\infty e^{ir} \subset$ $b\infty e^{ir} + \infty e^{is}\infty e^{ir}$ for $\dfrac{b}{e^{is}} \notin \lvert\mathbb{R}\rvert$

$(0+\infty)\infty \subset 0\cdot\infty + \infty\cdot\infty$	$(0+\infty e^{is})\infty e^{ir} \subset 0\cdot\infty e^{ir} + \infty e^{is}\infty e^{ir}$	$(0+\infty e^{is})\infty e^{ir} \subset 0\cdot\infty e^{ir} + \infty e^{is}\infty e^{ir}$
$b \subset \dfrac{b}{0}\cdot 0$	$b \subset \dfrac{b}{0}\cdot 0$	$b \subset \dfrac{b}{0}\cdot 0$
$b \subset \dfrac{b}{\infty}\cdot \infty$	$b \subset \dfrac{b}{\infty e^{ir}}\cdot \infty e^{ir}$	$b \subset \dfrac{b}{\infty e^{ir}}\cdot \infty e^{ir}$
$a \subset (a+\infty) - \infty$	$a \subset \left(a+\infty e^{ir}\right) - \infty e^{ir}$	$a \subset \left(a+\infty e^{ir}\right) - \infty e^{ir}$
$e^{\infty} = \{0, \infty\}$	$e^{\infty} = \{0, \infty e^{i\mathbb{R}}\}$	$e^{\infty} = \infty e^{i\mathbb{R}}$
$e^{-\infty} = \{0, \infty\}$	$e^{-\infty} = \{0, \infty e^{i\mathbb{R}}\}$	$e^{-\infty} = 0$
$e^{i\infty} = \{0, \infty\}$	$e^{i\infty} = \{0, \infty e^{i\mathbb{R}}\}$	$e^{i\infty} = \{0, \infty e^{i\mathbb{R}}\}$
$e^{e^{ir}\infty} = \{0, \infty\}$	$e^{e^{ir}\infty} = \{0, \infty e^{i\mathbb{R}}\}$	$e^{e^{ir}\infty} =$ $\infty e^{i\mathbb{R}}$ for $\operatorname{Re} e^{ir} > 0$, 0 for $\operatorname{Re} e^{ir} < 0$ $\{0, \infty e^{i\mathbb{R}}\}$ for $\operatorname{Re} e^{ir} = 0$
$\ln\infty = \infty$	$\ln\infty = \pm\infty$	$\ln\infty = \infty$
$\ln(-\infty) = \infty$	$\ln(-\infty) = \pm\infty$	$\ln(-\infty) = \infty$
$\ln(i\infty) = \infty$	$\ln(i\infty) = \pm\infty$	$\ln(i\infty) = \infty$
$\ln 0 = \infty$	$\ln 0 = \pm\infty$	$\ln 0 = -\infty$
$\ln 1 = 2\widetilde{\mathbb{Z}}\pi i$	$\ln 1 = 2\widehat{\mathbb{Z}}\pi i$	$\ln 1 = 2\overline{\mathbb{Z}}\pi i$
$\ln(-1) = (2\widetilde{\mathbb{Z}}+1)\pi i$	$\ln(-1) = (2\widehat{\mathbb{Z}}+1)\pi i$	$\ln(-1) = (2\overline{\mathbb{Z}}+1)\pi i$
$\sqrt{\infty} = \infty$	$\sqrt{\infty} = \{\pm\infty, \pm i\infty\}$	$\sqrt{\infty} = \pm\infty$
$\lvert\infty\rvert = \infty$	$\lvert\infty e^{ir}\rvert = \infty$	$\lvert\infty e^{ir}\rvert = \infty$

Extensions to other standard classes

We may classify ∞ as a natural number since it is the sum of other natural number s: $\infty = 1+1+1+\ldots$. In this case, we can add one or two infinite numbers to the natural numbers, integers, and rational numbers.

$$\widehat{\mathbb{N}} := \mathbb{N} \cup \infty$$

$$\overline{\mathbb{N}} := \mathbb{N} \cup \infty$$

$$\widehat{\mathbb{Z}} := \mathbb{Z} \cup \infty$$

$$\overline{\mathbb{Z}} := \mathbb{Z} \cup \pm\infty$$

$$\widehat{\mathbb{Q}} := \mathbb{Q} \cup \infty$$

$$\overline{\mathbb{Q}} := \mathbb{Q} \cup \pm\infty$$

Some properties of these numbers change when they are extended this way. For instance, in the extended integers, the sum of an integer and a noninteger may be an integer.

Indeterminate expressions and the full class

Conventional infinite element extensions leave indeterminate expressions such as $\frac{0}{0}$ and $\infty - \infty$ undefined, since they cannot handle multivalued expressions. Any assignment of such expressions to single values creates inconsistencies. For example, if we define $0 \cdot \infty$ as 1, then associativity of multiplication fails: $2 \cdot (0 \cdot \infty) = 2$, but $(2 \cdot 0) \cdot \infty = 1$. If we define $(+\infty) + (-\infty)$ as 0, then associativity of addition fails: $1 + [(+\infty) + (-\infty)] = 1$, but $[1 + (+\infty)] + (-\infty) = 0$.

This means that infinite element extensions, whether numerisitic or conventional, under addition or multiplication or both, do not satisfy the axioms of conventional algebraic structures such as group, ring, or field, since there is no single valued binary operation which satisfies the respective axioms and is defined for all elements. For example, the affinely extended real numbers are not even a semigroup under addition, since $(+\infty) + (-\infty)$ is either undefined (conventional) or $\not\infty$ (numeristic).

In numeristics, indeterminate expressions play an important role of connecting classes. For example, even though 0 is a natural number, $\frac{0}{0}$ includes nonintegral, irrational, and imaginary elements.

The above tables state that indeterminate expressions such as $\frac{0}{0}$ and $\infty - \infty$ are equal to $\not\infty$, but we must be aware that, while they include all values in the elementary classes $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ that we have considered so far, they may not include absolutely all numbers. For instance, there are classes in which there exist a such that $0a \neq 0$, so $a \notin \frac{0}{0}$ but $a \in \not\infty$.

To clarify this situation, we can use intersection: for example, we can say

$$\frac{0}{0} \cap \mathbb{R} = \not\infty \cap \mathbb{R} = \mathbb{R}.$$

A similar situation may occur with determinate expressions, such as the different interpretations of $\sqrt{-1}$ in the complex numbers $\mathbb{C}$ and the quaternions $\mathbb{H}$:

$$\begin{aligned} \sqrt{-1} \cap \mathbb{C} &= \pm i \\ \sqrt{-1} \cap \mathbb{H} &= \{ai + bj + ck \mid a, b, c \in \mathbb{R} \land a^2 + b^2 + c^2 = 1\} \\ &= ie^{k\mathbb{R}_{:1}}e^{i\mathbb{R}_{:2}} \end{aligned}$$

FURTHER NUMERISTIC CALCULATIONS

Signum function

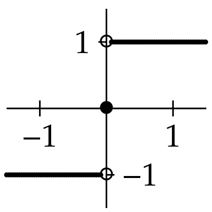

FIG. 12:
Conventional signum
function $f(x) = \operatorname{sgn} x$

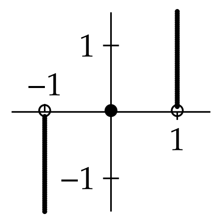

FIG. 13:
Conventional signum function
inverse $f^{-1}(x) = \operatorname{sgn}^{-1} x$

Figure 12 shows the usual form of the signum (or sign) function $\operatorname{sgn} x$, which can be defined by either

$$f(x) = \operatorname{sgn} x = \begin{cases} \dfrac{x}{|x|} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

or

$$f(x) = \operatorname{sgn} x = \begin{cases} -1 & \text{for } x < 0 \\ 0 & \text{for } x = 0 \\ +1 & \text{for } x > 0 \end{cases} .$$

Figure 13 shows the inverse $\operatorname{sgn}^{-1} x$, which is not single valued, and is therefore not a function in the conventional sense, but is a function in the numeristic sense. It can also be expressed as

$$f^{-1}(x) = \operatorname{sgn}^{-1} x = \begin{cases} \mathbb{R}^- & \text{for } x = -1 \\ 0 & \text{for } x = 0 \\ \mathbb{R}^+ & \text{for } x = +1 \\ \{\} & \text{otherwise} \end{cases} .$$

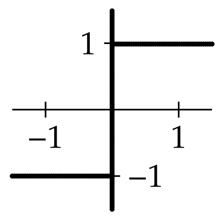

FIG. 14:
Alternate signum
function $f(x) = \mathrm{sgn}_2\, x$

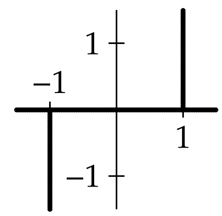

FIG. 15:
Alternate signum function
inverse $f^{-1}(x) = \mathrm{sgn}_2^{-1}\, x$

Figure 14 shows a revised form of the signum function, $\mathrm{sgn}_2\, x$, defined as

$$\mathrm{sgn}_2\, x = \frac{x}{|x|}$$

for all x, which can also be expressed as

$$g(x) = \mathrm{sgn}_2\, x = \begin{cases} -1 & \text{for } x < 0 \\ \mathbb{R} & \text{for } x = 0 \\ +1 & \text{for } x > 0 \end{cases}.$$

The vertical line at $x = 0$ shows that the value at this point is the indeterminate class $\mathrm{sgn}_2\, 0 = \dfrac{0}{0}$.

Figure 15 shows the inverse, $\mathrm{sgn}_2^{-1}\, x$, which can be expressed as

$$\mathrm{sgn}_2^{-1}(x) = \begin{cases} -|\mathbb{R}| & \text{for } x = -1 \\ |\mathbb{R}| & \text{for } x = +1 \\ 0 & \text{otherwise} \end{cases}.$$

Neither $\mathrm{sgn}_2\, x$ nor $\mathrm{sgn}_2^{-1}\, x$ are single valued and therefore cannot be conventional functions, but both are numeristic functions.

Solution of $x = rx$

As a demonstration of numerisitic techniques, we consider the equation $x = rx$. A conventional solution could run as follows:

$$x - rx = 0$$

$$x(1 - r) = 0,$$

from which we conclude that $x = 0$, except for $r = 1$, where x is indeterminate.

This is not a complete numeristic solution, since it assumes that for any a and b, $a - a = 0$, and $ab = 0$ implies $a = 0$ or $b = 0$. Both of these assumptions are valid only for finite a and b.

We now examine a numeristic solution, which adds all the neglected cases.

1. $r = 1$: x is unrestricted.
2. $r = 0$:
   a. x finite: $x = 0$.
   b. x infinite: $x = \infty$. Here we allow "=" to also mean "$\supseteq$" in the original equation.
3. Other finite r:
   a. x finite:

$$\begin{aligned} x - rx &= 0 \\ x(1-r) &= 0 \\ x &= 0 \end{aligned}$$

   b. x infinite: $x = \infty$.
4. Infinite r: Invert the equation and follow the case $r = 0$:

$$\begin{aligned} \frac{1}{x} &= 0\frac{1}{x} \\ \frac{1}{x} &= 0, \infty \\ x &= 0, \infty \end{aligned}$$

Singular matrices

The inverse of a 2×2 matrix is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

We can use infinite elements to apply this to a singular matrix:

$$\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} = \frac{1}{0} \begin{pmatrix} 2 & -2 \\ -1 & 1 \end{pmatrix}.$$

In the projectively extended real numbers, this yields

$$\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} \infty & \infty \\ \infty & \infty \end{pmatrix} = \infty \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

while in the affinely extended real numbers, this is

$$\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} \pm\infty & \mp\infty \\ \mp\infty & \pm\infty \end{pmatrix} = \pm\infty \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

The product of the original matrix and its inverse are

$$\infty \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 2 & -2 \\ -1 & 1 \end{pmatrix} = \infty \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \not\infty_{:11} & \not\infty_{:21} \\ \not\infty_{:12} & \not\infty_{:22} \end{pmatrix} = \mathbb{R}^{2\times 2}.$$

We can also use determinants of singular matrices to solve degenerate cases of simultaneous equations.

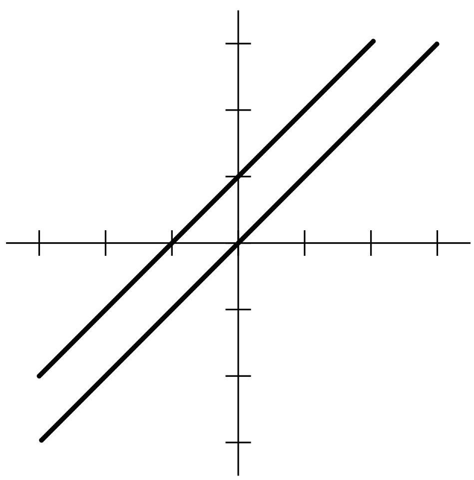

FIG. 16: Parallel simultaneous equations $x - y = -1$ and $x - y = 0$

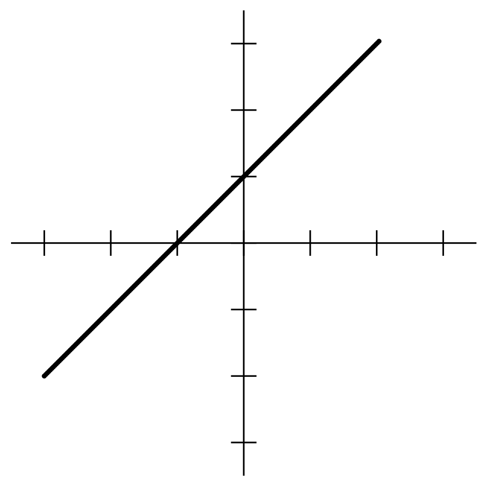

FIG. 17: Coincident simultaneous equations $x - y = -1$ and $2x - 2y = -2$

Figure 16 shows a system of two equations whose graphs are parallel. The solution by Cramer's rule is

$$x = \frac{\begin{vmatrix} -1 & -1 \\ 0 & -1 \end{vmatrix}}{\begin{vmatrix} 1 & -1 \\ 1 & -1 \end{vmatrix}} = \frac{2}{0} = \pm\infty$$

$$y = \frac{\begin{vmatrix} -1 & -1 \\ 1 & 0 \end{vmatrix}}{\begin{vmatrix} 1 & -1 \\ 1 & -1 \end{vmatrix}} = \frac{2}{0} = \pm\infty$$

The algebraic solution is one (projective) or two (affine) points at infinity, which agrees with the geometric solution as the meeting point of parallel lines.

Figure 17 shows a system of two equations whose graphs coincide. Again by Cramer's rule, the solution is

$$x = \frac{\begin{vmatrix} -1 & -1 \\ -2 & -2 \end{vmatrix}}{\begin{vmatrix} 1 & -1 \\ 2 & -2 \end{vmatrix}} = \frac{0}{0} = \mathbb{R}$$

$$y = \frac{\begin{vmatrix} -1 & -1 \\ 1 & -2 \end{vmatrix}}{\begin{vmatrix} 1 & -1 \\ 2 & -2 \end{vmatrix}} = \frac{0}{0} = \mathbb{R}$$

The algebratic solution shows that there is a solution for every value of x and y, which agrees with the geometric solution of the whole grapsh.

HOW NUMERISTICS HANDLES RUSSELL'S PARADOX

Russel's paradox (or antinomy) demonstrates a weakness of naive set theory, the predecessor to axiomatic set theories. In naive set theory, a set is allowed to be an element of itself. Russell's paradox considers the set S which contains all sets that do not contain themselves. The existence of such a set leads to a contradiction: if S contains itself, then by definition it does not contain itself, and if it does not contain itself, then again by definition it contains itself.

Axiomatic set theories avoid this problem in various ways. Zermelo-Fraenkel set theory restricts the elements that sets can contain. Bernays-Gödel set theory makes a distinction between a *class*, which can contain elements, and a *set*, which can be an element; all sets are classes, but not all classes are sets.

Numeristic classes differ fundamentally from sets and set-theoretic classes, primarily in their flat structure, by which a class containing a single element is identical to the element itself. Every numeristic class contains itself, and, if it is an element, it is an element of itself. There is no numeristic class that does not contain itself. The class of all classes that do not contain themselves is therefore the empty class, the class of all elements satisfying contradictory conditions.

A NUMERISTIC VIEW OF ABSTRACTION

The numeristic view of the ultraprimitives of infinity, unity, and zero establishes that there is nothing more abstract than these ultraprimitives. In this light, the attribute "abstract" in the term *abstract structures* is something of a misnomer.

We can consider an abstract structure to be a single valued operation or a pair of single valued operations which satisfies certain conditions. For example, the class of groups is the class of single valued operations satisfying the group axioms, each operation being restricted to an appropiate class of elements.

The operations in this type of abstract structure are always single valued and thus cannot include indeterminate forms, which as we have seen naturally arise in the arithmetic of even very simple classes.

Moreover, the class of axioms in any abstract structure is always finite, whereas Gödel's incompleteness theorem establishes that no finite class of axioms can give us complete knowledge of any system that includes the natural numbers, addition multiplication, and quantifiers. This is one of the reasons that numeristics does not use axioms, instead relying on subjective and objective observation to establish rigor.

While there is often useful information in an abstract structure, the vision of the whole is lost. Numeristic ultraprimitives restore this vision, when they are experienced on the level of pure subjectivity.

We identify two kinds of abstraction:

- ***Abstraction of rules***: Categorizing or classifying a structure based on a portion of its properties; the type of abstraction in abstract algebra.
- ***Abstraction of reference***: Realization of the full extent of a structure, by transcending from object referral to subject referral; the type of abstraction that evolves in numeristics with the subjective experience of ultraprimitives, coupled with the objective experience of applications of numeristics.

APPENDIX: OTHER FOUNDATIONAL THEORIES

Maharishi Vedic Mathematics

Vedic means referring to Veda, an ancient body of knowledge preserved in India. Vedic literature contains much that is scientific and mathematical. Maharishi refers to Maharishi Mahesh Yogi, who revived the knowlege and experience of Veda.

Maharishi Vedic Mathematics can be described as the mathematics of nature or the mathematics of pure consciousness. Pure consciousness is a state of pure subjectivity, independent of any objects of experience. Mathematically, it is focused on the ultraprimitives described above, especially zero.

The experience and understanding of pure consciousness is the inspiration for the subjective side of numeristics. The objective side is provided by modern mathematics. Numeristics is an attempt to bring these two together into a single compatible field of knowledge.

> Vedic Mathematics is the structuring dynamics of Natural Law; it spontaneously designs the source, course, and goal of Natural Law—the orderly theme of evolution.
>
> Vedic Mathematics, the system of maintaining absolute order, is the reality of self-referral consciousness, which, fully awake within itself, forms the structures of the Veda and Vedic Literature, and further proceeds to structure the fundamentals of creation in the most perfect, eternal, symmetrical order, and eternally glorifies creation on the ground of evolution.
>
> Vedic Mathematics is the quality of infinte organizing power inherent in the structure of self-referral consciousness—pure knowledge—the Veda.
>
> As Veda is structured in consciousness, Vedic Mathematics is the mathematics of consciousness; coexistence of simultaneity and sequence characterize Vedic Mathematics.
>
> As self-Referral consciousness is the Unity (Saṁhitā) of observer (Ṛishi), process of observation (Devatā), and observed (Chhandas), Vedic Mathematics, being the mathematics of self-referral consciousness, is the mathematics of the relationship between these four values—Saṁhitā, Ṛishi, Devatā, Chhandas.
>
> Vedic Mathematics is the mathematics of relationship; it is the science of relationship. Vedic Mathematics is the system of maintaining perfect order in all relationships.
>
> Vedic Mathematics, being the mathematics of the order-generating principle of pure consicousness, it itself the mathematician, the process of deriving results, and the

conclusion; whatever consciousness is and wherever consciousness is, there is the structure of Vedic Mathematics, the source of perfect order.

—Maharishi Mahesh Yogi, [M96, p. 338–340]

The mechanics of ordering have to be mathematically derived in order for the knowledge to be really complete, and also for the infinite organizing power of knowledge to be precisely, properly, and thoroughly applied so that life can be naturally lived on the ground of orderly evolution, so that nothing shadows life—nothing shadows the immortal, eternal continuum of bliss, which is the nature of the self-sufficient, self-referral quality of the Absolute Number, from where everything emerges, through which everything is sustained, and to which everything evolves.

Unless the Absolute Number is enlivened in conscious awareness, unless the all-dimensional value of the Absolute Number is lively on the level of *SmṚiti*—the lively level of memory that maintains order and steers the evolutionary process—the process of computation, the process of ordering, cannot be explained, and cannot be practically lived in life.

It is a joy to mention here that Transcendental Meditation is the process of maintaining connectedness with the Absolute Number—the source of the creative process—and through this programme, the precision of evolution and order in the process of creation is enlivened in human awareness, and is expressed in all thought, speech, and action.

—Maharishi Mahesh Yogi, [M96, p. 616–617]

We admire the achievement of scientists in every field of modern science—Physics, Chemistry, etc.—who have presented in one symbol the entire knowledge of the ever-expanding universe. What remains to be achieved is that every mathematical symbol is able to whisper *I am Totality—Ahaṁ Brahmāsmi.*

What remains to be acheived is that every physical expression of total knowledge (mathematical symbol), is awakened to feel and say and behave with the total competence of the WHOLENESS of knowledge; what remains to be achieved is the enlivenment of the structure of knowledge in which one single symbol of Mathematics is a self-sufficiently lively field of intelligence WHICH CAN OPERATE FROM WITHIN ITSELF and self-sufficiently perform with precision and order from the level of the entire creative potential of intelligence of Cosmic Life; what remains to be achieved is the realization of the reality "*Anoranīyān* is *mahato mahīyān*"—smaller than the smallest is bigger than the biggest; what remains to be realized is the enlivenment of the silent objectivity of the mathematical symbol into the lively dynamism of the intelligence within it; what remains to be achieved is just one step from the object to the subject—from the objectivity of the mathematical expression to the field of subjectivity within it, so that the mathematician can identify his self-referral intelligence with the structure of intelligence within the physical structure of the mathematical formula.

This last step of knowledge, evolving from the objective quality of its structure to its lively subjective basis, is provided by my Vedic Mathematics; therefore my Vedic Approach (subjective approach), my approach of knowledge, my science of knowledge, through its subjective approach, has competence to enliven the spark of knowledge contained in any mathematical symbol (formula) of total knowledge from every field of modern science.

—Maharishi Mahesh Yogi, [M95, p. 296–298]

Skolem's primitive recursive arithmetic

In the early 20th century, Thoralf Skolem developed a formal foundational system based on natural numbers, standard logic, and primitive recursion. Skolem's system later became known as primitive recursive arithmetic and was used by Kurt Gödel in the proofs of his famous incompleteness theorems. Skolem's primitive recursive arithmetic is developed in detail in [S23].

Skolem's primitives include the following:

- classical logic (first order logic with quantifiers)
- the natural number 1
- successor operation of a natural number
- equality of natural numbers
- primitive recursion

His definitions include:

- order relations (using the successor operation)
- multiplication of natural numbers (using recursion)
- divisibility of natural numbers
- subtraction and division of natural numbers (in those cases where the result is a natural number)
- greatest common divisor and least common multiple
- prime numbers

The theorems include:

- associative and commutative laws of addition
- trichotomy of order relations
- distributive law of multiplication over addition

- associative and commutative laws of multiplication
- prime factorization

Skolem constructs a formal foundational theory with numbers and without sets. Like Weyl, using only a very few primitives, he develops a substantial numeric theory. One of his primitives is primitive recursion, which is a partial value of the basic self referral property of consciousness.

Skolem's primary goal in this paper is to develop a theory of natural numbers. He regards this theory as finitistic, in the sense that it contains no infinite elements and thus avoids the transfinite numbers of set theory. As he remarks in the concluding section of the paper: "[O]ne can doubt that there is any justification for the actual infinite or the transfinite." [S23, p. 332].

However, his system does generate an infinite number of finite numbers, and the number of referents of his primitive natural number 1 is infinite. His system therefore cannot count the number of numbers or the number of referents to any of the numbers.

The two quotes below, from Skolem's other works of this period, describe some aspects of the thinking which went into his creation of primitive recursive arithmetic.

> 7. ...[T]he notion that really matters in these logical investigations, namely "proposition following from certain assumptions", also is an inductive (recursive) one: the propositions we consider are those that are derivable by means of an *arbitrary finite number* of applications of the axioms. Thus the idea of the *arbitrary finite* is essential, and it would necessarily lead to a vicious circle if the notion "finite" were itself based, as in set theory, on certain axioms whose consistency would then in turn have to be investigated.
>
> Set theoreticians are usually of the opinion that the notion of integer should be defined and that the principle of mathematical induction should be proved. But it is clear that we cannot define or prove ad infinitum; sooner or later we come to something that is not further definable or provable. Our only concern, then, should be that the initial foundations be something immediately clear, natural, and not open to question. This condition is satisfied by the notion of integer and by inductive inferences, but it is decidedly not satisfied by set-theoretic axioms of the type of Zermelo's or anything else of that kind; if we were to accept the reduction of the former notions to the latter, the set-theoretic notions would have to be simpler than mathematical induction, and reasoning with them less open to question, but this runs entirely counter to the actual state of affairs.
>
> In a paper [Hi22] Hilbert makes the following remark about Poincaré's assertion that the principle of mathematical induction is not provable: "His objection that this principle could not be proved in any way other than by mathematical induction itself is unjustified and is refuted by my theory." But then the big question is whether we can prove this principle by means of simpler principles and *without using any property of finite expressions or formulas that in turn rests upon mathematical induction or is equivalent to it.* It seems to me that this latter point was not sufficiently taken into consideration by Hilbert. For example, there is in his paper (bottom of page 170), for a lemma, a proof in which he makes use of the fact that in any arithmetic proof in which a certain sign occurs that sign must necessarily occur for a first time. Evident though this property may be on the basis of our perceptual

intuition of finite expressions, a formal proof of it can surely be given only by means of mathematical induction. In set theory, at any rate, we go to the trouble of proving that every ordered finite set is well-ordered, that is, that every subset has a first element. Now why should we carefully prove this last proposition, but not the one above, which asserts that the corresponding property holds of finite arithmetic expressions occurring in proofs? Or is the use of this property not equivalent to an induction inference?

I do not go into Hilbert's paper in more detail, especially since I have seen only his first communication. I just want to add the following remark: It is odd to see that, since the attempt to find a foundation for arithmetic in set theory has not been very successful because of the logical difficulties inherent in the latter, attempts, and indeed very contrived ones, are now being made to find a different foundation for it—as if arithmetic had not already an adequate foundation in inductive inferences and recursive definitions.

8. So long as we are on purely axiomatic ground there is, of course, nothing special to be remarked concerning the principle of choice (though, as a matter of fact, new sets are *not* generated *univocally* by applications of this axiom); but if many mathematicians—indeed, I believe, most of them—do not want to accept the principle of choice, it is because they do not have an axiomatic conception of set theory at all. They think of sets as given by specification of arbitrary collections; but then they also demand that every set be definable. We can, after all, ask: What does it mean for a set to exist if it can perhaps never be defined? It seems clear that this existence can only be a manner of speaking, which can lead only to purely formal propositions—perhaps made up of very beautiful *words*—about objects *called* sets. But most mathematicians want mathematics to deal, ultimately with performable computing operations and not to consist of formal propositions about objects called this or that.

Concluding remark

The most important result above is that set-theoretic notions are relative. I had already communicated it orally to F. Bernstein in Göttingen in the winter of 1915–16. There are two reasons why I have not published anything about it until now: first, I have in the meantime been occupied with other problems; second, I believed that it was so clear that axiomatization in terms of sets was not a satisfactory ultimate foundation of mathematics that mathematicians would, for the most part, not be very much concerned with it. But in recent times I have seen to my surprise that so many mathematicians think that these axioms of set theory provide the ideal foundation for mathematics; therefore it seemed to me that the time had come to publish a critique.

—Thoralf Skolem, [S22, p. 299–301], emphasis his

I here permit myself a remark about the relation between the fundamental notions of logic and those of arithmetic. No matter whether we introduce the notion of propositional function in the first or the second way, we are confronted with the idea of the integer. For, even when the notion of propositional function is introduced axiomatically, we shall have to consider (for instance, in investigations concerning consistency) what we can derive by using the axioms an arbitrary finite number of times. On the other hand, it is not possible to characterize the number sequence logically without the notion of propositional function. For such a characterization must be equivalent to the principle of mathematical induction, and this reads as follows: If a propositional function $A(x)$ holds for $x = 1$ and if

$A(x+1)$ is true whenever $A(x)$ is true, then $A(x)$ is true for every x. In signs, it takes the form

$$\prod_U \left(\overline{U(1)} + \sum_x U(x)\overline{U(x+1)} + \prod_y U(y) \right)$$

[in modern notation

$$(\forall U)\left[\neg U(1) \vee (\exists x)(U(x) \wedge U(x+1)) \vee (\forall y)U(y)\right].$$

This proposition clearly involves the totality of propositional functions. Therefore, the attempt to base the notions of logic upon those of arithmetic, or vice versa, seems to me to be mistaken. The foundations for both must be laid simultaneously and in an interrelated way.

—Thoralf Skolem, [S28, p. 517]

Weyl's foundational system of the continuum

Also in the early 20th century, Hermann Weyl developed a theory of the real numbers, which he intended as an alternative to set theory as a foundation of analysis (calculus). Weyl bases his theory of the real continuum on natural numbers, basic logical operations, and primitive recursion, without transfinite set theory or proof by contradiction. This system of the real continuum is developed in [W32].

Weyl's primitives include the following:

- classical logic (first order logic with quantifiers)
- sets which have only numbers, or ordered multiples of numbers, as elements
- the natural numbers
- successor operation of a natural number
- identity (equality)
- iteration (primitive recursion)

His definitions include:

- relations
- order relations (using the successor operation)
- multiplication of natural numbers (using recursion)

- cardinality of sets
- fractions and rational numbers
- zero and negative rational numbers
- addition, subtraction, and multiplication of rational numbers
- real numbers (as cuts of rational numbers)
- addition, subtraction, multiplication, and division of real numbers, excluding division by zero
- exponentiation of real numbers by natural numbers (by recursion)
- algebraic numbers
- complex numbers (as real number pairs)
- sequences, limits, and convergece
- infinite series and power series
- continuity
- function inverses

Weyl indicates that it is possible to define the exponential function, logarithms, differentiation, and integration in his system, but he does not actually define them.

The theorems include:

- associative and commutative laws of addition
- trichotomy of order relations
- distributive law of multiplication over addition
- associative and commutative laws of multiplication
- Cauchy convergence principle
- Heine-Borel theorem in the one dimensional case of real intervals

Like Skolem, Weyl develops a substantial numeric theory using only a few primitives, which include natural numbers and primitive recursion. His theory has sets, but these sets include only numbers and ordered multiples of numbers, so there are no transfinite numbers. He defines division of real numbers but excludes division by zero. His system is less formal than most other foundational theories, which seems to be the result of his stated aim of providing a firm foundation for analysis. He anticipates numeristics by he using the natural numbers as a primitive rather than defining them as sets.

The quotes below describe some aspects of the thinking which went into Weyl's creation of his theory of the continuum.

> It is not the purpose of this work to cover the "firm rock" on which the house of analysis is founded with a fake wooden structure of formalism—a structure which can fool the reader and, ultimately, the author into believing that it is the true foundation. Rather, I shall show that this house is to a large degree built on sand. I believe that I can replace this shifting foundation with pillars of enduring strength.
>
> —Hermann Weyl, [W87, p. 1]

> It is characteristic of every mathematical discipline that 1) it is based on a sphere of operation such as we have presupposed here from the beginning; that 2) the natural numbers along with the relation S [successor relation] which connects them are always associated with this sphere; and that 3) over and above this composite sphere of operations, a realm of new ideal objects, of sets and functional connections is erected by means of the mathematical process which may, if necessary, be repated arbitrarily often. The old explantion of mathematics as the doctrine of number and space has, in view of the more recent development of our science, been judged to be too narrow. But, clearly, even in such disciplines as pure geometry, analysis situs [topology], group theory, and so on, the natural numbers are, from the start, related to the objects under consideration. So from now on we shall assume that at least one category of object underlies our investigation and that at least one of these underlying categories is that of the natural numbers. If there is more than one such category, we should recall the observation in §1 that each blank of a judgment scheme of a primitive or derived relation is affiliated with its own definite category of object. If the underlying sphere of operation described at the beginning of ths paragraph is that of the natural numbers, without anything further being added, then we arrive at *pure number theory*, which forms the centerpiece of mathematics; its concepts and results are clearly of significance for *every* mathematical discipline.
>
> If the natural numbers belong to the sphere of operations, then a new, important, and specifically mathematical principle of definition joins those enumerated in §2; namely, the principle of it iteration (definition by complete induction) by virtue of which the natural numbers first come into contact with the objects of the remaining categories of the underlying sphere of operations (if there are any).
>
> —Hermann Weyl, [W87, p. 25–26], emphasis his

[W]e are less certain than ever about the ultimate foundations of (logic and) mathematics; like everybody and everything in the world today, we have our "crisis". We have had it for nearly fifty years. Outwardly it does not seem to hamper our daily work, and yet I for one confess that it has had a considerable practical influence on my mathematical life: it directed my interests to fields I considered relatively "safe", and it has been a constant drain on my enthusiasm and determination with which I pursued my research work. The experience is probably shared by other mathematicians who are not indifferent to what their scientific endeavours mean in the contexts of man's whole caring and knowing, suffering and creative existence in the world.

—Hermann Weyl, [W46, p. 13]

The *circulus vitiosis* [vicious circle, of circular reasoning in the foundations of mathematics], which is cloaked by the hazy nature of the usual concept of set and function, but which we reveal here, is surely not an easily dispatched formal defect in the construction of analysis. Knowledge of its foundamental significance is something which, at this particular moment, cannot be conveyed to the reader by a lot of words. But the more distinctly the logical fabric of analysis is brought to givenness and the more deeply and completely the glance of consciousness penetrates it, the clearer it becomes that, given the current approach to foundational matters, every cell (so to speak) of this mighty organism is permeated by the poison of contradiction and that a thorough revision is necessary to remedy the situation.

A "hierarchical" version of analysis is artificial and useless. It loses sight of its proper object, i.e. number (cf. note 24). Clearly we must take the other path—that is, we must restrict the existence concept to the basic categories (here, the natural and rational numbers) and must not apply it in connection with the system of properties and relations (or the sets, real numbers, and so on, corresponding to them). In other words, the only natural strategy is *to abide by the narrower iteration procedure.* Further, only this procedure guarantees too that all concepts and results, quantities and operations of such a "precision analysis" are to be grasped as idealizations of analogues in a mathematics of approximation operating with "round numbers." This is of crucial significance with regard to *applications.*

—Hermann Weyl, [W87, p. 32], emphasis his

The concept of function has two historical roots. *First*, this concept was suggested by the "natural dependencies" which prevail in the material world—the dependencies which consist, on the one hand, in the fact that conditions and states of real things are variable over *time*, the paradigmatic independent variable, on the other hand, in the *causal* connections between action and consequence. The arithmetical-algebraic operations form a *second*, and entirely independent, source of the concept "function." For, in bygone days, analysis regarded a *function* as an expression formed from the independent variables by finitely many applications of four primary rules of arithmetic and a few elementary transcendental ones. Of course, these elementary operations have never been clearly and fully defined. And the historical development of mathematics has again and again pushed beyond boundaries which were drawn much too narrowly (even though those responsible for this development were not always entirely aware of what they were doing).

These two independent sources of the concept of function join together in the concept "law of nature." For in a law of nature, a natural dependence is represented as a function constructed in a purely conceptual-arithmetical way. Galileo's laws of falling bodies are the first great example. The modern development of mathematics has revealed that the algebraic principles of construction of earlier versions of analysis are much too narrow either for a general and logically natural construction of analysis or for the role which the concept "function" has to play in the formulation of the laws which govern material events. General logical principles of construction must replace the earlier algebraic ones. Renouncing such a construction altogether, as modern analysis (judging by the wording of its definitions) seems to have done, would mean losing oneself entirely in the fog; and, at the same time, the general notion of natural law would evaporate into emptiness. (But, happily, here too what one says and what one does are two different things.)

I may or may not have managed to fully uncover the requisite general logical principles of construction—which are based, on the one hand, on the concepts "and," "or," "not," and "there is," on the other, on the specifically mathematical concepts of set, function, and natural number (of iteration). (In any case, assembling these principles is not a matter of convention, but of logical discernment.) The one entirely certain thing is that the negative part of my remarks, i.e., the critique of the previous foundations of analysis and, in particular, the indicication of the circularity in them, are all sound. And one must follow my path in order to discover a way out.

With the help of a tradition bound up with that complex of notions which even today enjoys absolute primacy in mathematics and which is connected above all with the names Dedekind and Cantor, I have discovered, traversed, and here set forth my own way out of this circle. Only after having done so did I become acquainted with the ideas of Frege and Russell which point in exactly the same direction. Both in his pioneering little treatise (1884) and in the detailed work (1893), Frege stresses emphatically that by a "set" he means merely the scope (i.e., extension) of a concept and by a "correspondence" merely the scope or, as he says, the "value-range" of a relation. Russell's theory of logical types corresponds to the formation pf levels mentioned in §6 and is motivated by his "vicious-circle principle": "No totality can contain members defined in terms of itself." Of course, Poincaré's very uncertain remarks about impredicative definitions should also be noted here. But Frege, Russell, and Poincarè all neglect to mention what I regard as the crucial point, namely, that the principles of definition must be used to give a precise account of the sphere of the properties and relations to which the sets and mappings correspond. Russell's definition of the natural numbers as equivalence classes (a technique which he borrows from Frege) and his "Axiom of Reducibility" indicate clearly that, in spite of our agreement on certain matters, Russell and I are separated by a veritable abyss. So it is only to be expected that he discusses neither the "narrower procedure" nor the concept of function introduced at the end of §6.

My investigations began with an examination of Zermelo's axioms for set theory, which constitute an exact and complete formulation of the foundations of the Dedekind-Cantor theory. Zermelo's explanation of the concept "definite set-theoretic predicate," which he employs in the crucial "Subset"-Axiom III, appeared unsatisfactory to me. And in my effort to fix this concept more precisely, I was led to the principles of definition of 2. My attempt to formulate these principles as axioms of set formation and to express the requirement that sets be formed only by finitely many applications of the principles of construction embodied in the axioms—and, indeed, to do this *without presupposing the concept of the natural numbers*—drove me to a vast and ever more complicated formulation but, unfortunately, not to any satisfactory result. Only when I had achieved certain general philosophical insights (which, incidentally, required that I renounce conventionalism), did I realize that I was wrestling with a scholastic pseudo-problem. And I became firmly convinced (in agreement with Poincaré, whose philosophical position I share in so few other respects) that *the idea of iteration, i.e., of the sequence of the natural numbers, is an ultimate*

> *foundation of mathematical thought*—in spite of Dedekind's "theory of chains" which seeks to give a logical foundation for definition and inference by complete induction without employing our intuition of the natural numbers. For if it is true that the basic concepts of set theory can be grasped only through this "pure" intuition, it is unnecessary and deceptive to turn around then and offer a set-theoretic foundation for the concept "natural number." Moreover, I must find the theory of chains guilty of a *circulus vitiosus*. If we are to use our principles to erect a mathematical theory , we need a foundation—i.e., a basic category and a fundamental relation. As I see it, mathematics owes its greatness precisely to the fact that in nearly all its theorems what is essentially infinite is given a finite resolution. But this "infinitude" of the mathematical problems springs from the very foundation of mathematics—namely, the infinite sequence of the natural numbers and the concept of existence relevant to it. "Fermat's last theorem," for example, is intrinsically meaningful and either true or false. But I cannot rule on its truth or falsity by employing a systematic procedure for sequentially inserting all numbers in both sides of Fermat's equation. Even though, viewed in this light, this task is infinite, it will be reduced to a finite one by the mathematical proof (which, of course, in this notorious case, still eludes us).
>
> —Hermann Weyl, [W87, p. 45–49], emphasis his

Set theories

Usually "set theory" is referred to only in the singular, but in fact there are several varieties. See [Ho12]. By far the most commonly used and de facto standard is ***Zermelo-Fraenkel*** set theory (ZF), or its extension by the Axiom of Choice (ZFC). The set-theoretic notion of class comes from an equivalent axiomatization called ***Von Neumann–Bernays–Gödel*** set theory (NBG).

Below we briefly describe some of significant alternative set theories. All of these alternative theories essentially suffer from the same problems as described above in Inadequacies of set theory.

Internal set theory wad develped by Edward Nelson as an alternative axiomatic basis for nonstandard analysis [N77]. It enriches ZFC by adding nonstandard sets to the standard sets of ZFC.

New Foundations was developed by Quine in 1937. It is a typed theory, and it has ***universal set*** (a set which includes all other sets), but it has no foundational elements and thereby allows infinite descent. It avoids Russel's antinomy by allowing only ***stratified*** formulae, e.g. $a \in b$ is stratified if a and b are of different types but not if they are of the same type. The axiom of choice can be shown to be false in this system. The axiom of infinity is a theorem, since the negation of the axiom of choice implies that there exists an infinite set.

Structural set theories contrast with ***material set theories***, which include ZF. Instead of being constructed from one or more atoms, a set in a structural theory is defined only through functions and relations that involve it [SST]. The canonical example of a structural set theory is the Elementary Theory of the Category of Sets (ETCS), an axiomatization of set theory designed to be congruent with category theory.

Reverse mathematics attempts to find axioms which are necessary to prove ordinary mathematical theorems. Proponents of this approach often reject set theory as too expressive, thus generating too much hierarchy and leaving the door open to poorly resolved issues which have little or no bearing on the rest of mathematics. Reverse mathematics often uses subsystems of second order arithmetic, in which quantifiers can range over sets of numbers in addition to individual numbers. A recent book of Simpson is often regarded as important [Si09].

Category theory

A ***category*** may be defined in set-theoretic terms as a collection of objects (elements) and arrows (functions), for example the category of groups and group homomorphisms. See [Ma14]. Similarly, most of the "abstract" structures investigated by modern mathematics are categories: rings, fields, vector spaces, topological spaces, etc.

Since a collection of objects and functions depends on the definition of set, this definition depends on set theory. Alternatively it is possible to define categories independently of set theory, by defining the category of all categories, in which sets are one category. This makes category theory an alternative foundational theory.

A certain type of category knows as a ***topos*** forms the basis of another foundational theory.

Category theory and topos theory suffer from many of the same problems as set theory, as discussed above in Inadequacies of set theory and A numeristic view of abstraction. From the numeristic perspective, the main problem with such theories is that they do not fully account for the structures they include, and as such they really only classify rather than define.

Type theory

Type theory has many variations; see [TT]. In most of them, every ***term*** (syntactic element) has a ***type***, for example the number 5 has the type of integer. This assignment is called a ***judgment*** ([J]). A function uses judgments to restrict its domain and range to specific types. ***Rules*** govern the transformation of terms through their types. Propositions can have their own type, so type theory can encode logic. Type theory can encode sets or conversely.

Type theory is naturally connected to typed programming in computer science. From a numeristic perspective, it is significant that the structure underlying computer data (machine code) is numeric, and thus typed programming is actually dependent on numbers rather than the reverse. Likewise, type theory, along with set theory and category theory, are dependent on number, since counting logically precedes all distinctions such as type, judgment, and rule. See Inadequacies of set theory.

Mereology

Mereology is the philosohical and mathematical study of the relationship between wholes and parts. See [V16]. This study is both ancient and modern. Mathematical mereology is similar to the study of the inclusion relation of numeristics and set theory. Mereology often calls this relation ***parthood***.

There are several axiomatic mereological systems, such as in [CV99]. The inclusion relation alone cannot yield set membership [HK16], but additional axioms can yield systems in which ZFC axioms are theorems. In this latter type of system, as in numeristics, inclusion is only one of several primitives.

A mereological collection, called a ***fusion*** or ***sum***, is very similar to a numeristic class, and its ultimate components, called ***atoms***, are similar to elements of a numeristic class. The terms it fusion and *sum* can also denote the union operator. Mereological fusions and sums are flat, like numeristic classes and unlike sets.

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