Numeristics: A Number-Based Foundational Theory of Mathematics

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ABSTRACT

Numeristics is a number-based alternative foundational theory of mathematics. Numeristics is inspired in part by the recent revival of the Vedic tradition of India, as expressed by Maharishi Mahesh Yogi in his Vedic Mathematics. This paper gives the fundamental ideas of numeristics as they apply to arithmetic and elementary algebra. Numeristics includes an alternative approach to analysis called *equinfinitesimal analysis*, described in a separate paper.

> All things that are known have number; for without this nothing whatever could possibly be thought of or known.—Philolaus, quoted in [St].

> I don't know what predominates in Cantor's [set] theory—philosophy or theology, but I am sure that there is no mathematics there.—Leopold Kronecker

> [T]his house [set theory] is to a large degree built on sand.—Hermann Weyl, quoted in [We].

> The infinite! No other question has ever moved so profoundly the spirit of man; no other idea has so fruitfully stimulated his intellect; yet no other concept stands in greater need of clarification than that of the infinite.—David Hilbert

DEFINITION AND SCOPE

Numeristics is a number-based alternative foundational theory of mathematics. It postulates that the real basis of mathematics is number, not sets.

Numeristics includes many ideas that are inspired by the recent revival of knowledge and experience in the Vedic tradition of India, as expressed by Maharishi Mahesh Yogi in his Vedic Mathematics. Source material in Maharishi Vedic Mathematics is summarized in [CaS] and includes especially [Ma]. A overview of the field is given by this author in [CaI]. The present paper does not assume any familiarity with Maharishi Vedic Mathematics.

This paper gives the fundamental ideas of numeristics as they apply to arithmetic and elementary algebra. Numeristics includes an alternative approach to analysis called *equinfinitesimal analysis*, described in [CaE], and applied to divergent series in [CaD].

WHY NUMERISTICS

Inadequate understanding of mathematical proof

What is called mathematical proof is really a derivation, a chain of logic connecting axioms and previously proven theorems to a new theorem. It cannot be considered complete proof because it is a purely subjective process and because it assumes axioms without proof. Since the Renaissance, the presentation of mathematics as a whole increasingly emphasizes formality and neglects objective verfication, agreement with nature, i.e. agreement of axioms and theorems to observable physical facts. This has led to an increasingly general belief among mathematicians that mathematics is a game that derives its authority from social consensus, rather than a science.

Inadequacies of set theory

Set theory is currently held by the vast majority of mathematicians to be a universal basis of mathematics, at least on a formal level. The modern neglect of objective verification in mathematics as a whole has had an important effect on the development of set theory. This development, starting in the late 19th century, has been purely subjective, focused on paper proofs only, and devoid of concern with objective verification.

The axioms of set theory have been notoriously controversial during this time. For instance, the axiom of infinity, which asserted the existence of an infinite set, encountered considerable controversy when it was introduced, which to this day has never been completely settled. Other axioms are even more controversial, such as the axiom of choice and the generalized continuum hypothesis.

As far as this author has been able to determine, the assertions of set theory about the infinite have never been proved by experiment. Robin Ticciati, for instance, the author of a well known reference work of the mathematics of quantum field theory [Ti99], when asked if he knew of any use of mathematics in quantum theory that depended on a set theoretical result, responded in the negative [Ti03].

Russell's paradox means that set theory must be constructed so that a set cannot be a member of itself. Since the only thing that sets can really do is include sets and be members of sets, this strikes a fatal blow to any aspiration of making set theory self-referent.

Moreover, although it is claimed that set theory defines numbers, this reasoning is circular. Set theory and the system of logic it is built upon are implicitly dependent on numbers. Both set theory and logic assume fundamental dualities and multiplicities. Dualities are implicit uses of the number 2, and multiplicities are implicit uses of higher numbers. Even this pales besides the implicit use of the number 1, which occurs each time we express or even think of any object of attention.

From this foregoing it should be clear that any system that explains number must account for the whole range of manifestation, from the subtlest thinking level to the most obvious, including both subjective and objective phenomena. It must also be clear that any such system cannot be based on intellectual values alone, since intellectual conception and expression necessarily take place in a field of multiplicity.

PHILOSOPHY OF NUMERISTICS

Numeristics bases mathmatics purely on number. Number, as with everything else, ultimately starts from the infinite. The infinite is inexhaustible and therefore only partly conceptualizable. An account of the infinite is always limited, but the infinite itself goes on without end.

Within the infinite is its point, zero, also called the absolute number. Zero is absolute because it is the unmanifest state from which all manifestation begins.

Vedic Mathematics shows how zero, the absolute number, the unmanifest, can be experienced as a fourth state of consciousness, distinct from waking, dreaming, and sleeping. The Vedic tradition of India is very familiar with this state of consciousness and gives it many names, including *samādhi*. It can be experienced in innumerable ways, but a systematic way of experiencing this fourth state of consciousness is through the TM (Transcendental Meditation) program.

At a more subtle level, the absolute number is also the field of lawfulness, orderliness, and precision which governs all of its expressions, which together form the manifest universe.

The number one is the first mathematical expression. It expresses the unified nature of infinity. In its pure state, it does not distinguish itself from zero or anything else.

When there is a concept of distinction, the number two arises to distinguish the number zero from the number one. Other numbers quickly arise in the same way. Other types of number, including nonintegral numbers and negative numbers, result from deeper consideration of this fundamental phenomenon of manifestation.

Whenever a number can be used to measure any object of experience, we consider the object to be an instance of the number. This means that the number one, since it is an identifiable object of our attention, is an instance of itself. Numbers therefore can refer to themselves.

The practical value of numeristics is in having a theory which fits closely with experience. It gives us a mathematics which is well grounded, since it is verifiable both subjectively, through the experience of the Absolute Number, and objectively, through empirical verification. Numeristics is intuitive and does justice to not only to numbers but also to functions, relations, and geometric figures.

BASIC MECHANICS OF NUMERISTICS

Freedom

In numeristics we are free to perform any arithmetic operation as long as we put it in correct context and use appropriate sensitivity. This means that every numeric operation has a numeric result. No numeric operation is undefined. A numeric operation may be multivalued, as described below.

Classes

A numeristic *class* is a potentially multivalued number or other numeric or number-like construction. Every number is a class. A class containing a single number is identical to the number. Classes have a flat structure: a class containing multiple numbers distributes operations on it and statements about it over each constituent number.

This flat structure does not allow us to use the same type of structure to describe functions or relations or other constructs as we use to describe numbers. The numeristic point of view is that this is neither necessary or sufficient; we take each of these constructs as primitive concepts, since both the concepts and the knoweledge of how to use them emerge in a natural way from the experience of the absolute number.

The flat structure of classes allows operations on them in a straightforward way. For example, if $x^2 = 1$ then x is a class with the two elements +1 and −1, and we say $x = \pm 1 = +1, -1$, and $x + 1 = \pm 1 + 1 = 0, 2$. In general, for a class c, $f(c) = \{f(a) \mid a \in c\}$.

When describing numeristic classes, we use several notations from set theory. It must be emphasized that these notations have somewhat different meanings in numeristics from those in set theory. Below are samples of notation we can use to describe the numeristic class ±1:

$$\begin{aligned}
\pm 1 &= +1, -1 \\
&= \{1, -1\} \\
&= \{a \mid a^2 = 1\} \\
&= 1 \cup -1 \\
&= \bigcup_{k=1}^{2} (-1)^k.
\end{aligned}$$

A *subclass* is a class that is completely included in another class. We use the subset symbol for subclasses, e.g. $1 \subset \pm 1$. We may also indicate inclusion through equality with a condition, for example:

$$\pm 1 = +1 \text{ when } \pm 1 \text{ is positive}$$
$$+1 = \pm 1 \text{ when } \pm 1 \text{ is positive}$$

We use the notation $\cap$ to denote the class of elements common to two classes, e.g. $1, 2, 3 \cap 3, 4, 5 = 3$.

In this paper, we do not attempt to investigate a "class of all numbers." To do this properly would require a merging of all fields, including logic, relations, operations, and numbers, and is beyond the scope of our current investigation. Instead we take the point of view that a given discussion will usually be restricted to a certain class of numbers, such as $\mathbb{R}$ or $\mathbb{C}$, which we call the *scope* or *context*.

We also use notations such as $\sim$ to denote removal of elements: $c \sim d := \{a \in c \mid a \notin d\}$. The unary use of $\sim$, e.g. $\sim c$, means $S \sim c$, where S is the current scope.

Elements

An *element* is a class which does not contain any smaller subclasses. If a is an element of b, then we use the notation $a \in b$ or $b \ni a$ e.g. $1 \in \pm 1$. Since an element is also a class, $a \in b$ implies $a \subseteq b$.

A class that is not an element is said to be *multivalued*. An element may also be called a *single valued* class.

An important feature of numeristics is that elements are relative to levels of sensitivity. A class that is an element at one level of sensitivity may be multivalued at another level of sensitivity. This is explained in [CaE].

Sentence distribution over a class

If $S(a)$ is a sentence about an element a, and c is a class, then $S(c)$ is the logical conjunction of $S(a)$ for all elements of c, i.e. $S(c)$ is equivalent to $(\forall a \in c)S(a)$. An example is $(\pm 1)^2 = 1$.

To indicate that $S(c)$ is true only for some of the elements of c, the logical disjunction of $S(a)$ for $a \in c$, we borrow a symbol from modal logic and say $S(\diamond c)$, equivalent to $(\exists a \in c)S(a)$. An example is $\diamond \pm 1 > 1$. This notation is ambiguous with multiple classes; for two classes c and d, $S(\diamond c, d)$ does not indicate whether it means $(\exists a \in c)(\forall b \in d)S(a, b)$ or $(\forall b \in d)(\exists a \in c)S(a, b)$. To distinguish these two, we can use a notation such as $(\diamond cd)S(c, d)$ and $(d \diamond c)S(c, d)$, or $\diamond c, d : S(c, d)$ and $d \diamond c : S(c, d)$.

The empty class

The empty class or null class, denoted $\emptyset$ or $\{\}$, stands for lack of any value. For example, $1 \cap 2 = \{\}$.

When a sentence distributes over the empty class, the result is the empty statement, no statement at all.

If a function f is undefined at a, we can say $f(a) = \emptyset$. The result of any arithmetic operation on $\emptyset$ is $\emptyset$, e.g. $1 + \emptyset = \emptyset$.

Class occasions

$\pm$ can be regarded as a class of operations +,-, whether unary or binary. Multiple occurences of $\pm$ represent positions in which an operation may be chosen, e.g. $\pm 3 \pm 1$. By itself, such a formula is ambiguous: Does it represent $\{4, 2, -2, -4\}$ or $\{4, -4\}$?

To help settle such ambiguities, we call an occurence of a class within a forumula an *occasion* or a *choice* of the class. Like a variable, an occasion may represent an independent choice of an element or operation, or it may represent a dependence of one element or operation on another. We denote the k-th occasion of a class c represented by ${}_k c$ or c:k.

The above ambiguous expressions can be expressed as independent occasions: $\{4, 2, -2, -4\} = {}_1{\pm}\, 3\, {}_2{\pm}\, 1 = (\pm{:}1)\, 3\, (\pm{:}2)\, 1$ and $\{4, -4\} = {}_1{\pm}\, 3\, {}_1{\pm}\, 1 = (\pm{:}1)\, 3\, (\pm{:}1)\, 1$. As an example of dependent occasions, we note that $- \pm 2 = \mp 2 \neq \pm 2$, which we can also express as $-\, {}_1{\pm}\, 2 = {}_2{\pm}\, 2 \neq {}_1{\pm}\, 2$, and so ± 2 is a solution of $-{}_1 x = {}_2 x$ but not of $-{}_1 x = {}_1 x$.

Inverses

One important consequence of the flat structure of numeristic classes is that the inverse of a function is always itself a function. For instance, the inverse of $f(x) = x^2$ is $f^{-1}(x) = \pm\sqrt{x}$.

For a general function f, $f^{-1}(x) = \{a \mid f(a) = x\}$. It follows that $f^{-1}(x) = \{\}$ is equivalent to $\sim (\exists a) f(a) = x$. It also follows that if $f(x) = y$, then $f^{-1}(y) \supseteq x$.

We may use the radical symbol as a multivalued inverse, i.e. $\sqrt[n]{b} := \{a \in C \mid a^n = b\}$.

That inverse functions give us the same type of value as the original function is an important feature of numeristics. It means we can always retrace our steps and return to the starting point with a minimum of formulaic overhead.

Infinity and division by zero

The principle of freedom of numeric operation includes division by zero. As a numeric value, we define infinity as $\dfrac{1}{0}$ and denote it ∞. There is only one infinite value, since $a \cdot \dfrac{1}{0} = \dfrac{1}{0}$ for any nonzero a, and $0 \cdot \dfrac{1}{0} \supseteq 1$.

Since $a \cdot 0 = 0$ for any finite a, $\dfrac{0}{0}$ includes all finite numbers. From this we deduce that $\infty + a = \dfrac{1}{0} + a \supseteq \dfrac{1}{0} + \dfrac{0}{0} = \dfrac{1}{0} = \infty$.

Similarly, we have $\infty - \infty = \dfrac{0}{0}$. Thus, $a - a = 0$ only for finite a, while for general a, $a - a \supseteq 0$.

Since $0 \cdot 0 = 0$ and $\infty \cdot \infty = \infty$, $0, \infty \in \dfrac{0}{0} = 0 \cdot \infty = \dfrac{\infty}{\infty}$, the *unrestricted class*. We avoid the use of the word "indeterminate" to describe such expressions, since they are fully determinate as the unrestricted class. Likewise, we avoid describing division by zero as "undefined," since it yields a well defined infinite or unrestricted class.

When $a = 0$ or ∞, we say it is *afinite*; otherwise we say it is *perfinite*. $\dfrac{a}{a} = 1$ only for perfinite a, while for afinite a, $\dfrac{a}{a} \supset 1$. Thus for general a, $\dfrac{a}{a} \supseteq 1$.

The numeristic handling of zero and infinity is a central feature of the numeristic theory of analysis, *equinfinitesimal analysis*, and is described in greater detail in [CaE].

Standard numeric classes

Letting $b := 0, 1$, we can define some standard numeric classes as follows:

$$\begin{aligned}
\mathbb{N} &:= \bigcup_{k=0}^{\infty} k \\
\mathbb{Z} &:= \pm\mathbb{N} \\
\mathbb{Q} &:= \frac{{}_1\mathbb{Z}}{{}_2\mathbb{Z}} \\
\mathbb{I} &:= \sum_{k=1}^{\infty} b_k 2^{-k} \\
\mathbb{R} &:= \bigcup_{k=-\infty}^{\infty} {}_k\mathbb{I} + k = \bigcup_{k=-\infty}^{\infty} b_k 2^k \\
\mathbb{C} &:= {}_1\mathbb{R} + {}_2\mathbb{R}i
\end{aligned}$$

By the above definitions, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ all include ∞.

We also define $\overline{\mathbb{R}} := \mathbb{R} \sim \infty$, the finite values of $\mathbb{R}$, and $\widehat{\mathbb{R}} := \mathbb{R} \sim \{0, \infty\}$, the perfinite values; similarly for $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{C}$.

Three dimensional real space $\mathbb{R}^3$, including points at infinity, can be expressed as $\{(a_1, a_2, a_3) \mid a_1, a_2, a_3 \in \mathbb{R}\}$ or more succinctly as $({}_1\mathbb{R}, {}_2\mathbb{R}, {}_3\mathbb{R})$. This is a slight inconsistency of notation, since numeristically $\mathbb{R}^3$ should mean $\{a^3 \mid a \in \mathbb{R}\}$, but since this is the same as $\mathbb{R}$, we use the former meaning for $\mathbb{R}^3$.

FURTHER NUMERISTIC CALCULATIONS

Signum function

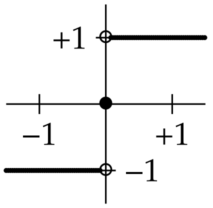

FIG. 1:
Conventional signum function $f(x) = \operatorname{sgn} x$

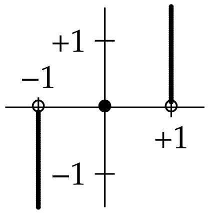

FIG. 2:
Conventional signum function inverse $f^{-1}(x) = \operatorname{sgn}^{-1} x$

Figure 1 shows the usual form of the signum (or sign) function $\operatorname{sgn} x$, which can be defined by either

$$f(x) = \operatorname{sgn} x = \begin{cases} \dfrac{x}{|x|} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

or

$$f(x) = \operatorname{sgn} x = \begin{cases} -1 & \text{for } x < 0 \\ 0 & \text{for } x = 0 \\ +1 & \text{for } x > 0 \end{cases} .$$

Figure 2 shows the inverse $\operatorname{sgn}^{-1} x$, which is not single valued, and is therefore not a function in the conventional sense, but is a function in the numeristic sense. It can also be expressed as

$$f^{-1}(x) = \operatorname{sgn}^{-1} x = \begin{cases} \mathbb{R}^{-} & \text{for } x = -1 \\ 0 & \text{for } x = 0 \\ \mathbb{R}^{+} & \text{for } x = +1 \\ \{\} & \text{otherwise} \end{cases} .$$

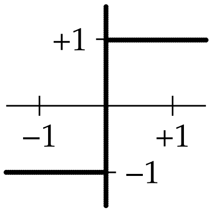

FIG. 3:
Alternate signum
function $g(x) = \operatorname{sgn}_2 x$

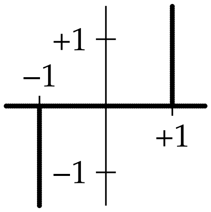

FIG. 4:
Alternate signum function
inverse $g^{-1}(x) = \operatorname{sgn}_2^{-1} x$

Figure 3 shows a revised form of the signum function, $\operatorname{sgn}_2 x$, defined as

$$\operatorname{sgn}_2 x = \frac{x}{|x|}$$

for all x, which can also be expressed as

$$g(x) = \operatorname{sgn}_2 x = \begin{cases} -1 & \text{for } x < 0 \\ \mathbb{R} & \text{for } x = 0 \\ +1 & \text{for } x > 0 \end{cases} .$$

The vertical line at $x = 0$ shows that the value at this point is the indeterminate class $\operatorname{sgn}_2 0 = \frac{0}{0}$.

Figure 4 shows the inverse, $\operatorname{sgn}_2^{-1} x$, which can be expressed as

$$\operatorname{sgn}_2^{-1}(x) = \begin{cases} -\mathbb{R}^* & \text{for } x = -1 \\ \mathbb{R}^* & \text{for } x = +1 \\ 0 & \text{otherwise} \end{cases} .$$

$\mathbb{R}^*$ here denotes class of nonnegative values of $\mathbb{R}$; by operator distribution over class values, $-\mathbb{R}^*$ is the class of nonpositive values of $\mathbb{R}$.

Neither $\operatorname{sgn}_2 x$ nor $\operatorname{sgn}_2^{-1} x$ are single valued and therefore cannot be conventional functions, but both are numeristic functions.

Solution of $x = rx$

As a demonstration of numerisitic techniques, we consider the equation $x = rx$. A conventional solution could run as follows:

$$\begin{aligned} x - rx &= 0 \\ x(1-r) &= 0, \end{aligned}$$

from which we conclude that $x = 0$, except for $r = 1$, where x is indeterminate.

This is not a complete numeristic solution, since it assumes that for any a and b, $a - a = 0$, and $ab = 0$ implies $a = 0$ or $b = 0$. Both of these assumptions are valid only for finite a and b.

We now examine a numeristic solution, which adds all the neglected cases.

1. $r = 1$: x is unrestricted.
2. $r = 0$:
   a. x finite: $x = 0$.
   b. x infinite: $x = \infty$. Here we allow "=" to also mean "$\supseteq$" in the original equation.
3. Other finite r:
   a. x finite:
   $$\begin{aligned} x - rx &= 0 \\ x(1-r) &= 0 \\ x &= 0 \end{aligned}$$
   b. x infinite: $x = \infty$.
4. Infinite r: Invert the equation and follow the case $r = 0$:

$$\begin{aligned} \frac{1}{x} &= 0\frac{1}{x} \\ \frac{1}{x} &= 0, \infty \\ x &= 0, \infty \end{aligned}$$

HOW NUMERISTICS HANDLES RUSSELL'S PARADOX

Russel's paradox (or antinomy) demonstrates a weakness of naive set theory, the predecessor to axiomatic set theories. In naive set theory, a set is allowed to be an element of itself. Russell's paradox considers the set S which contains all sets that do not contain themselves; if S contains itself, then by definition it does not contain itself, and if it does not contain itself, then again by definition it contains itself.

Axiomatic set theories avoid this problem in various ways. Zermelo-Fraenkel set theory restricts the elements that sets can contain. Bernays-Gödel set theory makes a distinction between a *class*, which can contain elements, and a *set*, which can be an element; all sets are classes, but not all classes are sets.

Numeristic classes differ fundamentally from sets, primarily in their flat structure, by which a class containing a single element is identical to the element. Every numeristic class contains itself, and, if it is an element, it is an element of itself. There is no numeristic class that does not contain itself. The class of all classes that do not contain themselves is therefore the empty class, the class of all elements satisfying contradictory conditions.

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REFERENCES

CaD K. Carmody, Divergent series: A numeristic approach, http://kevincarmody.com/math/divergentseries.pdf.

CaE K. Carmody, Equinfinitesimal analysis: An extension of numeristics, http://kevincarmody.com/math/equinfinitesimal.pdf.

CaI K. Carmody, Is consciousness a number? How Maharishi Vedic Mathematics resolves problems in the foundations and philosophy of mathematics, http://kevincarmody.com/math/mvmath.pdf.

CaS K. Carmody, Summaries of published sources by Maharishi Mahesh Yogi on mathematics, including modern mathematics and

Vedic Mathematics,
http://kevincarmody.com/math/maharishimathsummaries.pdf.

Ma Maharishi Mahesh Yogi, *Maharishi's Absolute Theory of Defence*, Age of Enlightenment Publications, India, 1996.

St Stanford Encyclopedia of Philosophy, http://plato.stanford.edu/entries/presocratics/#PytTra.

Ti99 R. Ticciati, *Quantum field theory for mathematicians*, Cambridge University Press, 1999.

Ti03 R. Ticciati, private conversation with the author, ca. 2003.

We H. Weyl, *Das Kontinuum; kritische Untersuchungen über die Grundlagen der Analysis* (The continuum; critical investigations on the foundations of analysis), Veit, Leipzig, 1918.