EQUIPOINT ANALYSIS

A NUMERISTIC
APPROACH TO CALCULUS

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CONTENTS

Non-standardanalysis frequently simplifies substantially the proofs, not only of elementary theorems, but also of deep results. This is true, e.g., also for the proof of the existence of invariant subspaces for compact operators, disregarding the improvement of the result; and it is true in an even higher degree in other cases. This state of affairs should prevent a rather common misinterpretation of non-standard analysis, namely the idea that it is some kind of extravagance or fad of mathematical logicians. Nothing could be farther from the truth. Rather, there are good reasons to believe that non-standard analysis, in some version or other, will be the analysis of the future.—Kurt Gödel, [G74]

And there is every reason to believe that the codification of intuitive concepts and the reinterpretation of accepted principles will continue also in future and will bring new advances, into territory still uncharted.—Abraham Robinson, [R68]

[Srinivasa Ramanujan] sometimes spoke of 'zero' as the symbol of the Absolute (Nirguna Brahman) of the extreme monistic school of Hindu Philosophy, that is, the reality to which no qualities can be attributed, which cannot be defined or described by words and is completely beyond the reach of the human mind; according to Ramanujan, the appropriate symbol was the number 'zero', which is the absolute negation of all attributes. He looked on the number 'infinity' as the totality of all possibilities which was capable of becoming manifest in reality and which was inexhaustible. According to Ramanujan, the product of infinity and zero would supply the whole set of finite numbers. Each act of creation, as far as I could understand, could be symbolized as a particular product of infinity and zero, and from each such product would emerge a particular individual of which the appropriate symbol was a particular finite number. . . . He spoke with such enthusiasm about the philosophical questions that sometimes I felt he would have been better pleased to have succeeded in establishing his philosophical theories than in supplying rigorous proofs of his mathematical conjectures.—P. C. Mahalanobis, [Mn]

Unlike my colleagues, I think that an attempt to reconsider the idea of an infinitesimal as a variable finite quantity is fully scientific, and that the proposal to replace variable infinitesimals by fixed ones, far from having purely pedagogical significance, has in its favor something immeasurably deeper, and that this idea is growing roots in modern analysis. . . .

I have a clear recollection of my ideas on infinitesimal analysis. I was a second-year student. When the professors announced that $\frac{dy}{dx}$ is the limit of a ratio, I thought: "What a bore! Strange and incomprehensible. No! They won't fool me: it's simply the ratio of infinitesimals, nothing else." . . .

Imagine what would happen in physics if physicists held on to earlier view on atoms, i.e. if they imagined a small sphere, a little ball of matter covered with a shell. . . . It is a sobering thought that if we had adhered to tradition we would not have modern quanta! *It is much the same in mathematics.* . . . I cannot but see a stark contradiction between the intuitively clear fundamental formulas of the integral calculus and the incomparably artificial and complex work of their "justification" and their "proofs". —N. N. Luzin, [Lu] (emphasis his)

I still remember the sight of [my high school caclculus teacher] standing in front of the blackboard w[h]ere she had drawn a wonderfully smooth parabola, inserting a secant and telling us that $\Delta y/\Delta x$ is its slope, until finally she convinced us that the slope of the tangent is dy/dx where dx is infinitesimally small and dy accordingly. . . . This, I admit, impressed me deeply. Until then our school Math had consisted largely of Euclidean geometry, with so many problems of constructing triangles from some given data. This was o.k. but in the long run that stuff did not strike me as more than boring exercises. But now, with those infinitesimals, Math seemed to have more interesting things in stock than I had met so far. ... [However, at the university,] we were told to my disappointment that my Math teacher had not been up to date after all. We were warned to beware of infinitesimals since they do not exist, and in any case they lead to contradictions. Instead, although one writes dy/dx . . . , this does not really mean a quotient of two entities, but it should be interpreted as a symbolic notation only, namely the limit of the quotient $\Delta y/\Delta x$. I survived this disappointment too. . . . [Later,] when I learned about Robinson's infinitesimals, my early school day experiences came to my mind again and I wondered whether that lady teacher had not been so wrong after all. The discussion with Abraham Robinson kindled my interest and I wished to know more about it. Some time later there arose the opportunity to invite him to visit us in Germany where he have lectures on his ideas, first in Tübingen and later in Heidelberg, after I had moved there.—P. Roquette, [R10]

अणोरणीयान् महतो महीयान् आत्मास्य जन्तोर्निहितो गुहायाम् ॥
Aṇoraṇīyān mahato mahīyān ātmāsya jantornihito guhāyām.
The Self is smaller than the smallest, bigger than the biggest, and is hidden in
a secret place of all creatures.—Katha Upanishad 2.20

यथा पिरडे तथा ब्रह्मारडे ॥
Yathā piṇḍe tathā brahmāṇḍe.
As is the point, so is the infinite.—Charaka Samhita

SUMMARY

This document extends the concepts of numeristics to analysis. Numeristics is introduced in a separate document. Here a theory of analysis is developed, based on infinitesimals which are all exactly equal to zero, and infinite values that are their reciprocals.

Fundamental concepts derive from Maharishi Mahesh Yogi's Vedic Mathematics, Charles Musès's analysis of zero and infinity, and Abraham Robinson's nonstandard analysis. This theory uses multiple levels of sensitivity to extend real and complex arithmetic and evaluate equality. It then defines derivatives and integrals solely in terms of elementary arithmetic operations in this extended arithmetic.

Topics include:

- **Levels of sensitivity** (p. 18), including multilevel numbers, functions, and relations.

- **The fundamental theorems of calculus** (p. 36).

- **Chain rule** (p. 39), **product rule** (p. 40), derivatives of **trigonometric** (p. 45) and **exponential** (p. 47) functions.

- **Limit** (p. 50) defined in terms of sensitivity levels, and **continuity** (p. 52) in terms of these limits.

- **The natural logarithm developed as a polynomial** (p. 48) in the extended arithmetic.

- **Singularities** (p. 68): jump singularities, removable singularities, poles, essential singularities.

- **Complex analysis** (p. 96): complex derivative, Cauchy integral formula, Taylor and Laurent series, complex poles, complex essential singularities.

- **Calculus of variations** (p. 110): functional derivative, product rule, chain rule, transfer rule, application to straight line theorem.

An **appendix** (p. 139) compares equipoint analysis to other theories of analysis: conventional analysis, nonstandard analysis, relative analysis, and smooth infinitesimal analysis.

HOW TO USE THIS DOCUMENT

This is not a textbook. This document describes a new system of calculus and analysis, equipoint analysis, and shows the differences between it and other systems of calculus and analysis. This document should therefore be used as a supplement to other mathematical texts at that level.

At a minimum, this text assumes familiarity with calculus. Some material is aimed at a more advanced level, such as complex analysis and functionals. Those who are not familiar with these areas can skip these sections.

To understand equipoint analysis, it is essential to understand its total, unrestricted arithmetic and the refinement of this arithmetic by expanding a point to a space.

- The unrestricted arithmetic is developed in a separate document on numeristics [CN].

- The refinement of this arithmetic is covered in this document in the chapter on **sensitivity** (p. 18).

- The refined arithmetic is then used to **redefine the derivative and integral** (p. 31).

Equipoint analysis is also used in an alternative theory of divergent series, described in [CD], and this alternative theory is applied to repeating decimals in [CR].

NON-CONVENTIONAL
THEORIES OF ANALYSIS

The conventional theory of analysis, based on set theory and limits, was first developed in the 19th century. Since 1960, the following theories of analysis have emerged as alternatives to classical analysis. Here we briefly describe the history of these theories. See the **appendix** (p. 139) for a more detailed description and comparison to equipoint analysis.

Nonstandard Analysis

Nonstandard analysis has its roots in the original development of calculus in terms of infinitesimals by Leibnitz in the 17th century. In the intervening centuries, calculus was found to be very useful, but the explanation of it in terms of infinitesimals did not satisfy very many mathematicians. With the increasing demand for rigor in the 19th century, the theory of infinitesimals was replaced by classical limits-based analysis.

In 1960, Abraham Robinson resurrected the theory of infinitesimals by developing it as a modern set theoretic system he called ***nonstandard analysis*** [R74]. Jerome Keisler used the principles of nonstandard analysis in his elementary calculus textbook [KE] and undergraduate analysis text [KF]. Nonstandard analysis is widely considered to be a significantly simpler and more elegant system than classical analysis, yet in the more than 50 years since its introduction, it has not achieved widespread use, either in teaching or in research.

Relative Analysis

More recently, ***relative analysis*** was developed by Karel Hrbáček, Oliver Lessman, and Richard O'Donovan [H10], and used by O'Donovan in high school instruction [OD09]. This theory uses the terms *ultrasmall* and *ultralarge*, whereas *infinitesimal* and *infinite* are used in nonstandard analysis. Like nonstandard analysis, relative analysis has not achieved widespread use.

Smooth Infinitesimal Analysis

Smooth infinitesimal analysis was developed by John L. Bell, in [BI] and [BP], as a branch of synthetic differential geometry. It was originally developed by F. William Lawvere from category theory starting in 1967, but it remained obscure until Lawvere's 1998 article [La]. Like other alternatives, smooth infinitesimal analysis has not achieved widespread use.

ORIGIN OF EQUIPOINT ARITHMETIC

As explained in [CN], numeristics is based on the infinite and the experience of the silent, unmanifest point of infinity, *samādhi* or zero. In numeristics this is conceptualized to give an arithmetic of 0 and ∞, including total, unrestricted multiplication and division by these quanities, such as $\frac{1}{0}$, $\frac{0}{0}$, and $\infty + 1$.

Some of these unrestricted operations, including $\frac{0}{0}$, $\infty - \infty$, and $\infty \cdot 0$, give rise to indeterminate expressions. Numeristics gives a value to these expressions. This value is called the ***full class*** (ϕ), a class which includes all numeric values.

In some cases, numeristic arithmetic yields an inderminate expression in response to a question which clearly has a determinate result. One example is the calculation of the slope of the tangent to a curve $y = f(x)$ at a point a. Numeristic arithmetic alone yields the result $\frac{f(x) - f(x)}{0} = \frac{0}{0} = \phi$. In such cases, numeristic arithmetic needs to be refined to yield a determinate result. This need is called the ***principle of determinacy***, and it is implemented through equipoint arithmetic.

In this and similar works, the principle of determinacy is used in the following:

- **Derivatives** (p. 31).

- **Integrals** (p. 33).

- **Offset derivatives** (p. 68).

- **Class count comparisons** (p. 125).

- Divergent series in [CD].

- Infinite left decimals in [CR].

Numeristics starts with the experience of infinity and zero as the point of infinity. Equipoint arithmetic extends with the experience of the point open-

ing up into a vast inner space distinct from and much richer than ordinary space, and the contraction of this space back into ordinary space.

Equipoint analysis conceptualizes some aspects of the expanded space by considering it as a space of points which all have the ordinary single value of zero from the perspective of ordinary space, but which form a class of distinct zeros from the perspective of the expanded space.

This allows us to do calculus with infinitesimals that are exactly zero, and with numeric infinites that are reciprocals of these zeros. In the next chapter, we formalize the multiple perspectives as *levels of sensitivity*.

SENSITIVITY

Unfolding zero

In equipoint analysis, every number has multiple *levels of sensitivity*. At the level of least sensitivity, we distinguish real numbers, integers, rational numbers, etc. as elements. When, as described in the previous chapter, zero opens up into a space of zeros, these zeros are all exactly equal to zero at the lowest level of sensitivity, but at a higher level of sensitivity, zero is multivalued, and the individual zeros are distinct elements. This is shown pictorially below.
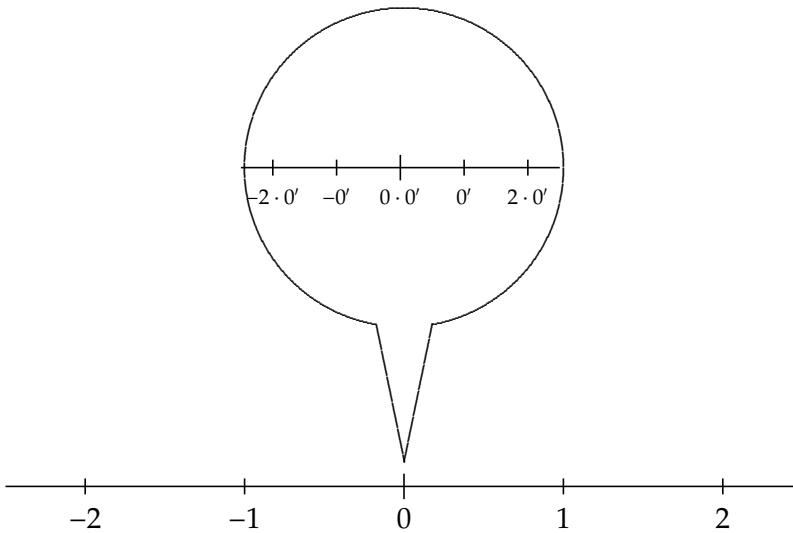

FIG. 1:
Real number line with
microscope view of unfolded 0

In Figure 1, we have the ordinary real number line with the real number 0 expanded into a space. When 0 is expanded into a space, we find multiple

distinct zeros in that space. We call the ordinary number line the *folded* space and the expanded space the *unfolded* space around 0. Folded space, the folded real number line, is the level of least sensitivity, and the unfolding shown in this figure is a level of greater sensitivity. The bubble showing the infinitely expanded space is called a *microscope*, and the original graph is called a *macroscope*.

The figure shows one of the unfolded zeros denoted as $0'$, and it also shows some multiples of $0'$. At the lowest of sensitivity, $0' = 0 \cdot 0' = 0$, but at the level of sensitivity of the unfolded space, $0' \neq 0 \cdot 0'$, and $0' \in 0$.

In the unfolded space of 0, each of the individual values of 0 in that space has a well defined ratio with every other point in that space. For instance, if $0'' := 3 \cdot 0'$, then $\dfrac{0''}{0'} = 3$. This also means that the unfolded space is ordered analogously to real space, e.g. $0'' > 0'$ at the unfolded level of sensitivity.

Finite multiples of each value in the unfolded space are distinct, but squares and higher powers of any value in this space end up at the origin of the unfolded space: $0'^2 = 0'^n = 0 \cdot 0'$ for any $0'$ in the space and any $n > 1$.

We characterize a level of sensitivity by a *sensitivity unit*, a number which does not lie at the origin. The sensitivity unit of the folded space is 1, while the sensitivity unit of the unfolded space in Figure 1 is $0'$.

Since equality may depend on the sensitivity unit, we use the notation $='$ for equality at the unfolded level, where $0'$ is the unit, and reserve the symbol $=$ for equality at the folded level. More generally, we use the notation $\sim u$, usually as a subscript, to indicate a sensitivity unit u, and refer to it as *sense u*. The default sense is 1: $a = b$ means $a =_{\sim 1} b$, and $a =_{\sim u} b$ is equivalent to $\dfrac{a}{u} = \dfrac{b}{u}$.

These unfolded zeros are the infinitesimals of this system of analysis. The name *equipoint* reflects the fact that these infinitesimals are all equal at sense 1.

To summarize, at the folded level, sense 1, there is only one 0:

$$0' = \mathbb{R} \cdot 0 = 0'^2 = 0.$$

At the unfoldedlevel of sense $0'$, there are multiple distinct values of 0:

$$0' \neq' 0'^2$$

$$0' \neq' 2 \cdot 0' \neq 0 \cdot 0'$$
$$0'^2 =' 0'^3 =' 0 \cdot 0'$$

Unfolding perfinite numbers
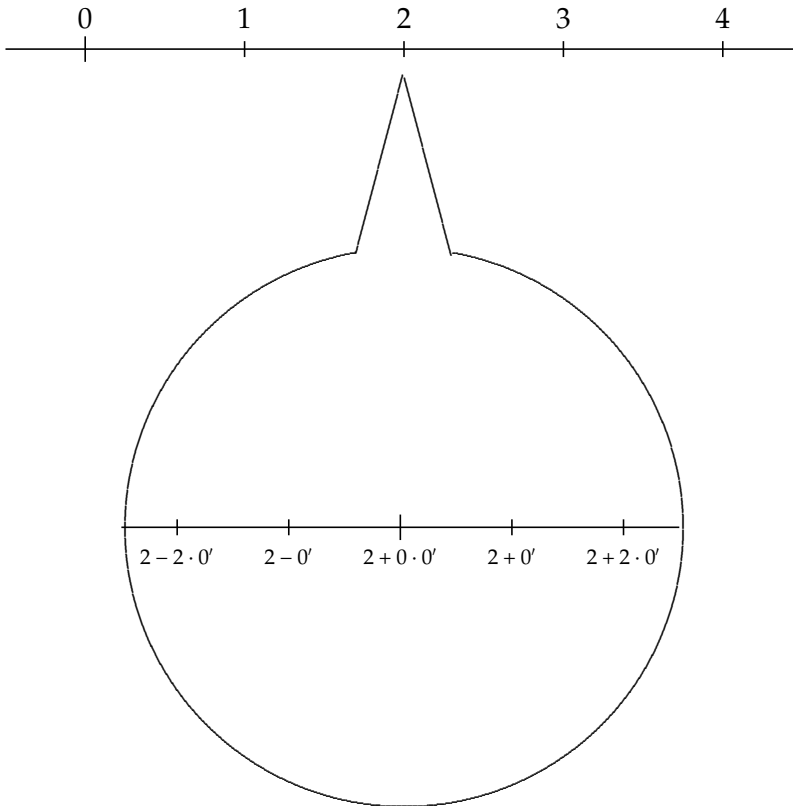
FIG. 2:
Real number line with
microscope view of unfolded +2

Since $a + 0 = a$ for any real a, every real number a can open up into a space consisting of a plus zeros. Figure 2 shows this for $a = 2$. Just as we locate $0'$ within 0 and use it to perform a sensitive arithmetic within 0, we can also locate $a' := a + 0'$ within a and use it as the base of a sensitive arithmetic within a.

Equipoint Analysis

Following are examples of arithmetic within a finite real a.

$$a - a = 0$$
$$a + 0' = a + 0 \cdot 0' = a$$
$$a + 0' \neq' a + 0 \cdot 0' \neq' a$$
$$(a + 0') - (a + 0 \cdot 0') =' (a + 0') - a =' 0'$$
$$a + 0' \in' a_{\sim 1}.$$

Unfolding infinite numbers
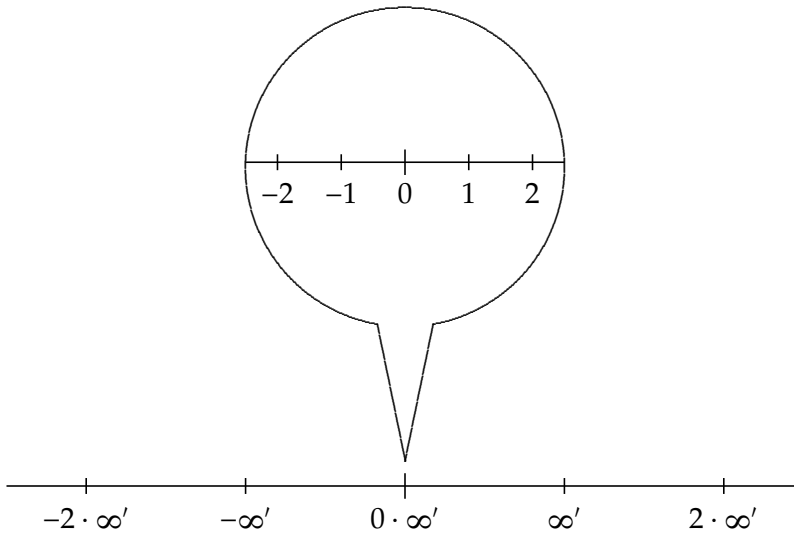
FIG. 3:
Line of infinities with microscope view of
real number line within $0 \cdot \infty'$

Inversely to the expansion of a point of real space, the real number line can collapse into a point within a space of infinities. This is shown in Figure 3.

When an infinite number is unfolded, the roles of microscope and macroscope are reversed: The microscope shows folded finite space, and the macroscope shows unfolded infinite space.

The class of folded infinite values, and the unfoldings derived from them, depend on the type of infinite element extension. In the projectively

extended real numbers, we defined $\infty := \dfrac{1}{0}$, while in the affinely extended reals, we defined $\infty := \left| \dfrac{1}{0} \right|$.

$$\infty' = |\mathbb{R}^*| \, \infty' = \infty'^2 = \infty$$
$$\infty' \neq' \infty'^2$$
$$\infty' \neq' 2 \cdot \infty' \neq' 0 \cdot \infty'$$
$$\infty'^2 =' \infty'^3 =' \infty'^\infty$$
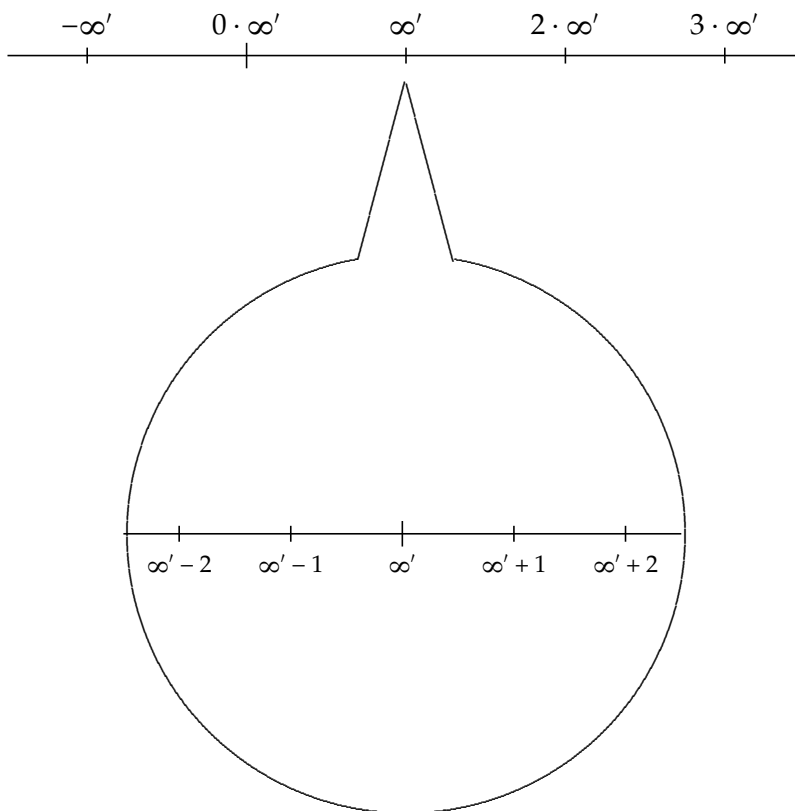
FIG. 4:
Line of infinities with
perfinite unfolding of ∞'

Figure 4 shows that each infinite element in the space of infinities unfolds into a space in which finite numbers added to the infinite element are distinct points.
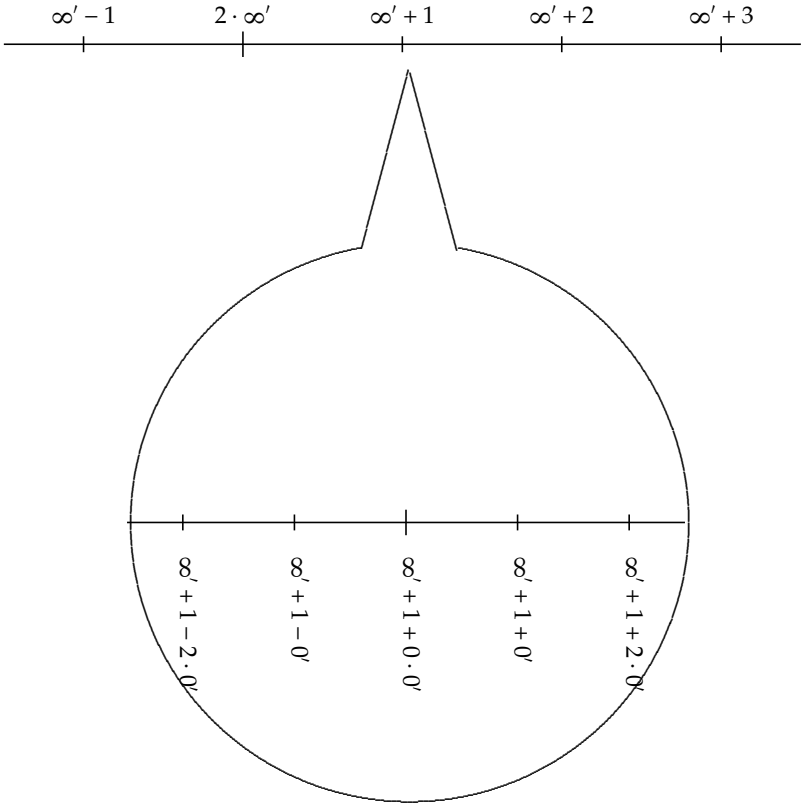
FIG. 5:
Line of infinities with
infinitesimal unfolding of $\infty' + 1$

Figure 5 shows that each point of infinite plus finite unfolds into a space with infinitesimals added.

Superunfolding
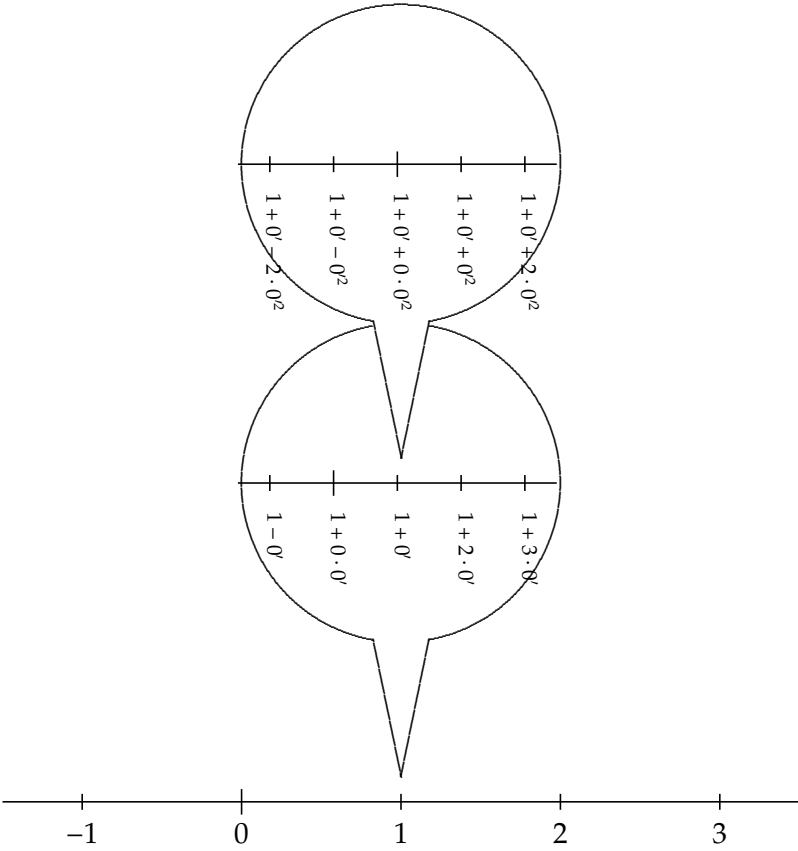
FIG. 6:
Unfolding of $1 + 0'$,
superunfolding of 1

Within an unfolded space, any point can be unfolded again. This **second unfolding** uses a sensitivity unit of $0'^2$ instead of $0'$. We can also call this an unfolding with respect to $0'^2$.

Figure 6 shows the superunfolding of $1 + 0'$, which is itself an element of the unfolded 1.

For each positive integral power $0'^n$, we can make an ***n-th unfolding*** or unfolding with respect to $0'^n$. This includes $n = \infty$, which is called the ***ultimate unfolding***. The **0-*th unfolding*** is the folded numbers. Any unfolding beyond the first is called a ***superunfolding***.

If two quantities are equal at all unfoldings, i.e. equal at the ultimate unfolding, we use the symbol $\equiv$, called ***equivalence***. To define a variable or expression as equivalent to another, we use the symbol $:\equiv$.

$$0'' :\equiv 0'^2$$
$$0'' = \mathbb{R} \cdot 0' = 0''^2 = 0$$
$$0'' =' \mathbb{R} \cdot 0'' =' 0''^2$$
$$0'' \neq'' 0''^2$$
$$0'' \neq'' 2 \cdot 0'' \neq'' 0 \cdot 0''$$

Power series enable us to evaluate various unfoldings of non-polynomial functions:

$$e^{0'x} =_{\sim_1} 1$$
$$=_{\sim_{0'}} 1 + 0'x$$
$$=_{\sim_{0'^2}} 1 + 0'x + \frac{0'^2 x^2}{2}$$
$$\cdots$$
$$\equiv \sum_{k=1}^{\infty} \frac{0'^k x^k}{k!}.$$

Sensitivity extensions of standard classes

Starting with the projectively extended real numbers $\widehat{\mathbb{R}}$, or the affinely extended real numbers $\overline{\mathbb{R}}$, the above sections have described the unfolding of this class to the $0'$ sensitivity level. We denote this unfolded real number class $\widehat{\mathbb{R}}'$ or $\overline{\mathbb{R}}'$. The unfolding was described in four steps:

1. Unfold 0 into the space of infinitesimals, $0'\mathbb{R}$ (Fig. 1).

2. Unfold every perfinite number by adding the infinitesimals to the perfinite (Fig. 2).

3. Unfold ∞ into the space of infinities, $\infty'\mathbb{R}$, the reciprocals of the infinitesimals (Fig. 3).

4. Unfold each element in the space of infinities by adding the finites, $\mathbb{R}$, to each element (Fig. 4).

5. Unfold each element in the space of infinites plus finites by adding the infinitesimals to each element (Fig. 5).

The result is

$$\mathbb{R}' =' \left(\underset{1}{\mathbb{R}} + \underset{2}{0'\mathbb{R}} \right) \cup \left(\underset{1}{\infty'\mathbb{R}} + \underset{2}{\mathbb{R}} + \underset{3}{0'\mathbb{R}} \right).$$

Entirely analogous procedures can be used to unfold $\mathbb{Q}$, $\mathbb{C}$, or higher dimensional classes.

Unfolding relations and functions

We have unfolded real numbers and the equality and membership relations at various sensitivity units. We have also implicitly unfolded addition and multiplication. We now unfold relations and functions more formally. For every folded relation or function, and every unfolding of number space, we postulate that there exists a unique extension of the relation or function in the unfolded number space, which we call the *unfolded relation* or *unfolded function*.

We further postulate that unfolded relations and functions follow the *transfer principle*: Any statement or expression using functions and relations in the folded space is equivalent to the corresponding statement or expression in the unfolded space.

In other words, an unfolded function or relation inherits its behavior from its folded original. For example, $a' +' b' =' c'$ if and only if $a + b = c$. The transfer principle applies only to points at the origin of the unfolded space: if a', for example, is the origin of the unfolded space around a. The transfer

principle does not apply to other points in the unfolded space, even though the unfolded function or relation exists at those points.

A function or relation may be defined in an unfolded space but not be the unfolding of any folded function or relation; in other words, it is not defined solely at the folded level. In this case, we call it a ***proper unfolded*** function or relation; otherwise it is ***canonical unfolded***. We postulate that a canonical unfolding is unique: a folded function or relation has only one canonical unfolding within unfolded numbers.

A proper unfolded function or relation can be folded, but some information will be lost:

$$f(x)_{\sim 1} \equiv \{ f(x + 0') \mid 0' \in' 0_{\sim 1} \},$$

that is, take all the values in the unfolding $f(x + \mathbb{R}0')'$ and put them into a single class that is assigned to the folded $f(x)$. If f is single valued at the unfolded level, and there is any $0'$ such that $f(x) \neq' f(x + 0')$, then f will be multivalued at the folded level.

A canonical unfolded function or relation does not lose any information when folded. Unfolded polynomial functions are canonical unfolded, but the **Dirac delta function** (p. 79) is a proper unfolded function.

If f is a single valued canonical unfolded function, then for every x:

- $f(x)'$ is single valued.

- $f(x)$ is single valued.

- If $x =' y$, then
  $f(x)' =' f(y)$.

- If $x = y$ but $x \neq' y$, then
  $f(x) = f(y)$
  and usually
  $f(x) \neq' f(y)$.

If f is a proper unfolded function that is single valued at the unfolded level, then for every x:

- $f(x)'$ is single valued.

- $f(x)$ is usually multivalued.

- If $x =' y$, then
  $f(x) =' f(y)$.

- If $x = y$ but $x \neq' y$, then
  $f(x) = f(y)$
  and usually
  $f(x) \neq' f(y)$.

"Usually" here is used in the sense that if $x \neq y$, then usually $f(x) \neq f(y)$.

The n-th unfolding of a relation distributes that unfolding to its arguments, i.e. the arguments are all taken at the n-th unfolding. The default unfolding number of a relation is 0, i.e. if no unfolding is indicated, its arguments are presumed to be folded. We have seen how this works with equality, which by default is folded, and which in turn assumes its two arguments are folded.

The n-th unfolding of a function is the maximum unfolding used by its arguments and results. Exactly how this works depends on the function. With addition and subtraction, all the arguments must be at the same unfolding, so the unfolding of the function is the maximum unfolding of the arguments. For multiplication and division, the arguments and result can be at a variety of unfolding levels: the unfolding number n of the function is the maximum of the unfolding numbers of the two arguments and the result.

If a number's unfolding needs to be increased to be compatible with other function or relation arguments, it can be unfolded to that level and remain equivalent. The unfolded number is at the origin of the unfolded space.

If the unfolding needs to be decreased, it can be folded into the unique element it belongs to at the lower unfolding, but equivalence can only be guaranteed if the unfolded number is at the origin of the unfolded space.

We now examine some examples of these principles.

EXAMPLE 1. $2 + 0 = 2$. Everything in these expressions is in folded arithmetic.

EXAMPLE 2. $2 + 0' = 2 + 0 = 2$. The addition in the first expression is unfolded, and the addition in the middle expression is folded.

EXAMPLE 3. $2 + 0' >' 2$. The unfolded order relations are defined in much the same way as the equality relation.

EXAMPLE 4. $0' \in' 0$, and $0'^2 =' 0 \cdot 0'$, but $0'^2 \in_{\sim 0'^2} 0 \cdot 0'$. The unit in the superunfolded infinitesimals is $0'^2$.

EXAMPLE 5. $\infty + 1 = \infty$. The arithmetic of ∞, like other numbers, defaults to folded.

EXAMPLE 6. $\dfrac{0}{0} = \cancel{\infty}$. Again, the default arithmetic is folded.

EXAMPLE 7. $\dfrac{0'}{0'} = 1$. The arguments and value of division can be at any levels of unfolding. In this case, the maximum unfolding is the first unfolding, and the division is the inverse of $1 \cdot 0' =' 0'$.

EXAMPLE 8. For finite x, $x - x \equiv 0$. For folded perfinite p and unfolded infinite x such that $x ='$ $p\infty'$, $x - x \equiv 0$. For folded infinite $x = \infty$, $x - x \equiv \cancel{0}$. The operations that work in folded arithmetic work identically in unfolded arithmetics, provided every item is unfolded. Therefore, expressions such as these can be evaluated in unfolded arithmetic whether x is folded or unfolded.

EXAMPLE 9. Similarly, for perfinite x, $\dfrac{x}{x} \equiv 1$, and for folded perfinite p and unfolded afinite x such that $x ='$ $p \cdot \{0', \infty'\}$, $\dfrac{x}{x} \equiv \dfrac{p\{0', \infty'\}}{p\{0', \infty'\}} \equiv \dfrac{\{0', \infty'\}}{\{0', \infty'\}} \equiv 1$, but for folded afinite $x :\equiv p \cdot \{0, \infty\}$, $\dfrac{x}{x} \equiv \cancel{0}$.

EXAMPLE 10. In unfolded arithmetic, if we use $x ='$ 0 instead of $x ='$ $0'$, then $\dfrac{x}{x} \equiv \dfrac{\underset{1}{\mathbb{R}} \cdot 0'}{\underset{2}{\mathbb{R}} \cdot 0'} = \cancel{0}$.

EXAMPLE 11. For finite folded r,
$$\begin{aligned}
\frac{0'r + 0'^2}{0'} &\equiv \frac{0'r}{0'} + \frac{0'^2}{0'} \\
&\equiv r + 0' \\
&= r.
\end{aligned}$$

Nonintegral superunfoldings

A **superunfolding** (p. 24) normally uses a sensitivity unit of $0'^p$, where p is an integer. The microscope of such a superunfolded space magnifies unfolded space by a factor of ∞'^p, where $\infty' :\equiv \frac{1}{0'}$.

Any positive perfinite p, including noninteger p, results in a magnification by an infinite amount, since ∞'^p is infinite. For any positive perfinite $q < p$, the unfolding with $0'^p$ is a superunfolding of the unfolding with $0'q$, since $p - q$ is also positive perfinite and ∞'^{p-q} is infinite. Thus there is a continuum of superunfoldings corresponding to the real continuum, each with its own sensitivity level.

A positive infinitesimal p, such as $p :\equiv 0'$, results in a finite magnification, since ∞'^p is finite. In this case, we do not obtain an unfolding or a separate sensitivity level.

DEFINITIONS OF
DERIVATIVE AND INTEGRAL

Definition of derivative

The equipoint derivative directly calculates the rate of change at a point using sensitivity levels and the transfer principle.
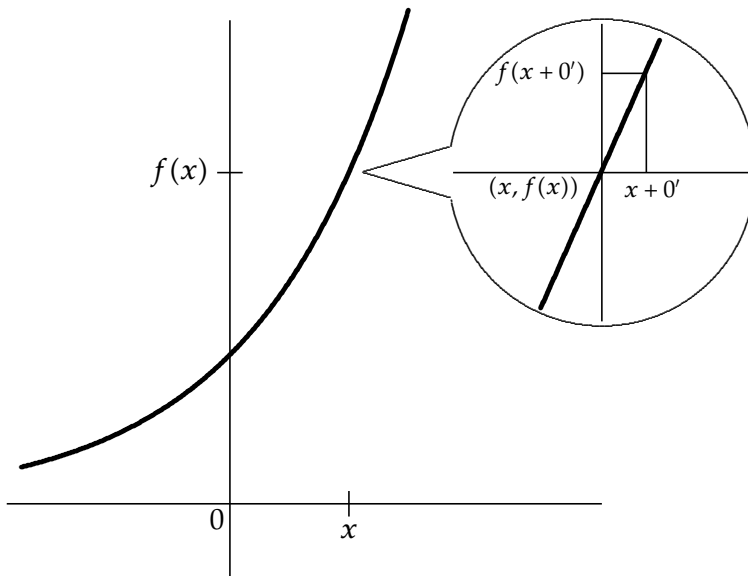
FIG. 7:
Calculation of derivative
as slope within a point

Figure 7 shows a curve $y = f(x)$ and a microscope view of the point $(x, f(x))$. Within the point, the curve is infinitely magnified and becomes a straight line. The Δx of this line is an infinitesimal $0'$, the sensitivity level of the microscope, and the Δy of this line is $f(x + 0') - f(x)$. In unfolded space, we denote Δx and Δy as dx and dy. The slope of the line, and the ***derivative*** of $f(x)$ at x, is

$$f'(x) := \frac{df(x)}{dx} :\equiv \frac{f(x + 0') - f(x)}{0'}.$$

As an example of this calculation:

$$f(x) = x^2$$

$$\frac{df(x)}{dx} \equiv \frac{(x + 0')^2 - x^2}{0'}$$

$$\equiv \frac{x^2 + 2 \cdot 0'x + 0'^2 - x^2}{0'}$$

$$\equiv \frac{2 \cdot 0'x}{0'}$$

$$= 2x.$$

Since $\dfrac{0'^2}{0'} \equiv 0'$ and $\dfrac{x^2 - x^2}{0'} \equiv \dfrac{0'^2}{0'} \equiv 0'$, these terms vanish from the final result.

For a comparison of this definition of the derivative with that in other systems of analysis, see the **Appendix** (p. 139).

Definition of definite integral

The equipoint integral directly calculates an area as an infinite sum of zero width rectangles.
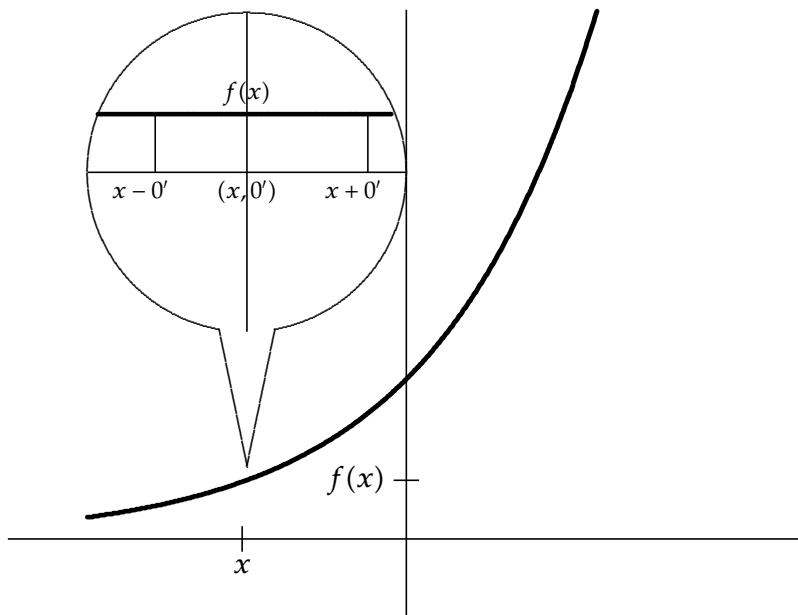
FIG. 8:
Calculation of integral
as sum of zero width rectangles

Figure 8 shows a curve $y = f(x)$ and a microscope view of the sliver at x an infinitely thin area under the curve $f()$ at the point x. The microscope expands the sliver in the x direction but not in the y direction. Within the sliver, the curve becomes a flat line, and sliver is a rectangle with height $f(x)$ and width $x + 0' - x = 0'$.

The total area under the curve from $x = a$ to $x = b$ is the sum of the areas of these slivers, the number of these slivers is $\infty' :\equiv \dfrac{b-a}{0'}$, and the width of each sliver is $0' \equiv \frac{b-a}{\infty'}$. The total area from a to b, and the **definite integral** of $f(x)$ from a to b, is

$$\int_a^b f(x)\, dx :\equiv \sum_{k=1}^{\infty'} f\left(a + \frac{k(b-a)}{\infty'}\right) \frac{b-a}{\infty'}.$$

As an example of this calculation:

$$\int_0^u 2x\, dx \equiv \sum_{k=1}^{\infty'} 2\frac{ku}{\infty'}\frac{u}{\infty'}$$

$$\equiv \frac{2u^2}{\infty'^2} \sum_{k=1}^{\infty'} k$$

$$\equiv \frac{2u^2}{\infty'^2} \infty' \frac{\infty' + 1}{2}$$

$$\equiv u^2\left(1 + \frac{1}{\infty'}\right)$$

$$\equiv u^2(1 + 0')$$

$$= u^2.$$

For a comparison of this definition of the definite integral with that in other systems of analysis, see the **Appendix** (p. 139).

Infinite bounds on integrals and path integral

In the equipoint definition of integral, $b - a$ may be infinite if either of the limits a or b is infinite. In this case, we simply choose a $0'$ of a high enough sensitivity that $\frac{b-a}{0'}$ is infinitesimal, e.g. $\frac{1}{(b-a)^2}$.

Equipoint analysis can be used with any infinite element extension discussed in [CN]. With a projectively extended system, bounds of integration may appear ambiguous, since $+\infty$ and $-\infty$ are identical. In this case, it is helpful to remember that bounds of integration implicitly establish a *path* of integration: integrating from $-\infty$ to $+\infty$ integrates through 0 and all the finite values, integrating from 0 to $+\infty$ integrates through all the positive finite values, etc. The equipoint integral along a path $x = P(t)$, where t runs from a to b, is given by

$$\int_P f(x)\, dx = \int_{t=a}^b f(P(t))\, dP(t) = \int_a^b f(P(t))\frac{dP(t)}{dt}\, dt.$$

Differentiability and integrability

The **Singularities** (p. 68) chapter discusses several **types of singularity** (p. 72) which may present difficulties using the above definitions of derivative and integral.

Briefly, at a jump discontinuity, the derivative is infinite, and the integral can be calculated straighforwardly through the singularity. See the discussions of the **absolute value function** (p. 73), the **Kronecker delta function** (p. 78), and the **Dirac delta function** (p. 79).

At punctured functions, poles, and essential singularity singularities, it is necessary to use an **offset derivative** (p. 68), and attempts to integrate through these singularities may be incorrect. See the discussions of the **punctured constant function** (p. 75), the **axial function** (p. 87), **poles** (p. 85), and the **function $\sin \frac{1}{x}$.** (p. 89)

THE FUNDAMENTAL THEOREMS OF CALCULUS

For the following equipoint proofs of the first and second fundamental theorems of calculus, we assume the following:

1. The splitting property $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ for all c, which is easily proved from the definition of the definite integral.

2. A corollary, the zero property $\int_a^a f(x)\,dx = 0$ for all a.

3. Another corollary, the reversal property $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$.

4. The function f is **continuous** (p. 52): $f(x + 0') = f(x)$ for the endpoints $x = a$ and $x = b$ in the first theorem, and for all x in the second theorem. Cases where this condition does not hold are discussed at the end of this seciton.

We do *not* assume the mean value theorem.

We recall that the definite integral is defined as

$$\int_a^b f(x)\,dx \equiv \sum_{k=1}^{\infty'} f\left(a + \frac{k(b-a)}{\infty'}\right)\frac{b-a}{\infty'},$$

and the derivative as

$$\frac{df(x)}{dx} \equiv \frac{f(x + 0') - f(x)}{0'}.$$

THE FIRST FUNDAMENTAL THEOREM OF CALCULUS:

$$\int_a^b \frac{df(x)}{dx}\,dx = f(b) - f(a).$$

PROOF.

$$\int_a^b \frac{df(x)}{dx}\, dx \equiv \int_a^b \frac{f\left(x - \frac{b-a}{\infty'}\right) - f(x)}{\frac{b-a}{\infty'}}\, dx$$

$$\equiv \frac{b-a}{\infty'} \frac{\infty'}{b-a} \sum_{k=1}^{\infty'} f\left(a + \frac{k(b-a)}{\infty'} + \frac{b-a}{\infty'}\right) - f\left(a + \frac{k(b-a)}{\infty'}\right)$$

$$\equiv \sum_{k=1}^{\infty'} f\left(a + (k+1)\frac{b-a}{\infty'}\right) - f\left(a + k\frac{b-a}{\infty'}\right)$$

$$\equiv f\left(a + (\infty'+1)\frac{b-a}{\infty'}\right) - f\left(a + \frac{b-a}{\infty'}\right)$$

$$\equiv f\left(a + (b-a) + 0'\right) - f\left(a + 0'\right)$$

$$= f(b) - f(a). \ \square$$

THE SECOND FUNDAMENTAL THEOREM OF CALCULUS:

$$\frac{d}{dx}\int_c^x f(u)\, du = f(x).$$

PROOF.

$$\frac{d}{dx}\int_c^x f(u)\, du \equiv \frac{\int_c^{x+0'} f(u)\, du - \int_c^x f(u)\, du}{0'}$$

$$\equiv \frac{\int_c^x f(u)\, du + \int_x^{x+0'} f(u)\, du - \int_c^x f(u)\, du}{0'}$$

$$\equiv \frac{\int_x^{x+0'} f(u)\, du}{0'}$$

$$\equiv \frac{f(x+0')0'}{0'}$$

$$\equiv f(x+0')$$

$$= f(x). \ \square$$

We then have

$$\int_c^x f(u)\, du = F(x) + k,$$

where $F(x)$ is any function such that

$$\frac{dF(x)}{dx} = f(x),$$

and k is a constant that depends on c. Then we have

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$
$$= \int_c^b f(x)\,dx - \int_c^a f(x)\,dx$$
$$= F(b) - F(a).$$

Numeristics and equipoint analysis allow us to apply these definitions and theorems to a wide range of functions. A function that is conventionally considered **discontinuous** (p. 52) may have an infinite equipoint derivative at the point of discontinuity. A similarly wide net is cast for integration. Abscissas and ordinates may be finite or infinite, single valued or multivalued.

There are few types of singularity where these theorems do not apply completely, since the type of derivative used here at such points is not determinate. In these cases, it may be necessary to use an **offset derivative** (p. 68) and restrict the range of integration. Singularities where this occurs include **poles** (p. 85) and **essential singularities** (p. 89). This consideration is discussed in detail in the **Singularities** (p. 68) chapter.

DERIVATIVE THEOREMS

Chain rule

THE CHAIN RULE:

$$\frac{d}{dx}f(g(x)) = \left[\frac{d\,f(g(x))}{d\,g(x)}\right]\left[\frac{d}{dx}g(x)\right]$$

It might appear that $d\,g(x)$ can simply be cancelled, but since the differentials on the left and right sides have slightly different interpretations, we must proceed more carefully.

PROOF. Define

$$y :\equiv g(x)$$
$$0'' :\equiv g(x+0') - g(x)$$

Then

$$\frac{df(g(x))}{dx} \equiv \frac{f(g(x+0')) - f(g(x))}{0'}$$
$$\equiv \frac{f(y+0'') - f(y)}{0'}$$
$$\equiv \frac{f(y+0'') - f(y)}{0''} \cdot \frac{0''}{0'}$$
$$\equiv \frac{f(y+0'') - f(y)}{0''} \cdot \frac{g(x+0') - g(x)}{0'}$$
$$\equiv \frac{df(y)}{dy} \cdot \frac{dg(x)}{dx}$$
$$\equiv \frac{df(g(x))}{dg(x)} \cdot \frac{dg(x)}{dx}. \,\square$$

Product rule

THE BASIC PRODUCT RULE:
$$\frac{d}{dx}f(x)g(x) = f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x)$$

PROOF.
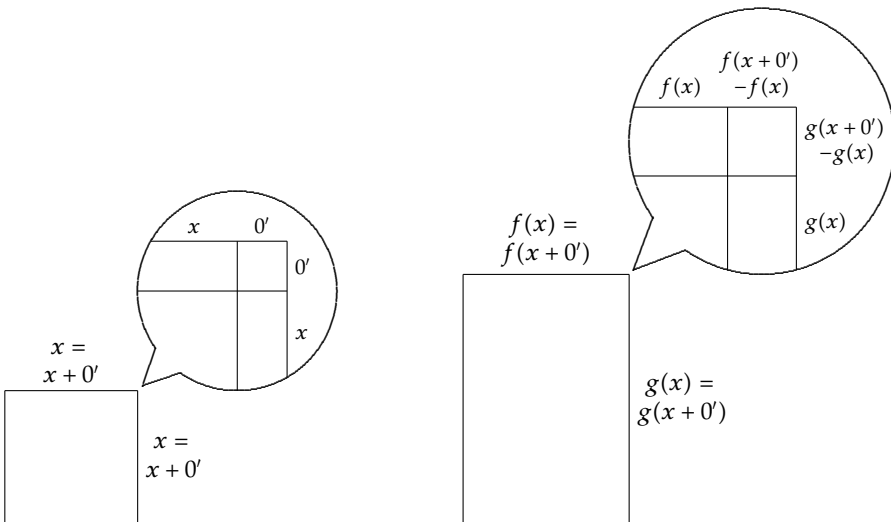
FIG. 9:
Calculation of derivative of
product of $f(x)$ and $g(x)$

Figure 9 shows, on the left, a square with sides of length $x + 0'$ and, on the right, a rectangle which is the transform of this square by $f(x)$ in the horizontal direction and $g(x)$ in the vertical direction. The rectangle has sides
$$f(x + 0') \equiv f(x) + [f(x + 0') - f(x)]$$
$$g(x + 0') \equiv g(x) + [g(x + 0') - g(x)].$$

The two strips on the sides of the left figure, with area $x \cdot 0'$, are infinitesimally small compared to the large portion of the left figure, with area

x^2. These strips are transformed to the two strips on the sides of the right figure, with areas $f(x) \cdot [f(x + 0') - f(x)]$ and $g(x) \cdot [g(x + 0') - g(x)]$, which are infinitesimally small compared to the large portion of the right figure, with area $f(x) \cdot g(x)$.

The small square in the upper right corner of the left figure, with area $0'^2$, is transformed to a small rectangle in the upper right corner of the right figure, with area $[f(x + 0') - f(x)] \cdot [g(x + 0') - g(x)]$. Both are infinitesimally small compared to the strips on the sides:

$$[f(x + 0') - f(x)] \cdot [g(x + 0') - g(x)] =' 0$$

or

$$\frac{[f(x + 0') - f(x)] \cdot [g(x + 0') - g(x)]}{0'} = 0.$$

Then
$$\frac{d}{dx}f(x)g(x) \equiv \frac{f(x + 0')g(x + 0') - f(x)g(x)}{0'}$$

$$\equiv \frac{1}{0'}\Big[f(x)g(x)$$
$$+ f(x)\left[g(x + 0') - g(x)\right]$$
$$+ \left[f(x + 0') - f(x)\right]g(x)$$
$$+ \left[f(x + 0') - f(x)\right]\left[g(x + 0') - g(x)\right]$$
$$- f(x)g(x)\Big]$$

$$= \frac{1}{0'}\Big[f(x)g(x)$$
$$+ f(x)\left[g(x + 0') - g(x)\right]$$
$$+ \left[f(x + 0') - f(x)\right]g(x)$$
$$- f(x)g(x)\Big]$$

$$\equiv \frac{f(x)\left[g(x + 0') - g(x)\right] + g(x)\left[f(x + 0') - f(x)\right]}{0'}$$

$$\equiv f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x). \ \square$$

THE MULTIPRODUCT RULE:

$$\frac{d}{dx}\prod_{k=1}^{n} f_k(x) = \sum_{j=1}^{n}\left(\prod_{\substack{k=1 \\ k \neq n}}^{n} f_k(x)\right)\left(\frac{d}{dx}f_j(x)\right)$$

PROOF. By induction. The case $n = 2$ is the basic product rule proved above. Assuming the multiproduct rule for n, then:

$$\frac{d}{dx} \prod_{k=1}^{n+1} f_n(x) \equiv \frac{d}{dx}\left[\left(\prod_{k=1}^{n} f_n(x) \right) f_{n+1}(x) \right]$$

$$= f_{n+1}(x) \left(\frac{d}{dx} \prod_{k=1}^{n} f_k(x) \right) + \left(\prod_{k=1}^{n} f_k(x) \right) \frac{d}{dx} f_{n+1}(x)$$

$$= f_{n+1}(x) \sum_{j=1}^{n} \left(\prod_{\substack{k=1 \\ k \neq n}}^{n} f_k(x) \right) \left(\frac{d}{dx} f_j(x) \right) + \frac{d}{dx} f_{n+1}(x) \left(\prod_{k=1}^{n} f_k(x) \right)$$

$$\equiv \sum_{j=1}^{n} \left(\frac{d}{dx} f_j(x) \right) \left(\prod_{\substack{k=1 \\ k \neq j}}^{n} f_k(x) \right) + \left(\frac{d}{dx} f_{n+1}(x) \right) \left(\prod_{\substack{k=1 \\ k \neq n+1}}^{n} f_k(x) \right)$$

$$\equiv \sum_{j=1}^{n+1} \left(\prod_{\substack{k=1 \\ k \neq n}}^{n+1} f_k(x) \right) \left(\frac{d}{dx} f_j(x) \right)$$

which is the rule for $n + 1$. $\square$

Inverse rule

THE INVERSE RULE: If f is single valued, and $y = f(x)$, then
$$\frac{df^{-1}(y)}{dy} \frac{df(x)}{dx} \supseteq 1,$$
with equality holding if f is injective.

PROOF. If f is single valued and injective, then $f^{-1}(f(x)) = f^{-1}(y) = x$, and by the chain rule,
$$\frac{df^{-1}(y)}{dy} \frac{df(x)}{dx} = \frac{df^{-1}(f(x))}{df(x)} \frac{df(x)}{dx} = \frac{dx}{df(x)} \frac{df(x)}{dx} = 1.$$

If f is only single valued, then $f^{-1}(y)$ may be multivalued, and $f^{-1}(f(x)) \supseteq x$, and
$$\frac{df^{-1}(y)}{dy} \frac{df(x)}{dx} \supseteq 1. \square$$

A simple example of function with a multivalued inverse is $f(x) := x^2$.

$$f^{-1}(y) = y^{\frac{1}{2}} = \pm\sqrt{y}$$

$$(x^2)^{\frac{1}{2}} = \pm x$$

$$\frac{d}{dy}y^{\frac{1}{2}} = \frac{d}{dy}(\pm x) = \frac{\pm 1}{\frac{dx^2}{dx}}$$

$$= \frac{\pm 1}{2x} = \frac{\pm 1}{2\sqrt{y}}$$

Power rule

THE POWER RULE: **For any complex n,**

$$\frac{d}{dx}x^n = nx^{n-1}$$

PROOF. FOR $n = 1$:

$$\frac{d}{dx}x \equiv \frac{x + 0' - x}{0'} \equiv 1.$$

FOR $n \in \mathbb{Z}^+$: By the **multiproduct rule** (p. 40) with $f_k(x) := x$ and this rule for $n = 1$,

$$\frac{d}{dx}x^n = \frac{d}{dx}\prod_{k=1}^{n}x = \sum_{j=1}^{n}\left(\prod_{\substack{k=1 \\ k \neq j}}^{n}x\right) \cdot 1 = nx^{n-1}.$$

FOR $n = -1$:

$$\frac{d}{dx}\frac{1}{x} \equiv \frac{1}{0'(x + 0')} - \frac{1}{0'x} \equiv \frac{x - x - 0'}{0'x(x + 0')} \equiv \frac{-0'}{0'x(x + 0')} = \frac{-1}{x^2}.$$

FOR $n \in \mathbb{Z}^-$: By the multiproduct rule with $f_k(x) := x^{-1}$ and $m := -n$, and this rule for $n = -1$,

$$\frac{d}{dx}x^n = \frac{d}{dx}x^{-m} = \frac{d}{dx}\prod_{k=1}^{m}x^{-1} = \sum_{j=1}^{m}\left(\prod_{\substack{k=1 \\ k \neq j}}^{m}x^{-1}\right)(-x^{-2}) = -mx^{-m-1} = nx^{n-1}.$$

FOR $n \in \frac{1}{\mathbb{Z}^*}$: By the **inverse rule** (p. 42) with $q := \frac{1}{n}$ and $y := x^{\frac{1}{q}}$,

$$\frac{d}{dx}x^n = \frac{d}{dx}x^{\frac{1}{q}} = \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{\frac{dy^q}{dy}} = \frac{1}{qy^{q-1}} = \frac{1}{q}y^{1-q} = \frac{1}{q}x^{\frac{1}{q}-1} = nx^{n-1}.$$

FOR $n \in \mathbb{Q}^*$: By the multiproduct rule with $f_k(x) := x^{\frac{1}{q}}$, and this rule for $n \in \frac{1}{\mathbb{Z}}^*$, and given any $p \in \mathbb{Z}^+$ and $q \in \mathbb{Z}^*$ such that $n = \frac{p}{q}$,

$$\frac{d}{dx}x^n = \frac{d}{dx}x^{\frac{p}{q}} = \frac{d}{dx}\prod k = 1^p x^{\frac{1}{q}} = \sum_{j=1}^{p}\left(\prod_{\substack{k=1 \\ k \neq j}}^{p} x^{\frac{1}{q}}\right)\left(\frac{1}{q}x^{\frac{1}{q}-1}\right) = \frac{p}{q}x^{\frac{p}{q}-1} = nx^{n-1}.$$

FOR $n \in \mathbb{R}^*$: n has at least one decimal representation $\sum_{k=-\infty}^{+\infty} a_k 10^k$, where each a_k is a decimal digit $0,1,\ldots,9$. As shown in [CR], this representation is unique only when the decimal representation is not repeating and infinite left decimals are not allowed. We do not require uniqueness here, only that there be at least one such representation.

The following uses an infinite case of the **multiproduct rule** (p. 40). As discussed in [CD], the numeristic theory of infinite series, including equipoint summation, does not have the inconsistencies of the conventional theory of infinite series, so we feel confident using such infinite methods without any special proofs.

Therefore, by the multiproduct rule and this rule for positive and negative integer n,

$$\frac{d}{dx}x^n = \frac{d}{dx}x^{\sum_{k=-\infty}^{+\infty} a_k 10^k} = \frac{d}{dx}\prod_{k=-\infty}^{+\infty} x^{a_k 10^k}$$

$$= \sum_{j=-\infty}^{+\infty}\left(\prod_{\substack{k=-\infty \\ k \neq j}}^{+\infty} xa_k 10^k\right)\left(a_j x^{a_j 10^j - 1}\right)$$

$$= \sum_{j=-\infty}^{+\infty} a_j 10^j \frac{1}{10}\left(\prod_{k=-\infty}^{+\infty} x^{a_k 10^k}\right)$$

$$= \left(\sum_{j=-\infty}^{+\infty} a_j 10^j\right)x^{\left(\sum_{k=-\infty}^{+\infty} a_k 10^k\right)-1} = nx^{n-1}.$$

FOR $n \in \mathbb{C}^*$: Let $r := \operatorname{Re} n$ and $s := \operatorname{Im} n$; and let b_k and c_k be the digits of respective digital representations of r and s; and a_k be the complex digits:

$$r = \sum_{k=-\infty}^{+\infty} b_k 10^k$$

$$s = \sum_{k=-\infty}^{+\infty} c_k 10^k$$

$$a_k := b_k + ic_k$$

Then $n = r + is = \sum_{k=-\infty}^{+\infty} a_k 10^k$. By the multiproduct rule and the same calculation as above for real n,

$$\frac{d}{dx} x^n = \frac{d}{dx} x^{r+is} = \frac{d}{dx} x^{\sum_{k=-\infty}^{+\infty} a_k 10^k} = nx^{n-1}. \; \Box$$

THE QUOTIENT RULE:

$$\frac{d}{dx} \frac{f(x)}{g(x)} == \frac{g(x)\frac{d}{dx}f(x) - f(x)\frac{d}{dx}g(x)}{g(x)^2}$$

PROOF. By the product rule, the power rule for $n = -1$, and the chain rule,

$$\frac{d}{dx} \frac{f(x)}{g(x)} = \frac{1}{g(x)} \frac{d}{dx} f(x) + f(x) \frac{d}{dx} \frac{1}{g(x)}$$

$$= \frac{1}{g(x)} \frac{d}{dx} f(x) - f(x) \frac{\frac{d}{dx}g(x)}{g(x)^2}$$

$$= \frac{g(x)\frac{d}{dx}f(x) - f(x)\frac{d}{dx}g(x)}{g(x)^2}. \; \Box$$

Derivatives of sine and cosine

$$\frac{d\cos\theta}{d\theta} = -\sin\theta$$

$$\frac{d\sin\theta}{d\theta} = \cos\theta$$
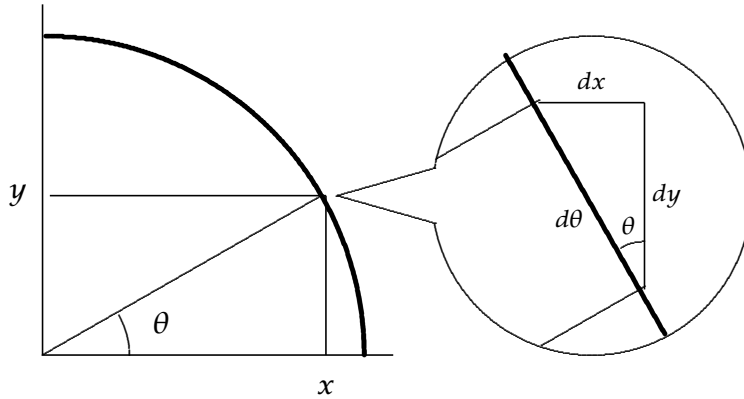
FIG. 10:
Calculation of derivatives
of sine and cosine

PROOF. Figure 10 depicts the calculation of the derivatives of the sine and cosine functions. In this figure, we have

$$x = \cos\theta$$
$$y = \sin\theta$$

and a microscope picture of the point (x, y).

In the microscope, the circle has become a straight line, coincident with the tangent to the circle at (x, y). Outside the microscope, the radius is a single line, but within the microscope, the radius is the class of all lines normal to the tangent. We show two such radius lines that are separated by the distance $d\theta$. The units of this distance must match the units in the tangent and radius, so we must measure $d\theta$, and thus θ itself, in radians.

The line segment along the tangent bounded by the two radii forms a triangle with legs dx and dy and hypotenuse $d\theta$. We then have

$$\frac{y}{x} = -\frac{dx}{dy}$$
$$\frac{dx}{d\theta} = -\sin\theta$$
$$\frac{dy}{d\theta} = \cos\theta. \ \square$$

Derivative of exponential function

$$\frac{d}{dx}e^x = e^x$$

PROOF. We start with an equipoint definition of e and compute e^x.

$$e = \left(1 + \frac{1}{\infty'}\right)^{\infty'}$$

$$e^x = \left(1 + \frac{1}{\infty'}\right)^{\infty'x}$$

$$= \left(1 + \frac{x}{\infty''}\right)^{\infty''}.$$

The last line comes from the substitution $\infty'x \to \infty''$, or $\infty' \to \dfrac{\infty''}{x}$. Adding to this the substitution $\dfrac{1}{\infty''} \to 0''$, we then have

$$e^{0''x} \equiv (e^x)^{0''}$$

$$\equiv \left(1 + \frac{x}{\infty''}\right)^{\infty''0''}$$

$$\equiv \left(1 + \frac{x}{\infty''}\right)^{1}$$

$$\equiv \left(1 + \frac{x}{\infty''}\right)$$

$$\equiv (1 + 0''x).$$

Solving for x we have

$$x \equiv \frac{e^{0''x} - 1}{0''}$$

$$1 \equiv \frac{e^{0''} - 1}{0''}$$

$$e^x \equiv e^x \frac{e^{0''} - 1}{0''}$$

$$\equiv \frac{e^{x+0''} - e^x}{0''}$$

$$= \frac{d}{dx}e^x.$$

Natural logarithm as a polynomial

$$\ln t = \frac{t^{0'} - 1}{0'}$$

PROOF. Start with a result from the previous section and substitute $x = \ln t$.

$$x = \frac{e^{0'x} - 1}{0'}$$

$$\ln t = \frac{t^{0'} - 1}{0'}.$$

$\int_1^t t^{-1}\, dt = \ln t$ is an instance of the general law $\int_0^t t^n\, dt = \dfrac{t^{n+1}}{n+1}$, not an exception.

PROOF. Integrate t^{-1} with $\int_0^t t^n\, dt = \dfrac{t^{n+1}}{n+1}$ and obtain

$$\int_1^t t^{-1}\, dt = \frac{t^{0'}}{0'} - \frac{1}{0'}$$

$$= \frac{t^{0'} - 1}{0'}$$

$$= \ln t. \ \square$$

This result can be verified with L'Hôpital's rule, which is proved below. For real t, this result is also verified by the following.

$$\int_1^x t^{-1}\, dt = \ln |x|$$

$$\frac{d}{dx} \ln |x| = \frac{\operatorname{sgn} x}{x} = \frac{x}{|x|^2} = \frac{1}{x}.$$

$\dfrac{y^{0'} - 1}{0'}$ is a zeroth-order polynomial in unfolded arithmetic, or more accurately a polynomial of order $0'$. Its integrals are unfolded polynomals of higher degrees:

$$\int \ln x \, dx = x \ln x - x$$

$$= \frac{x^{1+0'} - x}{0'} - x$$

$$\int (x \ln x - x) \, dx = \frac{x^2}{2} \ln x - \frac{3x^2}{4}$$

$$= \frac{x^{2+0'} - x^2}{2 \cdot 0'} - \frac{3x^2}{4}$$

LIMITS AND CONTINUITY

Limits

Definition of limit forms

A limit can be defined with an unfolded expression which gives results similar to those given by conventional definitions. In many cases, these expressions can be evaluated where a conventional limit fails to exist. Any syntactically correct statement is meaningful, and so these expressions always have a meaning, which may include multivalued classes or the empty class. We will later see several examples of this.

$$\lim_{x \to a} f(x) := f(a + 0'), \text{where } a \text{ is finite and } 0' \neq' 0'^2$$

$$\lim_{x \to a+} f(x) := f(a + 0'), \text{where } a \text{ is finite and } 0' >' 0'^2$$

$$\lim_{x \to a-} f(x) := f(a + 0'), \text{where } a \text{ is finite and } 0' <' 0'^2$$

$$\lim_{x \to \infty} f(x) := f(\infty'), \text{where } \infty' \neq' \infty'^2$$

$$\lim_{x \to +\infty} f(x) := f(\infty'), \text{where } \infty' <' \infty'^2$$

$$\lim_{x \to -\infty} f(x) := f(\infty'), \text{where } \infty' >' \infty'^2$$

Example

$$\lim_{x \to 0} \frac{e^x - 1}{x} := \frac{e^{0'} - 1}{0'} = 1$$

Offset expressions

A limit in the form $f(a + 0')$ or $f(\infty')$ will also be called an *offset expression*. It is not the result of a process but simply a value of a function at an unfolded point. An expression in the form of $f(a)$ or $f(\infty)$, which is at the origin of the unfolding, will be called an *original expression*.

Any of the above expressions may be multivalued and/or depend on $0'$ or ∞'. In such cases, we may wish to restrict our attention to those cases in which the expression is single valued and/or independent of $0'$ or ∞'.

Uniform offset expressions

An offset expression $f(a + 0')$ is *uniform* if it has the same folded value for all $0'$, i.e. if $f(a + 0') = f(a + 0'')$ for any $0', 0'' \in' 0$, even when $0'$ and $0''$ have different signs. If f is single valued and $f(a + 0')$ is uniform, then the class $f(a + \mathbb{R}0')$ or $f(a + \mathbb{C}0')$ is single valued. A derivative $f'_{0'}(a) \equiv \dfrac{f(a + 0') - f(a)}{0'}$ is uniform if it has the same value for all $0'$.

An offset expression $f(a + 0')$ is *semiuniform* if it is the same for every $0'$ of the same sign. If f is single valued and $f(a + 0')$ is semiuniform, then the class $f(a + |\mathbb{R}|0')$ or $f(a + |\mathbb{C}|0')$ is single valued.

An offset expression $f(a + 0')$ is *disuniform* if it is neither uniform nor semiuniform.

Continuity
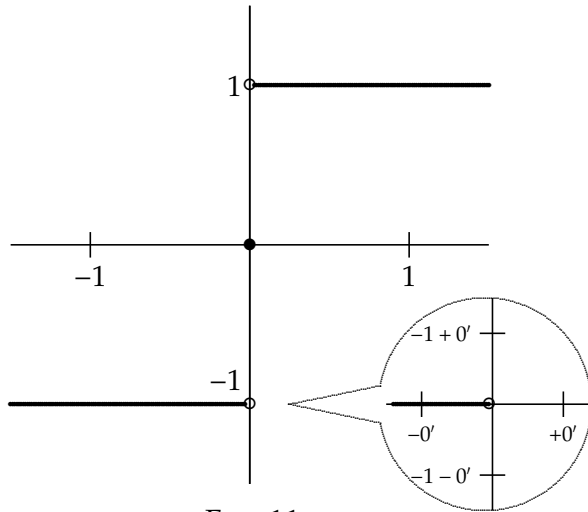
FIG. 11:
Discontinuity of signum function
sgn x at $x = 0$

Definition

A function f is ***continuous*** at x if the offset values are uniform and equal to the original value, i.e. $f(x + a0') = f(x)$ for every $a \in \mathbb{R}$, or $f(x + \mathbb{R}0') = f(x)$.

A function f is ***semicontinuous*** at x if the offset values are semiuniform and equal to the original value, i.e. $f(x + a0') = f(x + b0')$ for every $a, b \in \mathbb{R}$ and $\operatorname{sgn} a = \operatorname{sgn} b$. In this case, if $f(x + a0') = f(x)$ for positive a, then f is ***right continuous***; for negative a, ***left continuous***.

Figure 11 shows an example of a discontinuity in the signumfunction
$$\operatorname{sgn} x := \begin{cases} -1 & \text{for } x < 0 \\ 0 & \text{for } x = 0 \\ +1 & \text{for } x > 0. \end{cases}$$
This function is discontinuous at $x = 0$ because $\operatorname{sgn}(x) = 0$ while $\operatorname{sgn}(x - 0') = -1$ and $\operatorname{sgn}(x + 0') = +1$ for $0' >' 0^2$. It is semicontinuous but neither left continuous nor right continuous.

If f is continuous at x, then f is locally linear: $f(x + 0') - f(x) =' f(x + 0'(k + 1)) - f(x + 0'k)$ for real k.

If f has a finite derivative at x, then it is continuous at x:

$$f'(x) := \frac{f(x + 0') - f(x)}{0'}$$

$$f(x + 0') =' 0'f'(x) + f(x)$$

Since $f'(x)$ is finite, $0'f'(x) =' 0$

$$f(x + 0') = f(x).$$

Continuity for infinite values

Continuity involving infinite values may depend on the choice of infinite element extension. Consider the reciprocal function $f(x) :\equiv \frac{1}{x}$. At $x = 0$, $f(x + a0') \equiv \frac{1}{0 + a0'} = \frac{1}{a0'}$. In the projectively extended real numbers,

$$\frac{1}{a0'} = \infty$$

for all real a, so $f(x)$ is continuous at $x = 0$. But in the affinely extended real numbers,

$$\frac{1}{a0'} = \begin{cases} +\infty \neq -\infty & \text{for } a > 0 \\ -\infty \neq +\infty & \text{for } a < 0, \end{cases}$$

so $f(x)$ is not continuous at $x = 0$, only semicontinuous.

It can be shown that the characteristic function of the rational numbers $[\mathbb{Q}](x)$ is continuous at irrational x and discontinuous at rational x. See **Using class counts in derivatives and integrals** (p. 130).

Continuity of multivalued functions

One approach to the continuity of multivalued functions is to examine a single valued branch. A branch can be often be defined by intersecting the multivalued function with a subset of the range. For example, in $\mathbb{R}$, if we take $x^{\frac{1}{2}}$ to mean the multivalued inverse of x^2, we can define $\sqrt{x}$ to be the nonnegative branch of $x^{\frac{1}{2}}$, i.e. $\sqrt{x} := x^{\frac{1}{2}} \cap |\mathbb{R}|$. Then $\sqrt{x}$ is continuous, since $\sqrt{x + \mathbb{R}0'} = \sqrt{x}$.

Another approach to multivalued continuity is distributed continuity. We define a multivalued function $f(x)$ to be ***conjunctively continuous*** if $(\forall a \in f(x+\mathbb{R}0'))\ a \in f(x)$, and ***disjunctively continuous*** if $(\exists a \in f(x+\mathbb{R}0'))\ a \in f(x)$. Then $x^{\frac{1}{2}}$ is conjunctively continuous everywhere, since $(\forall a \in (x+\mathbb{R}0')^{\frac{1}{2}})\ a \in x^{\frac{1}{2}}$ for all x.

Subcontinuity of multivalued functions

We define a multivalued function to be $f(x)$ ***subcontinuous*** at x if $f(x+\mathbb{R}0') \subseteq f(x)$. We say it is ***semisubcontinuous*** if $f(x+a0') \subseteq f(x)$ for all a of a given sign. Assuming $0'$ is positive, i.e. $0' >' 0 \cdot 0'$, then if f is semisubcontinuous, we say that it is ***right subcontinous*** if a is positive, and ***left subcontinuous*** if a is negative.

In the projectively extended real numbers, we found above that $\dfrac{1}{x}$ is continuous at $x = 0$, but the same does not hold for $e^{\frac{1}{x}}$. At $x = 0$, this function has two values:

$$e^{\frac{1}{0}} = e^{\infty} = \left\{e^{+\infty}, e^{-\infty}\right\} = \{\infty, 0\}$$

and, assuming $0'$ is positive,

$$e^{\frac{1}{a0'}} = \begin{cases} +\infty & \text{for } a > 0 \\ 0 & \text{for } a < 0, \end{cases}$$

so $e^{\frac{1}{x}}$ is both left and right subcontinuous at $x = 0$.

DIFFERENTIAL AND INTEGRAL OPERATORS

Differentials and integrants

Differentials

As the Leibnitz notation $\dfrac{dy}{dx}$ indicates, a derivative is an arithmetic quotient of differentials. The differential of an independent variable is an infinitesimal, as is the differential of a dependent variable when the derivative is finite. Infinitesimals are unfolded members of folded zero, which are exactly equal to zero in folded aritmetic but distinct in unfolded arithmetic.

A **differential** is an operator on a function with respect to a member of zero. We define

$$^{0'}d_a f(x) :\equiv f(a + 0') - f(a),$$

from which follows

$$^{0'}d_a x :\equiv a + 0' - a \equiv 0'.$$

A derivative with respect to an infinitesimal $0'$ can therefore be defined as:

$$f'_{0'}(a) :\equiv \frac{^{0'}d_a f(x)}{^{0'}d_a x} \equiv \frac{f(a + 0') - f(a)}{0'}.$$

If the derivative is independent of the infinitesimal, we write:

$$f'(a) :\equiv \frac{d_a f(x)}{d_a x} \equiv \frac{f(a + 0') - f(a)}{0'}.$$

This occurs when $f(x)$ is analytic, since, for $f(x) = x^n$,

$$^{0'}d_a f(x) \equiv nx^{n-1}0' + \sum_{k=2}^{n} \binom{n}{k} a^{n-k}0'^k =' nx^{n-1}0'.$$

The notations $f'(x)$ and $\dfrac{df(x)}{dx}$, of course, mean the function $\dfrac{d_x f(x)}{d_x x}$, and $\dfrac{d_a f(x)}{d_a x}$ can be written $f'(x)\big|_a$ or $\dfrac{df(x)}{dx}\bigg|_a$.

Integrants

We also define an *integrant* as an operator on a function:

$$\overset{0'}{\int}{}^{a} f(x) :\equiv \sum_{k=1}^{\frac{a}{0'}} f(0'k).$$

An integrant is infinite whenever the corresponding integral is nonzero.

The definite integral can be defined in terms of an integrant and a differential:

$$\overset{0'}{\int}{}_{a}^{b} f(x)\,dx :\equiv \overset{0'}{\int}{}^{b} f(x)0' - \overset{0'}{\int}{}^{a} f(x)0' \equiv \sum_{k=1}^{\frac{b}{0'}} f(0'k)0' - \sum_{k=1}^{\frac{a}{0'}} f(0'k)0'$$

$$\left(\text{substituting } \infty' :\equiv \tfrac{b-a}{0'}\right)$$

$$\equiv \sum_{k=1}^{\frac{\infty' b}{b-a}} f\left(k\frac{b-a}{\infty'}\right)\frac{b-a}{\infty'} - \sum_{k=1}^{\frac{\infty' a}{b-a}} f\left(k\frac{b-a}{\infty'}\right)\frac{b-a}{\infty'}$$

$$\equiv \sum_{k=\frac{\infty' a}{b-a}+1}^{\frac{\infty' b}{b-a}} f\left(k\frac{b-a}{\infty'}\right)\frac{b-a}{\infty'} \qquad \left(\text{substituting } j :\equiv k - \tfrac{\infty' a}{b-a}\right)$$

$$\equiv \sum_{j=1}^{\infty'} f\left(a + j\frac{b-a}{\infty'}\right)\frac{b-a}{\infty'}$$

Again, if the integral is independent of the infinitesimal, we write:

$$\int_{a}^{b} f(x)\,dx :\equiv \sum_{k=1}^{\frac{b}{0'}} f(0'k)0' - \sum_{k=1}^{\frac{a}{0'}} f(0'k)0'.$$

We can define the indefinite integral operator in terms of the definite integral in two ways. The first way is as a definite integral plus an arbitrary constant:

$$\overset{0'}{\int} f(x)\,dx :\equiv \overset{0'}{\int}{}^{x} f(t)\,dt + \mathbb{R} \equiv \left\{\overset{0'}{\int}{}^{x} f(t)\,dt + a \mid a \in \mathbb{R}\right\}$$

or

$$\int f(x)\,dx := \int^x f(t)\,dt + \mathbb{R} \equiv \left\{ \int^x f(t)\,dt + a \mid a \in \mathbb{R} \right\}.$$

The second way to define the indefinite integral is as a class of definite integrals with an arbitrary lower limit:

$$\overset{0'}{\int} f(x)\,dx := \overset{0'}{\underset{\mathbb{R}}{\int}}^x f(t)\,dt \equiv \left\{ \overset{0'}{\underset{a}{\int}}^x f(t)\,dt \mid a \in \mathbb{R} \right\}$$

or

$$\int f(x)\,dx := \underset{\mathbb{R}}{\int}^x f(t)\,dt \equiv \left\{ \underset{a}{\int}^x f(t)\,dt \mid a \in \mathbb{R} \right\}.$$

Either of these is a class of functions. If we denote the first A and the second as B, then given any two $F_1, F_2 \in A$, we have $F_2(x) = F_1(x) + c$, where c is a constant, and conversely. Similarly, given any two $F_1, F_2 \in B$ and their corresponding a_1, a_2, we have $F_2(x) = F_1(x) - F(a_1) + F(a_2)$. Thus $A \supseteq B$, with equality holding if all the members of A are surjective.

The integrant is the left inverse of the differential, which is essentially the first fundamental theorem of calculus:

$$\int_0^x df(x) \equiv \sum_{k=1}^{\frac{x}{0'}} f(0'k + 0') - f(0'k)$$

$$\equiv \sum_{k=1}^{\frac{x}{0'}} f\left(0'(k+1)\right) - f(0'k)$$

$$\equiv f\left(0'\left(\frac{x}{0'} + 1\right)\right) - f(0')$$

$$\equiv f(x + 0') - f(0')$$

$$= f(x) - f(0').$$

The integrant is also the right inverse of the differential, which is essentially the second fundamental theorem of calculus:

$$d\int_0^x f(x) \equiv \int^{x+0'} f(x) + \int^x f(x)$$

$$\equiv \int^x f(x) + \int_x^{x+0'} f(x) + \int^x f(x)$$

$$\equiv \sum_{k=1}^{\frac{x+0'-x}{0'}} f(x)$$

$$= f(x).$$

Partial differentials

The *partial differential* is defined analogously to the differential. Here we define a partial differential on a function of two independent variables:
$$^{0'}\partial_x f(x,y) :\equiv f(x + 0', y) - f(x, y),$$
or, if the result is independent of $0'$:
$$\partial_x f(x,y) :\equiv f(x + 0', y) - f(x, y).$$

The total differential is then easily seen to be the sum of partial differentials:
$$\begin{aligned}
df(x,y) &= d_{x,y} f(x,y) \\
&= f(x + 0', y + 0') - f(x, y) \\
&= \left[f(x + 0', y + 0') - f(x, y + 0') \right] + \left[f(x, y + 0') - f(x, y) \right] \\
&= \left[f(x + 0', y) - f(x, y) \right] + \left[f(x, y + 0') - f(x, y) \right] \\
&= \partial_x f(x,y) + \partial_y f(x,y) \\
&= (\partial_x + \partial_y) f(x,y)
\end{aligned}$$

Quotiential and prodegrant operators

Closely related to the differential is its multiplicative equivalent, the *quotiential*:
$$^{0'}q_a f(x) :\equiv \frac{f(x + 0')}{f(x)} \equiv e^{d \ln f(x)}.$$

The inverse of the quotiential is the *prodegrant*:
$$^{0'}\!\!\int^a f(x) :\equiv \prod_{k=1}^{\frac{a}{0'}} f(0'k) \equiv e^{\int \ln x}.$$

From the quotiential and differential we derive two *quotient derivatives*, the *geometric derivative* and the *bilogarithmic derivative*:
$$\sqrt[dx]{qf(x)} \equiv qf(x)^{\frac{1}{dx}} \equiv e^{\frac{d \ln f(x)}{dx}} \equiv e^{\frac{df(x)}{f(x)dx}},$$
$$\log_{qx} qf(x) \equiv e^{\frac{df(x)}{d \ln x}} \equiv e^{\frac{xdf(x)}{dx}}.$$

We also derive two **product integrals**: the **geometric integral** or **type 1 product integral**, and the **bilogarithmic integral**:

$$\oint_a^b f(x)^{dx} :\equiv \frac{\displaystyle\oint^b f(x)^{dx}}{\displaystyle\oint^a f(x)^{dx}}$$

$$\equiv e^{\int_a^b \ln f(x)\, dx}$$

$$\oint_a^b qx^{f(x)} :\equiv \frac{\displaystyle\oint^b qx^{f(x)}}{\displaystyle\oint^a qx^{f(x)}}$$

$$\equiv e^{\int_a^b f(x)\, d\ln x}$$

Somewhat ambiguously, the symbol $\prod$ is sometimes used elsewhere instead of $\oint$.

Volterra, who first investigated product integrals [V87], originally defined what is now called the **type 2 product integral**:

$$\prod_a^b [1 + f(x)\, dx] :\equiv e^{\int_a^b f(x)\, dx} \equiv \oint_a^b e^{f(x)\, dx}$$

The inverse of the type 2 product integral is the **logarithmic derivative**:

$$\frac{f'(x)}{f(x)} \equiv \frac{d \ln f(x)}{dx} \equiv \frac{df(x)}{x\, dx}.$$

The **partial quotiential** is given by:

$$^{0'}\!\varphi_x f(x, y) :\equiv \frac{f(x + 0', y)}{f(x, y)} \equiv e^{0'\partial_x \ln f(x, y)}.$$

Higher order derivatives and integrals

The simple form of the equipoint derivative lends itself to direct calculaton of higher order derivatives. These derivatives are are also simple quotients, with $dx^n := (dx)^n$ in the denominator.

HIGHER ORDER DERIVATIVE FORMULA:

$$f^{(n)}(x) = \frac{\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(x + 0'k)}{0'^n}$$

PROOF. Computing higher order derivatives is mainly a matter of computing the numerator $d^n f(x)$, which is an iterated application of the differential operator:

$$
\begin{aligned}
f''(x) &= \frac{d^2}{dx^2} f(x) \\
&= \frac{d[d[f(x)]]}{0'^2} \\
&= \frac{d[f(x + 0') - f(x)]}{0'^2} \\
&= \frac{[f(x + 2 \cdot 0') - f(x + 0')] - [f(x + 0') - f(x)]}{0'^2} \\
&= \frac{f(x + 2 \cdot 0') - 2f(x + 0') + f(x)}{0'^2}.
\end{aligned}
$$

The expansion of these operators is similar to expansion of the binomial power

$$
\begin{aligned}
(a - b)^n &= \sum_{k=0}^{n} (-1)^k \binom{n}{k} a^{n-k} b^k \\
&= \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} a^k b^{n-k}.
\end{aligned}
$$

In derivatives, $f(x + 0'k)$ corresponds to $a^k b^{n-k}$: the n-th derivative is

$$f^{(n)}(x) = \frac{d^n}{dx^n} f(x)$$

$$= \frac{\sum_{k=0}^{n} (-1)^k \binom{n}{k} f\left(x + [n-k]0'\right)}{0'^n}$$

$$= \frac{\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(x + 0'k)}{0'^n},$$

which can be proved by induction:

$$f^{(0)}(x) = f(x) = \binom{0}{0}(-1)^0 f(x - 0'),$$

and

$$\frac{d^{n+1}}{dx^{n+1}} f(x) = \frac{d}{dx} \frac{d^n}{dx^n} f(x) = \frac{1}{0'} \left[\frac{d^n}{dx^n} f(x - 0') - \frac{d^n}{dx^n} f(x) \right]$$

$$= \frac{1}{0'^{n+1}} \left[\sum_{k=0}^{n} (-1)^k \binom{n}{k} f\left(x + [n+1-k]0'\right) \right.$$

$$\left. - \sum_{k=0}^{n} (-1)^k \binom{n}{k} f\left(x + [n-k]0'\right) \right]$$

$$= \frac{1}{0'^{n+1}} \left[\binom{n}{0} f\left(x + [n+1]0'\right) \right.$$

$$+ \sum_{k=1}^{n} (-1)^k \left[\binom{n}{k} + \binom{n}{k-1} \right] f\left(x + [n+1-k]0'\right)$$

$$\left. + \binom{n}{n} f(x) \right]$$

$$= \frac{1}{0'^{n+1}} \left[\binom{n+1}{0} f\left(x + [n+1]0'\right) \right.$$

$$+ \sum_{k=1}^{n} (-1)^k \binom{n+1}{k} f\left(x + [n+1-k]0'\right)$$

$$\left. + \binom{n+1}{n+1} f(x) \right]$$

$$= \frac{1}{0'^{n+1}} \sum_{k=0}^{n+1} (-1)^k \binom{n+1}{k} f\left(x + [n+1-k]0'\right). \ \square$$

A simple example of this theorem:

$$\frac{d^2(x^3)}{dx^2} \equiv \frac{(x + 2 \cdot 0')^3 - 2(x + 0')^3 + x^3}{0'^2}$$

$$\equiv \frac{x^3 + 6 \cdot 0'x^2 + 12 \cdot 0'^2 x + 0'^3 - 2x^3 - 6 \cdot 0'x^2 - 6 \cdot 0'^2 x - 0'^3 + x^3}{0'^2}$$

$$= 6x$$

HIGHER ORDER INTEGRAL FORMULA:

$$\underbrace{\int \cdots \int f(x) \, dx^n}_{n} = f^{(-n)}(x) = \sum_{k=n}^{\infty'} \binom{k-1}{k-n} f(x - 0'k) \, 0'^n.$$

PROOF. Since the binomial theorem extends to negative exponents, we can extend the previous theorem to integrals. In this case, the upper limit on the summation is infinite:

$$(a - b)^{-n} = \sum_{k=0}^{\infty'} (-1)^k \binom{-n}{k} a^{-n-k} b^k$$

$$= \sum_{k=0}^{\infty'} (-1)^{2k} \binom{n+k-1}{k} a^{-n-k} b^k$$

$$= \sum_{k=0}^{\infty'} \binom{n+k-1}{k} a^{-n-k} b^k.$$

For $n = 1$, this becomes

$$(a - b)^{-1} = \sum_{k=0}^{\infty'} a^{-1-k} b^k$$

and

$$f^{(-1)}(x) = \frac{d^{-1}}{dx^{-1}} f(x)$$

$$= \sum_{k=0}^{\infty'} f(x - (k+1)0') \, 0'$$

$$= \sum_{k=1}^{\infty'} f(x - 0'k) \, 0'.$$

Taking $0' = \frac{a-x}{\infty'}$, the above summation matches the definition of the definite integral:

$$f^{(-1)}(x) = \sum_{k=1}^{\infty'} f(x - 0'k) \, 0' = \int_a^x f(t) \, dt.$$

Since ∞' is independent of $0'$, a is arbitrary, and this expression is actually a class of functions of x, each expressed as a definite integral with a fixed lower limit and a variable upper limit. This matches the second definition of the indefinite integral $\int f(x) \, dx$ given in **Differentials and integrants** (p. 55) above.

Higher order integrals are obtained through other negative powers of binomials:

$$\underbrace{\int \cdots \int}_{n} f(x) \, dx^n = f^{(-n)}(x) = \frac{d^{-n}}{dx^{-n}} f(x)$$

$$= \sum_{k=0}^{\infty'} (-1)^k \binom{-n}{k} f(x - (k+n)0') \, 0'^n$$

$$= \sum_{k=0}^{\infty'} \binom{k+n-1}{k} f(x - (k+n)0') \, 0'^n$$

$$= \sum_{k=n}^{\infty'} \binom{k-1}{k-n} f(x - 0'k) \, 0'^n. \;\; \square$$

L'Hôpital's rule

L'HÔPITAL'S RULE FOR 0/0: If functions f and g are continuous at c and $f(c) = g(c) = 0$, then $\dfrac{f(c)}{g(c)} = \dfrac{f'(c)}{g'(c)}$.

The equipoint version of L'Hôpital's rule evaluates the function $\dfrac{f(x)}{g(x)}$ at c in unfolded arithmetic, since $f(c) = g(c) = 0$ in folded arithmetic is insufficient to compute $\dfrac{f(c)}{g(c)}$ as a single value.

PROOF. Since f is continuous at c, we have $f(c + d) = f(c + 0') = f(c) = 0 = f(d) - f(c)$, and similarly for g. Then

$$\frac{f(c)}{g(c)} = \frac{f(c + 0')}{g(c + 0')}$$

$$= \frac{f(c + 0') - f(c)}{g(c + 0') - g(c)}$$

$$\equiv \frac{\frac{f(c+0')-f(c)}{0'}}{\frac{g(c+0')-g(c)}{0'}}$$

$$\equiv \frac{f'(c)}{g'(c)}.$$

If both $f'(c)$ and $g'(c)$ are zero, then we can iterate the rule until we find some n for which either $f^{(n)}(c)$ or $g^{(n)}(c)$ or both are nonzero. If both derivatives are zero for all n, then the rule does not give a single value for the quotient. $\square$

L'Hôpital's rule for ∞/∞: If functions f and g are continuous at c, $f(c) = g(c) = \infty$, and f and g are finite in some punctured neighborhood around c, then $\dfrac{f(c)}{g(c)} = \dfrac{f'(c)}{g'(c)}$.

PROOF. Let $c' :\equiv c + 0'$. Since f and g are continuous and infinite at an isolated point c, the unfolded f and g must take on every unfolded infinite value within the unfolded space around c.

Let $\infty' :\equiv \frac{1}{0'}$ and let $c+0''$ be an arbitrary point within the unfolded space around c. Within this space, the magnitudes of f and g strictly decrease monotonically as the magnitude of $0''$ increases. It is therefore possible to choose $0''$ so that both

$$\log_{\infty'} |f(c' + 0'')| < \log_{\infty'} |g(c')|$$
$$\log_{\infty'} |g(c' + 0'')| < \log_{\infty'} |g(c')|$$

This means that $f(c' + 0'')$ and $g(c')$ are distinguishable from finite multiples of themselves at different sensitivity levels, and similarly for $g(c' + 0'')$ and $g(c')$. Since the magnitudes of $f(c' + 0'')$ and $g(c' + 0'')$ are less than that of

$g(c')$, we have

$$\frac{f(c' + 0'')}{g(c')} =' 0$$

$$\frac{g(c' + 0'')}{g(c')} =' 0$$

We then compute

$$\frac{f(c)}{g(c)} = \frac{f(c')}{g(c')} \equiv \frac{\frac{f(c')}{g(c')}}{1}$$

$$= \frac{\frac{f(c')}{g(c')} - \frac{f(c'+0'')}{g(c')}}{1 - \frac{g(c'+0'')}{g(c')}}$$

$$\equiv \frac{f(c' + 0'') - f(c')}{g(c' + 0'') - g(c')}$$

$$\equiv \frac{f'(c')}{g'(c')} = \frac{f'(c)}{g'(c)}. \quad \square$$

This proof only requires that $g(c)$ be infinite. If $f(c)$ is finite, then the rule still applies but is not needed, since $\frac{f(c)}{g(c)} = 0$ by ordinary extended arithmetic.

Since numeristic division and logarithms are unrestricted, it is easy to extend the rule to other indeterminate forms.

- If $f(c)g(c)$ is of the form $0 \cdot \infty$, then use the rule on $\frac{f(x)}{\frac{1}{g(x)}}$ or

$$\frac{\frac{1}{f(x)}}{g(x)}.$$

- If $f(c) - g(c)$ is of the form $\infty - \infty$, then use the rule on $e^{f(c)-g(c)} = \frac{e^{f(c)}}{e^{g(c)}}$.

- If $f(c)^{g(c)}$ is of the form 0^0, 1^∞, or ∞^0, then use the rule on $\ln f(c)^{g(c)} = g(c) \ln f(c)$.

Power series

In the following, we define σ as an integration operator with a fixed lower bound and a variable upper bound:

$$\sigma_a f(t) := \int_a^t f(t)\, dt$$

and

$$\sigma f(t) := \sigma_0 f(t) = \int_0^t f(t)\, dt$$

Powers of σ denote repeated integration or differentiation:

$$\sigma^n f(t) = \underbrace{\int_0^t \int_0^u \cdots \int_0^u}_{n} f(u)\, du^n$$

$$\sigma^{-n} f(t) = \left. \frac{d^n}{du^n} f(u) \right|_{u=t}$$

$$\sigma^0 f(t) = f(t).$$

We are now ready to derive a compact formula for power series of an analytic function.

POWER SERIES: For an analytic function f,
$$f(t) = e^{(t-a)\sigma^{-1}} f(a).$$

PROOF. We start by integrating and differentiating f repeatedly.

$$\sigma_a \sigma^{-1} f(t) = f(t) - f(a)$$
$$\sigma_a \sigma^{-2} f(t) = \sigma^{-1} f(t) - \sigma^{-1} f(a)$$
$$\sigma_a^2 \sigma^{-2} f(t) = f(t) - f(a) - (t-a)\sigma^{-1} f(a)$$
$$\sigma_a^2 \sigma^{-3} f(t) = \sigma^{-1} f(t) - \sigma^{-1} f(a) - (t-a)\sigma^{-2} f(a)$$
$$\sigma_a^3 \sigma^{-3} f(t) = f(t) - f(a) - (t-a)\sigma^{-1} f(a) - \frac{1}{2}(t-a)^2 \sigma^{-2} f(a)$$
$$\cdots$$

$$\sigma_a^n \sigma^{-n} f(t) = f(t) - f(a) - (t-a)\sigma^{-1}f(a) - \frac{1}{2}(t-a)^2\sigma^{-2}f(a) - \ldots$$
$$- \frac{1}{n!}(t-a)^n\sigma^{-n}f(a).$$

We then take the infinite case of this series and regard it as an operator ψ on f. We do similar operations on this series and find that it leaves the series unchanged.

$$\psi f(t) := \sigma_a^\infty \sigma^{-\infty} f(t)$$
$$= f(t) - f(a) - (t-a)\sigma^{-1}f(a) - \frac{1}{2}(t-a)^2\sigma^{-2}f(a) - \ldots$$
$$\psi\sigma f(t) = \sigma f(t) - \sigma f(a) - (t-a)\sigma^{-2}f(a) - \frac{1}{2}(t-a)^2\sigma^{-3}f(a) - \ldots$$
$$\sigma_a\psi\sigma f(t) = f(t) - f(a) - (t-a)\sigma^{-1}f(a) - \frac{1}{2}(t-a)^2\sigma^{-2}f(a) - \ldots$$
$$= \psi f(t).$$

For all infinitely differentiable f and all a, we now have $\psi f(t) = \sigma_a\psi\sigma f(t)$, or $\psi\sigma f(t) = \sigma_a\psi f(t)$. Since $\sigma_a\sigma f(t) = \sigma\sigma_a f(t)$, by the definition of ψ we have $\psi\sigma f(t) = \sigma\psi f(t) = \sigma_a\psi f(t) = \sigma\psi f(t) - \sigma\psi f(a)$. Subtracting, $\sigma\psi f(a) = 0$ for all a, i.e. $\sigma\psi f$ is the zero function. Hence $\psi f(t)$ must also be the zero function for all f, i.e. ψ is the zero operator. So

$$f(t) = f(a) - (t-a)\sigma^{-1}f(a) - \frac{1}{2}(t-a)^2\sigma^{-2}f(a) - \ldots$$
$$= \sum_{n=0}^{\infty} \frac{(t-a)^n}{n!}\sigma^{-n}f(a)$$
$$= e^{(t-a)\sigma^{-1}}f(a). \ \square$$

SINGULARITIES

Offset derivatives

In **Differentials and integrants** (p. 55), we defined the differential of a function f at a finite point x with respect to a zero $0'$, denoted $^{0'}df(x)$, is the difference $f(x + 0') - f(x)$. The equipoint handling of singularities sometimes requires a variant of this differential.

If f is continuous at x, the differential is zero: By continuity, $f(x + 0') = f(x)$, so $^{0'}df(x) :\equiv f(x + 0') - f(x) = 0$. Since dx is the differential of the identity function $f(x) = x$, it too is always zero.

If $f(x)$ is infinite or discontinuous, $df(x)$ may be nonzero. Previous chapters have assumed that differentials of dependent variables are zero, but most results continue to hold if they are nonzero. Exceptions include the two **fundamental theorems of calculus** (p. 36), which do not hold at **poles** (p. 85), as described below.
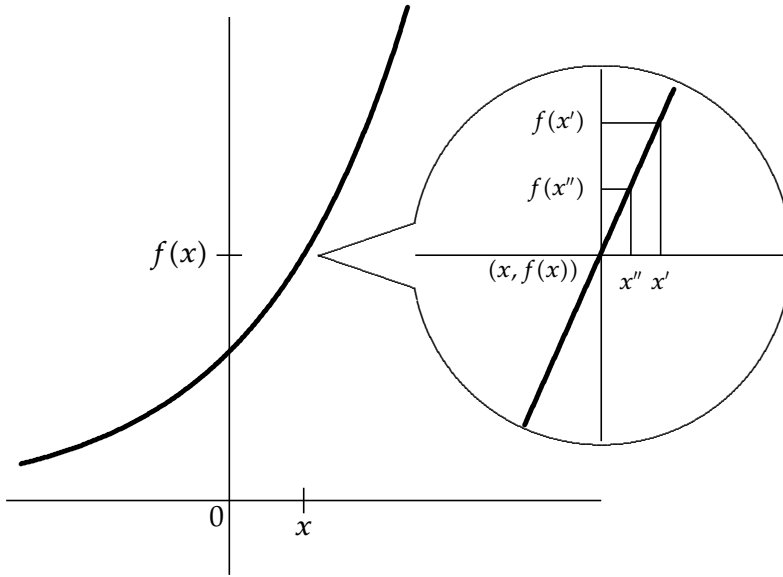
FIG. 12:
Calculation of derivative
with offset differentials

Occasionally the differential $^{0'}df(x) :\equiv f(x + 0') - f(x)$, or derivatives that use it, do not yield a determinate result. In such cases, we may use the fact that the slope of an analytic curve at a finite point x can be computed with any two points within the microscope. This is shown in Figure 12, where we use the two points

$$(x', f(x')) :\equiv (x + 0', f(x + 0'))$$
$$(x'', f(x'')) :\equiv (x + 0'', f(x + 0''))$$

The differentials along the two axes in the microscope are called **offset differentials**, and the derivative using them is called an **offset derivative**, with the following notations:

$$^{0'}_{0''}df(x) :\equiv f(x + 0') - f(x + 0'')$$
$$^{0'}_{0''}f'(x) :\equiv \frac{^{0'}_{0''}df(x)}{^{0'}_{0''}dx} \equiv \frac{f(x + 0') - f(x + 0'')}{0' - 0''}$$

The quantity $0'$ is called the **upper displacement** and $0''$ the **lower displacement**. The first type of differential, with only an upper displacement, is called a **original differential**, since the lower displacement is the origin of the microscope. As shown in Figure 12, for a finite analytic function, the curve

becomes a straight line in the microscope, so a original derivative and an offset derivative yield the same result.

Letting $0''' :\equiv 0' - 0''$, we have

$$\frac{\substack{0'\\0''}df(x)}{\substack{0'\\0''}dx} \equiv \frac{f(x+0') - f(x+0'')}{0'-0''} \equiv \frac{f(x+0''+0''') - f(x+0'')}{0'''} \equiv {}^{0'''}\!f'(x+0'').$$

This form of an offset derivative shows that it can be considered as the derivative of an **offset** (p. 50). As with original derivatives, if an offset derivative in this form is independent of its upper displacement, we omit it and write $f'(x+0'')$.

The above definitions apply only to finite x. For infinte x, we use the fact that for $x = 0$, $\substack{0'\\0''}df(x) :\equiv f(0') - f(0'')$. For infinite x, then, we define

$$\infty''' :\equiv \frac{1}{0'''} :\equiv \frac{1}{0'} - \frac{1}{0''} \equiv: \infty' - \infty''$$

$$\substack{0'\\0''}df(x) :\equiv f\left(\frac{1}{0'}\right) - f\left(\frac{1}{0''}\right) \equiv f(\infty') - f(\infty'')$$

$$\equiv f\left(\frac{1}{0''} + \frac{1}{0'''}\right) - f\left(\frac{1}{0''}\right) \equiv f(\infty'' + \infty''') - f(\infty'')$$

$$\substack{0'\\0''}f'(x) :\equiv \frac{\substack{0'\\0''}df(x)}{\substack{0'\\0''}dx} \equiv \frac{f\left(\frac{1}{0'}\right) - f\left(\frac{1}{0''}\right)}{\frac{1}{0'} - \frac{1}{0''}} \equiv \frac{f(\infty') - f(\infty'')}{\infty' - \infty''}$$

$$\equiv \frac{f\left(\frac{1}{0''} + \frac{1}{0'''}\right) - f\left(\frac{1}{0''}\right)}{\frac{1}{0'''}} \equiv \frac{f(\infty'' + \infty''') - f(\infty'')}{\infty'''} \equiv {}^{0'''}\!f'\left(\frac{1}{0''}\right) \equiv {}^{0'''}\!f'(\infty'')$$

$$\equiv f'\left(\frac{1}{0''}\right) \equiv f'(\infty'') \text{ if independent of } 0'''$$

Offset derivatives are not always inverse with integrals and should only be used when original derivatives do not yield a determinate result. This is clarified further in following sections, especially **Poles** (p. 85).

Definition of singularity

A class x is *integrous* if there is a bijection between the elements of x and some subset of the integers. Examples are 5, ±1 and $2\pi\mathbb{N}$. This concept is further discussed in **Class count comparisons** (p. 125).

A class x is *determinate* if it is nonempty and integrous.

A class is *semideterminate* if it is not empty, not determinate, and not full. An example is the interval $[-1, +1]$.

A class is *indeterminate* if it is full.

A function f is *regular* or *analytic* on a region A if:

- $f(x)$ and its original derivatives $f^{(n)}(x)$ are determinate and continuous for every $x \in A$;

- $f(x)$ is equal to some value of the power series $e^{(x-a)\sigma^{-1}} f(a)$ for every $x, a \in A$.

The numeristic theory of infinite series shows how most infinite series, even convergent ones, are multivalued. See [CD].

An *ordinary point* of a function f is any point x in a region where where f is regular. A *singularity* of f is any other point, i.e. where any of the above conditions fails.

A function f is *semiregular* on a region A if:

- $f(x)$ and $f^{(n)}(x)$ are determinate and continuous for every nonsingular x in A;

- The offset $f(x + 0')$ and offset derivatives $f^{(n)}(x + 0')$ are semideterminate and semiuniform for every singular x in A;

- $f(x)$ is equal to some value of the power series $e^{(x-a)\sigma^{-1}} f(a)$ for every $x, a \in A$, where the power series is calculated with original values and derivatives for nonsingular a and offset values and derivatives for singular a.

A *semiordinary point* of a function f is any point x in a region where f is semiregular. An *irregularity* of f is any other point.

Types of singularity

A singularity is *isolated* if there is a punctured perfinite-size neighborhood that contains no singularies. This means that the unfolding of the singularity contains only one singularity. In this chapter we discuss the following four types of isolated singularity:

- *Removable discontinuity*: f has a removable discontinuity at p if the offset values $f(p + 0')$ is uniform, but the function is discontinuous, i.e. $f(p+0') \neq f(p)$. Examples discussed below are the **punctured constant function** (p. 75), the **Kronecker delta function** (p. 78), and the **Dirac delta function** (p. 79).

- *Jump discontinuity*: f has a jump discontinuity at p if the offset value $f(p + 0')$ is semiuniform but not uniform, and the function is discontinuous. Examples discussed below are the **absolute value function** (p. 73) and its derivative, the step function.

- *Pole*: f has a pole at p if $f(x) = \dfrac{g(x)}{h(x)}$, g and h are regular, h has a root (zero) at p, and the multiplicity of the root p of h is finite. An example is the reciprocal function, discussed below in **Poles** (p. 85).

- *Essential singularity*: f has an essential singularity at p if it has a singularity that is not any of the above three types. An example is the function $\sin \frac{1}{x}$, discussed below in **Function $\sin \frac{1}{x}$** (p. 89).

There are many types of nonisolated singularities. Some examples are given in **Other singularities** (p. 94), but they are not analyzed in detail.

This chapter also gives an example of a function which is singular everywhere in conventional analysis but is regular in equipoint analysis. See **Weierstrass function** (p. 91).
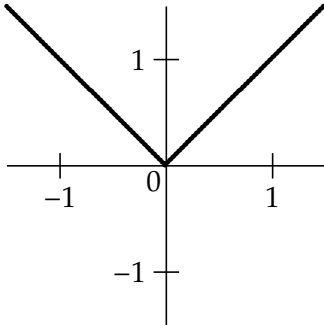
Absolute value function
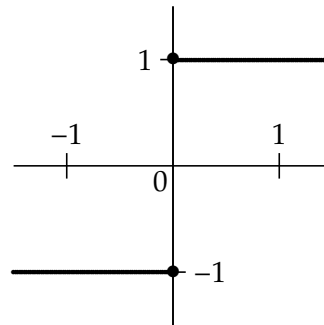

FIG. 13: Absolute
value function $a(x) := |x|$

FIG. 14: Derivative of
absolute value function $a(x) := |x|$,
$a'(x) = \text{sgn}_3 x$

The absolute value function $a(x) := |x|$ is shown in Figure 13. Its derivative is the step function shown in Figure 14. The derivative has a **jump discontinuity** (p. 72) at 0.

In the region $x > 0$, we have $a(x) = x$, $a'(x) = 1$, and the power series about any p in this region is $e^{(x-p)\sigma^{-1}} a(p) = p + (x - p) = x$. The function is therefore regular in this region. Similarly, it is regular in the region $x < 0$.

For any region that includes $x = 0$, the derivative is not uniform, since it has two values at 0:

$$a'_{0'}(0) \equiv \frac{a(0') - a(0)}{0'} \equiv \frac{0'}{0'} = 1$$

$$a'_{-0'}(0) \equiv \frac{a(-0') - a(0)}{-0'} \equiv \frac{0'}{-0'} = -1$$

$$a'(0) = \pm 1.$$

$a(x)$ is therefore not regular for any region which includes $x = 0$. However, $a'(x)$ is semiuniform, and $a(x)$ is therefore semiregular everywhere.

The derivative of a similar step function is discussed below in **Dirac delta function** (p. 79). As discussed in that section, a step function can be made

analytic at the unfolded level. In the same way, the absolute value function, as the integral of a step function, can also be made unfolded analytic.

As a complex function, the derivative of a is the unit circle: for any $0' \in' 0$,

$$a'_{0'}(0) \equiv \frac{a(0') - a(0)}{0'} \equiv \frac{|0'|}{0'} = \mathrm{sgn}\, 0'$$

$$a_{\mathbb{C}0'} = e^{i\mathbb{R}}.$$

Multivalued complex derivatives are discussed further in **Complex derivative** (p. 96).
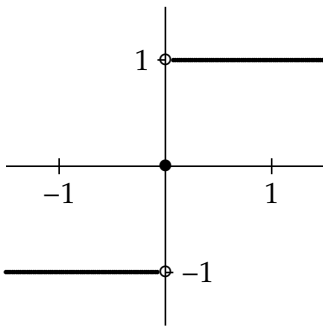
FIG. 15:
Conventional signum
function $f(x) = \mathrm{sgn}_1 x$
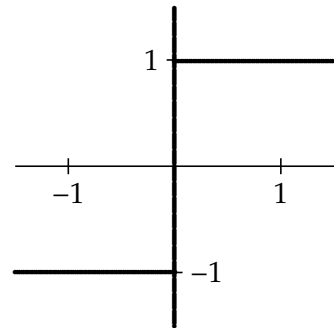
FIG. 16:
Alternate signum
function $f(x) = \mathrm{sgn}_2 x$

The derivative a' is an alternate form of the signum function. The standard form, shown in Figure 15, is

$$\mathrm{sgn}_1 x := \begin{cases} -1 & \text{for } x < 0 \\ 0 & \text{for } x = 0 \\ +1 & \text{for } x > 0. \end{cases}$$

In [CN] we developed an alternate form, shown in Figure 16:

$$\mathrm{sgn}_2 x := \frac{|x|}{x}$$

From the above derivative we can define

$$\mathrm{sgn}_3 x := f'(x) = \frac{|x \pm 0'|}{x \pm 0'}$$

$$\mathrm{sgn}_3 0 = \pm 1.$$

This third form allows us to calculate signum for infinite numbers:

$$\text{Projectively extended real numbers } (\hat{\mathbb{R}}): \quad \text{sgn}_3 \infty = \pm 1$$

$$\text{Affinely extended real numbers } (\overline{\mathbb{R}}): \quad \text{sgn}_3(+\infty) = +1$$

$$\text{sgn}_3(-\infty) = -1$$

$$\text{Single projectively extended complex numbers } (\widetilde{\mathbb{C}}): \quad \text{sgn}_3 \infty = e^{i\mathbb{R}}$$

$$\text{Double projectively extended complex numbers } (\hat{\mathbb{C}}): \quad \text{sgn}_3\left(\infty e^{ir}\right) = \pm e^{ir}$$

$$\text{Affinely extended complex numbers } (\overline{\mathbb{C}}): \quad \text{sgn}_3\left(\infty e^{ir}\right) = e^{ir}$$

Punctured constant function
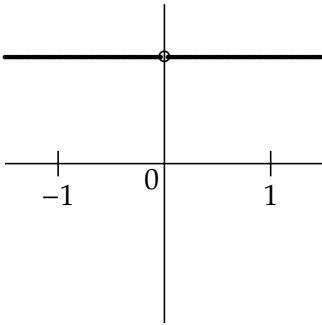
FIG. 17: Punctured
constant function $p(x)$
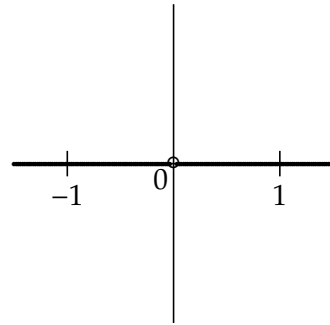
FIG. 18: Derivative of
punctured constant function $p'(x)$

A function with a missing point, a point where the function has no value, is shown in Figure 17. This is a punctured constant function:

$$p(x) := \begin{cases} 1 & \text{for } x \neq 0 \\ \emptyset & \text{for } x = 0. \end{cases}$$

The function p has a **removable discontinuity** (p. 72) at 0, since $p(0') = p(0'') = 1$ for all unfolded elements $0'$ and $0''$, but $1 = p(0') \neq p(0) = \emptyset$.

The derivative p', shown in Figure 18, also has a missing point:

$$p'(0) \equiv \frac{p(0') - p(0)}{0'} \equiv \frac{1 - \emptyset}{0'} = \emptyset.$$

The offset derivative, as defined in **Offset derivatives** (p. 68), yields a value everywhere:

$$_{0''}p'(0) \equiv \frac{p(0') - p(0'')}{0'} \equiv \frac{1 - 1}{0' - 0''} = 0.$$

$p(x)$ is irregular for any any region that includes the singularity at $x = 0$. Since $p(0)$ is empty, it is not determinate, and $p(x)$ cannot be regular or semiregular.

Singularities at infinity
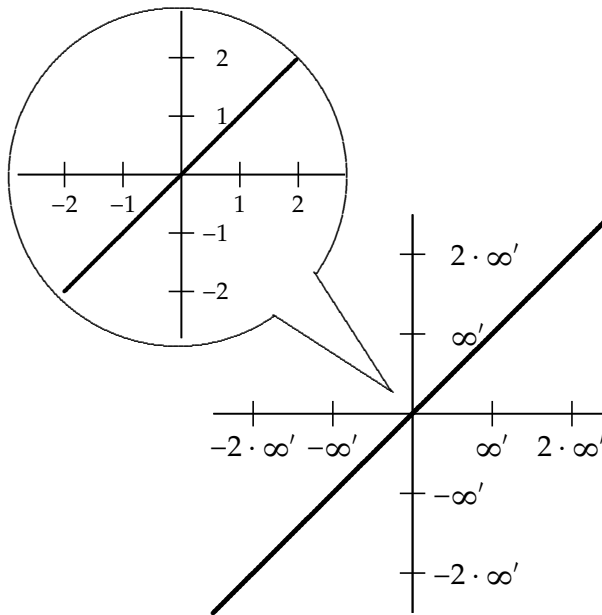
FIG. 19: Identity function $I(x) := x$
with microscope view of finite plane
within origin of unfolded infinite plane

Even a very simple function such as $I(x) := x$ has a singularity at infinite values. This function is shown in Figure 19, which shows the finite plane in a microscope and the unfolded infinite plane in the macroscope. For clarity, we use the affinely extended real numbers, and set $\infty' := \frac{1}{0'}$. The origin of the

unfolded infinite line is $\dfrac{1}{0 \cdot 0'} = \infty \cdot (\pm\infty')$. This is a pair of points infinitely removed from the origin of the macroscope.

The original derivative at $x = +\infty$ therefore uses $\infty \cdot \infty'$ as a lower displacement, but this yields an indeterminate result:

$$\infty' I'(x) \equiv \frac{(\infty + \infty') - \infty}{\infty'} \equiv \frac{(\infty - \infty) + \infty'}{\infty'} = \varnothing.$$

An offset derivative yields

$$\infty'_{\infty''} I'(x) \equiv \frac{\infty'' - \infty'}{\infty'' - \infty'} = 1.$$

Since the original derivative is indeterminate but the offset derivative is determinate, $I(x)$ is only semiregular in any region that includes an infinite value.

For another example, we take the exponential function $\exp(x) := e^x$. At $x = -\infty$, an original derivative is sufficient:

$$\infty' \exp'(x) \equiv \frac{e^{-\infty+\infty'} - e^{-\infty}}{\infty'} =' \frac{e^{-\infty} - e^{-\infty}}{\infty'} =' \frac{0 - 0}{\infty'} = 0.$$

But at $x = +\infty$, an offset derivative is required:

$$\infty' \exp'(x) \equiv \frac{e^{\infty+\infty'} - e^{\infty}}{\infty'} =' \frac{e^{\infty} - e^{\infty}}{\infty'} =' \frac{\infty - \infty}{\infty'} = \varnothing,$$

$$\infty'_{\infty''} \exp'(x) \equiv \frac{e^{\infty''} - e^{\infty'}}{\infty'' - \infty'} =' \frac{\infty'''}{\infty'' - \infty'} = \infty.$$

Therefore, in the affinely extended real numbers, $\exp(x)$ has a singularity at $+\infty$ but not at $-\infty$, and is regular in any region that includes $-\infty$ but only semiregular in a region that includes $+\infty$.

Every singularity at an infinite value is nonisolated, since $\infty + r = \infty$ for all perfinite r, and any punctured perfinite size neighborhood of the infinite value is still within the same infinite value.
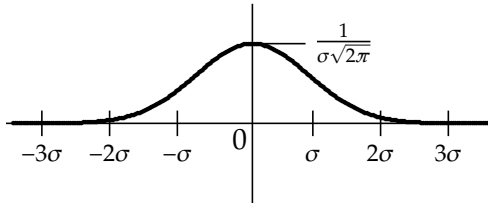
Kronecker delta function

FIG. 20: Normal distribution
function $\phi(x)$ with
standard deviation σ
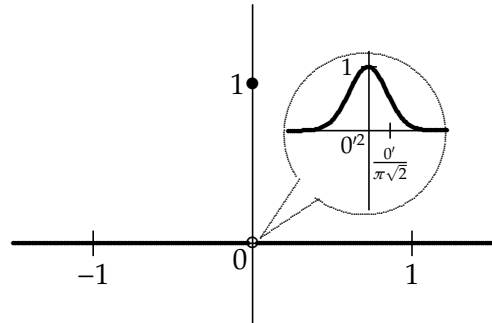
FIG. 21: Kronecker delta
function $\delta_{0,x}$ as proper unfolded
normal distribution

The *Kronecker delta function* has a very simple definition:

$$\delta_{a,b} := \begin{cases} 1 & \text{for } a = b \\ 0 & \text{for } a \neq b. \end{cases}$$

The function $\delta_{x,0}$ has a **removable discontinuity** (p. 72) at 0, since $f(0') = 0$ for all unfolded elements $0'$, but $f(0) = 1$. Put another way, $\lim_{x \to 0} \delta_{x,0}$ exists and is 0. In equipoint terms, $\lim_{x \to 0}$ means $f(x + 0')$, and to say it exists means that $f(x + 0')$ is single valued and independent of $0'$. See **Limits** (p. 50) above.

The Kronecker delta function is not regular function in any region that includes the singularity at $x = 0$, since the original and offset derivatives there do not agree: the original derivative is infinite while the offset derivatives are zero. The function is not semiregular in these regions, since the power series using offset derivatives at the singularity do not equal the function. Hence the function is irregular in these regions.

The Kronecker delta function, and any function with this type of discontinuity, can be made regular at the unfolded level, by constructing a proper unfolded regular function which folds into this function. Figure 21 shows one

way of doing this, by constructing a normal distribution with an infinitesimal standard deviation.

Figure 20 shows the standard normal distribution $\phi(x)$ with standard deviation σ:

$$\phi_\sigma(x) := \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \sum_{n=0}^{\infty} \frac{x^{2n}}{n!2^n\sigma^{2n}}$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \left(1 - \frac{x^2}{2\sigma^2} + \frac{x^4}{8\sigma^4} - \frac{x^6}{48\sigma^6} + \dots \right).$$

We can then define the Kronecker delta in terms of $\phi(x)$, as graphed in Figure 21:

$$\delta_{x,0} :\equiv 0'\sqrt{\pi}\ \phi_{\frac{0'}{\pi\sqrt{2}}}(x)$$

$$\equiv e^{-\frac{x^2}{0'^2}}$$

$$\equiv \sum_{n=0}^{\infty} \frac{x^{2n}}{n!0'^{2n}}$$

$$\equiv 1 - \frac{x^2}{0'^2} + \frac{x^4}{2 \cdot 0'^4} - \frac{x^6}{6 \cdot 0'^6} + \dots.$$

Dirac delta function

Definitions of the Dirac delta function

The *Dirac delta function* or *unit impulse function* $\delta(\cdot)$ has many definitions. Two qualities of δ which should follow from any definition are:

$$\delta(\mathbb{R}^*) = 0 \qquad (\mathbb{R}^* := \mathbb{R} \setminus 0)$$

$$\int_{-\infty}^{+\infty} \delta(x)\,dx = 1.$$

In equipoint terms, we should refine these conditions as follows:

$$\delta(\mathbb{R}^*) =' 0'^2$$

$$\int_{-\sqrt{\frac{1}{\infty'}}}^{+\sqrt{\frac{1}{\infty'}}} \delta(x)\,dx = 1.$$

These two conditions imply an infinite value for $\delta(0)$. In conventional analysis, this does not allow δ to be a function. δ is instead defined as a distribution or generalized function. Here we consider the Dirac delta to be a function with an infinite value at 0, a **removable discontinuity** (p. 72).

Here we give three definitions of the Dirac delta.

1. $\delta_{\infty'}(x)$ is the class of **proper unfolded functions** (p. 26) such that

$$\max \delta_{\infty'}(x) =' \infty'$$

$$\delta_{\infty'}(\mathbb{R}^*) =' \frac{1}{\infty'^2}$$

$$\int_{-\infty'}^{+\infty'} \delta_{\infty'}(x)\, dx = \int_{-\sqrt{\frac{1}{\infty'}}}^{+\sqrt{\frac{1}{\infty'}}} \delta_{\infty'}(x)\, dx = 1$$

2. $\delta_{\infty'}(x) := \int_{-\infty'}^{+\infty'} e^{2\pi i x y}\, d_{\frac{1}{2\infty'}} y$. This integral yields the class of unfolded functions in definition 1.

3. $\delta_{\frac{1}{0'}}(x)$ is the derivative of the *Heaviside step function*, also called the *unit step function*. This also has several definitions, but for the moment, we will use the left-continuous form:

$$H(x) := \begin{cases} 0 & \text{for } x \le 0 \\ 1 & \text{for } x > 0. \end{cases}$$

The derivative is

$$\delta_{\frac{1}{0'}}(x) :\equiv H'_{0'}(x) \equiv \frac{{}^{0'}dH(x)}{{}^{0'}dx}.$$

From any of the definitions, it easily follows that, for any finite function $f(x)$,

$$\int_{-\infty}^{+\infty} f(x)\, \delta(x)\, dx = f(0)$$

In particular,

$$\int_{-\infty}^{+\infty} 0\, \delta(x)\, dx = 0.$$

Definition 3

We now examine definition 3 in more detail. In the derivative expression, $^0dH(x) \equiv 1$ for any $0' >' 0'^2$, i.e. for any $0'$ on the right side of unfolded 0. When it is divided by $^0dx \equiv 0'$, the result is infinite.

<div align="center">
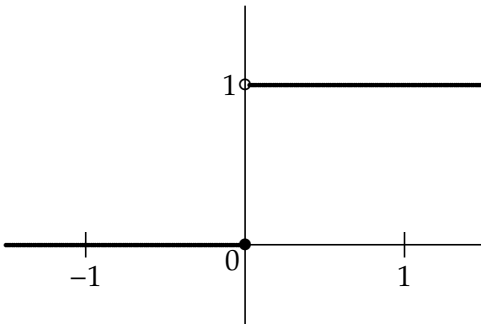
FIG. 22:
Heaviside step function $H(x)$
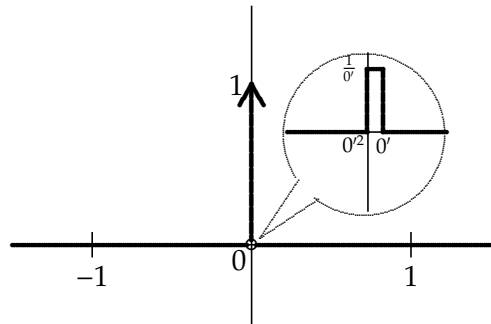
FIG. 23:
Dirac delta function $\delta(x)$

</div>

Figure 23 shows the infinite value at $\delta(0)$. The microscope in this figure expands infinitely in the x direction and *contracts* infinitely in the y direction. The rectangle in the microscope is infinitely tall and infinitely narrow, and its total area is 1.

At the unfolded level, $\delta(x)$ is a class of single valued functions, but at the folded level, it becomes multivalued and loses other properties:

$$\delta(\mathbb{R}^*) = 0$$
$$\delta(0) = \widehat{\mathbb{R}}^+ \text{ or } \overline{\mathbb{R}}^+$$

In the unfolded form, the properties of $\delta_0(x)$ are independent of $0'$. We can regard δ as a class of proper unfolded functions, and we can drop the subscripts and write

$$\delta(x) :\equiv H'(x) \equiv \frac{dH(x)}{dx}.$$

Figures 22 and 23 show the left-continuous form of $H(x)$ and the corresponding $\delta(x)$. There are several alternatives, a few of which are these:

1. In the right-continuous form of $H(x)$, $H(0) = 1$, and the rect-
   angle in the microscope of $\delta(0)$ is to the left of $0'^2$ instead of
   the right. The difference is only in the unfolded arithmetic;
   the folded properties of $\delta(x)$ remain the same.

2. If we define $H(0) = \frac{1}{2}$, then $H(x) = \dfrac{1 + \operatorname{sgn} x}{2}$, and the micro-
   scope rectangle of $\delta(0)$ is half on the left and half on the right
   of $0'^2$. Again, this makes no difference to the folded properties
   of $\delta(x)$.

3. If we allow $H(x)$ to be multivalued and set $H(0) = [0, 1]$, the
   unit interval, then the graph of $H(x)$ is a continuous path and
   can be parameterized with a single valued function. Since
   $H(0)$ is a multivalued class, then $\delta(0)$ is multivalued also,
   the class $\{[0, 1]\delta_1(0)\}$, where $\delta_1(x)$ is the single valued $\delta(x)$
   defined above. One of the members of this class, $0\delta_1(0)$, is
   itself multivalued, since $0\delta_1(0) \equiv 0'\mathbb{R}\frac{1}{0'} \equiv \mathbb{R}$. The other val-
   ues, $\{(0, 1]\delta_1(0)\}$, yield all the infinite multiples of $\delta_1(0)$ up to
   $\delta_1(0)$ itself. Therefore the graph of $\delta(x)$ is also a continuous
   path and can be parameterized with a single valued function.

4. Define $\delta(x)$ as a proper unfolded normal distribution, and
   $H(x)$ as its integral, as discussed below.

Under the first definition, the derivative of $\delta(x)$ can be computed in
superunfolded arithmetic. We must compute the derivative at the two sides of
the rectangle, first at the infinitely increasing step function at $0'^2$, and secondly
at the infinitely decreasing step function at $\dfrac{1}{0'}$. The result, $\dfrac{d\delta(x)}{dx}$, is a second
order proper unfolded function, and $\dfrac{d^{(n)}\delta(x)}{dx^n} = \dfrac{d^{(n+1)}H(x)}{dx^{n+1}}$, is $(n + 1)$-th order
proper unfolded.
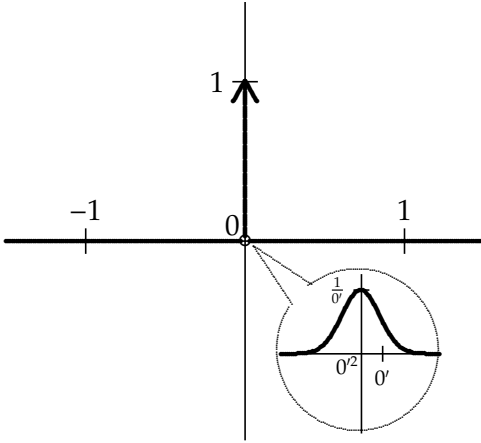
FIG. 24: Dirac delta
function $\delta(x)$ as proper unfolded
normal distribution
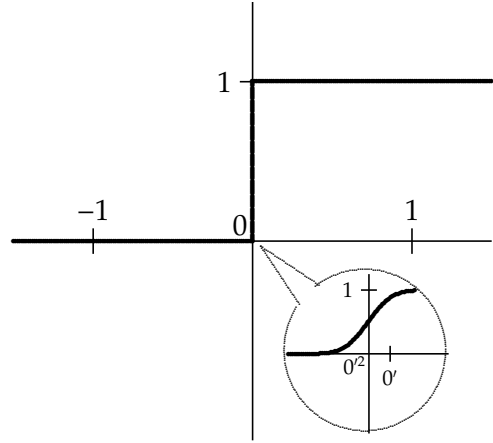
FIG. 25: Heaviside step
function $H(x)$ as proper unfolded
cumulative normal distribution

Unfolded regularity of the Dirac delta function

Like the **Kronecker delta function** (p. 78), the Dirac delta function is
not a regular in any region that includes the singularity at $x = 0$, since the
original and offset derivatives there do not agree, and it is not semiregular,
since power series using offset derivatives at the singularity do not equal the
function. Hence the function is irregular in these regions.

We made the Kronecker delta function regular at the unfolded level
by constructing it as a normal distribution with an infinitely small standard
deviation. A similar technique can be used with the Dirac delta function, as
shown in Figure 24. In this case, we want the integral under the function to
remain unity, so again we use $\sigma = \dfrac{0'}{\pi\sqrt{2}}$, but without any additional scaling:

$$\delta(x) := \phi_{\frac{0'}{\pi\sqrt{2}}}(x)$$

$$= \frac{1}{0'\sqrt{\pi}} e^{-\frac{x^2}{0'^2}}$$

$$= \frac{1}{0'\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{x^{2n}}{n!0'^{2n}}$$

$$= \frac{1}{0'\sqrt{\pi}} \left(1 - \frac{x^2}{0'^2} + \frac{x^4}{2 \cdot 0'^4} - \frac{x^6}{6 \cdot 0'^6} + \dots \right).$$

To do the same for the Heavisidestep function, we naturally choose the integral of the normal distribution, the cumulative normal distribution:

$$\Phi_\sigma(x) := \int_{-\infty}^{x} \phi_\sigma(u)\, du$$

$$= \int_{-\infty}^{x} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{u^2}{2\sigma^2}}\, du$$

$$= \frac{1}{2} + \frac{1}{\sigma\sqrt{2\pi}} \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)n!2^n\sigma^{2n}}$$

$$= \frac{1}{2} + \frac{1}{\sigma\sqrt{2\pi}} \left(x - \frac{x^3}{6\sigma^2} + \frac{x^5}{40\sigma^4} - \frac{x^7}{336\sigma^6} + \dots \right).$$

We can then redefine the Heaviside step function in terms of $\Phi(x)$, as graphed in Figure 25:

$$H(x) := 0'\sqrt{\pi}\, \Phi_{\frac{0'}{\pi\sqrt{2}}}(x)$$

$$\equiv \frac{1}{0'\sqrt{\pi}} \int_{-\infty}^{x} e^{-\frac{u^2}{0'^2}}\, du$$

$$\equiv \frac{1}{2} + \frac{1}{0'\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)n!0'^{2n}}$$

$$\equiv \frac{1}{2} + \frac{1}{0'\sqrt{\pi}} \left(x - \frac{x^3}{3 \cdot 0'^2} + \frac{x^5}{10 \cdot 0'^4} - \frac{x^7}{42 \cdot 0'^6} + \dots \right).$$

Poles
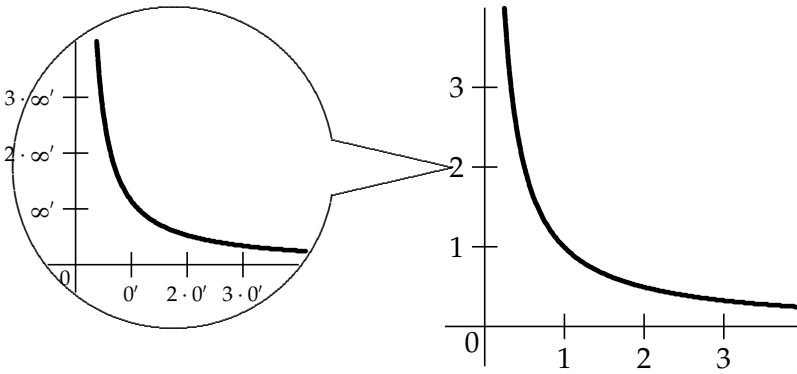
FIG. 26: Reciprocal function $r(x) := \frac{1}{x}$
with microscope view of y axis infinitely expanded
in x direction and infinitely compressed in y direction

In **Types of singularity** (p. 72), we defined a pole of a function f as a point p such that $f(x) = \dfrac{g(x)}{h(x)}$, g and h are analytic, $h(x)$ has a root (zero) at p, and the multiplicity of the root is finite.

Here we discuss the simplest pole, the function $r(x) := \frac{1}{x}$ at the point $x = 0$. Figure 26 shows a graph of $r(x)$ and a microscope of the y-axis, which is infinitely expanded in the x direction and infinitely compressed in the y direction. In curves like the one in Figure 7, the curve becomes straight in the microscope, but in Figure 26, the curve keeps its asymptote along the vertical axis. This remains the case no matter how many times the curve is superunfolded.

This leads to an indeterminacy in the derivative:

$$r'(0) :\equiv \frac{r(0 + 0') - r(0)}{0'} \equiv \frac{\frac{1}{0+0'} - \frac{1}{0}}{0'}$$

$$=' \frac{\frac{1}{0} - \frac{1}{0}}{0'} \equiv \frac{\infty - \infty}{0'} = \cancel{\infty}.$$

The **offset derivative** (p. 68) however is determinate:

$$r'(x + 0'') :\equiv \frac{r(0'' + 0') - r(0'')}{0'} \equiv \frac{\frac{1}{0'' + 0'} - \frac{1}{0''}}{0'}$$

$$\equiv \frac{0'' - 0'' - 0'}{0'0''(0'' + 0'')} \equiv \frac{-1}{0''(0'' + 0')} =' \frac{-1}{0''^2} = \begin{cases} \infty & \text{in } \widehat{\mathbb{R}} \\ -\infty & \text{in } \mathbb{R} \end{cases}.$$

$r(x)$ is therefore not regular in any region that includes the pole, but it is semiregular, since the offset derivatives are determinate and semiuniform, and the power series using them yields the function. For perfinite a and a zero $0'$ we have:

$$f(x) = e^{(x-a)\sigma^{-1}} f(a) = \frac{1}{a} - \frac{x - a}{a^2} + \frac{(x - a)^2}{a^3} - \ldots = \sum_{k=0}^{\infty} \frac{(-1)^k (x - a)^k}{a^{k+1}}$$

$$f(x) = e^{(x-0')\sigma^{-1}} f(0') = \frac{1}{a} - \frac{x - 0'}{0'^2} + \frac{(x - 0')^2}{0'^3} - \ldots = \sum_{k=0}^{\infty} \frac{(-1)^k (x - 0')^k}{0'^{k+1}}$$

An indeterminacy problem also occurs in the integral. A curve like the one in Figure 8 becomes flat in the microscope, but $r(x)$ keeps its asymptote in all unfoldings. If we integrate $r(x)$ from $0'$ to a positive point p, we obtain the infinite result $\ln p - \ln 0' = +\infty$ without a problem, but if we try to integrate through the pole, we run into an indeterminacy. An attempt to integrate from $-p$ to $+p$, for example, would lead to the following:

$$\int_{-p}^{+p} r(x)dx \equiv \int_{-p}^{-0'} r(x)dx + \int_{-0'}^{0\cdot0'} r(x)dx + \int_{0\cdot0'}^{+0'} r(x)dx + \int_{+0}^{+p} r(x)dx$$

The indeterminacy occurs with either of the middle two pieces, $\int_{-0'}^{0\cdot0'} r(x)dx$ and $\int_{0\cdot0'}^{+0'} r(x)dx$. The second of these two we can see in the microscope of Figure 26 as the area under the curve from the origin $0 \cdot 0'$ to $0'$. If the curve were flat, we could use a rectangle with the right side as the height, $0' \cdot \infty' \equiv 1$, but this value is clearly too small in this case. Using the left side as the height gives $0' \cdot \infty \cdot \infty' \equiv 0' \cdot \infty \equiv \phi$ in the projectively extended real numbers and $0' \cdot \infty \cdot \infty' \equiv 0' \cdot \infty \equiv |\phi|$ in the affinely extended real numbers. If we use the trapezoidal estimate, we still obtain an indeterminacy: $0' \left(\frac{\infty \cdot \infty' + \infty'}{2} \right) \equiv 0' \left(\frac{(\infty + 1)\infty'}{2} \right) \equiv 0' \cdot \infty = \phi$ or $|\phi|$. Further unfoldings yield the same indeterminate result, since $0'^n \frac{1}{0 \cdot 0'^n} = \frac{1}{0} = \infty$ for any n.

Any approximation to this area that involves the left endpoint gives in an indeterminacy, and any approximation that does not is inaccurate. Therefore we cannot integrate directly through this pole. The same problem occurs with any other pole.

This leaves us with two alternatives:

1. In real space, integrate piecewise, once to the left of the pole, and once to the right.

2. In complex space, integrate around the pole.

The antiderivative of $\dfrac{1}{x}$ is $\ln x$, but since this is imaginary for negative x, it cannot be used as an integral in real analysis. Instead we use the fact that $\ln x = \ln(|x| \operatorname{sgn} x) = \ln |x| + \ln \operatorname{sgn} x$ and integrate either completely on the positive side of the real axis or completely on the negative side. In this case, the $\ln \operatorname{sgn} x$ terms cancel, and the effective antiderivative is $\ln |x|$.

In complex analysis, the antiderivative is $\ln x$, and the path of integration is connected. This is discussed in detail in **Complex poles** (p. 105).
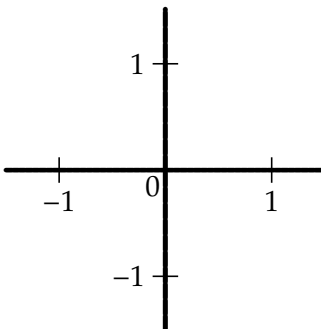
Axial function

FIG. 27:
Axial function $A(x)$
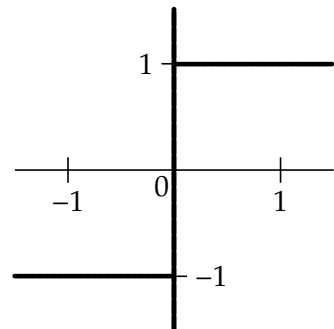
FIG. 28:
Sample of integral of
axial function $A^{(-1)}(x)$

Figure 27 shows the ***axial function*** $A(x) := \dfrac{0}{x}$, a multivalued function whose graph coincides with both the horizontal and vertical axes:

$$A(x) = \begin{cases} 0 & \text{for } x \neq 0 \\ \cancel{\varphi} & \text{for } x = 0 \end{cases}$$

As a multivalued function, the continuity of $A(x)$ is of three types, as defined in **Continuity** (p. 52):

- ***Classwise continuity***: The axial function is classwise discontinuous at 0 because $A(0') = \{0\}$ and $A(0) = \cancel{\varphi}$.

- ***Conjunctive continuity***: The axial function is conjunctively discontinuous at 0 because $\{0\}$ cannot be mapped bijectively to $\cancel{\varphi}$.

- ***Disjunctive continuity***: The axial function is disjunctively continuous at 0 because $A(0') = 0 \in \cancel{\varphi} = A(0)$.

The singularity of $A(x)$ at 0 is a **removable singularity** (p. 72). $A(x)$ is not regular in any region that includes the singularity at $x = 0$, since $A(0)$ is indeterminate. The function is not semiregular in these regions, since the power series using offset derivatives at the singularity equal zero. Hence the function is irregular in these regions.

The derivative of $A(x)$ is $A(x)$, since $A'(x) = \dfrac{dA(x)}{dx} = \dfrac{d}{dx}\dfrac{0}{x} = \dfrac{0}{-x^2} = \dfrac{0}{x^2} = \dfrac{0}{x}$. Since the original derivative is indeterminate at 0, as it is at a pole, the Fundamental Theorems of Calculus do not hold here. See **Poles** (p. 85) for a detailed discussion of this point.

To compute the integral of $A(x)$:

- In real space, integrate piecewise on the left and right sides. This allows us to choose independent constants of integration for the right and left integrals. One possible integral of $A(x)$ is shown in Figure 28.

- In complex space, integrate around the singularity. See **Complex axial function** (p. 107).
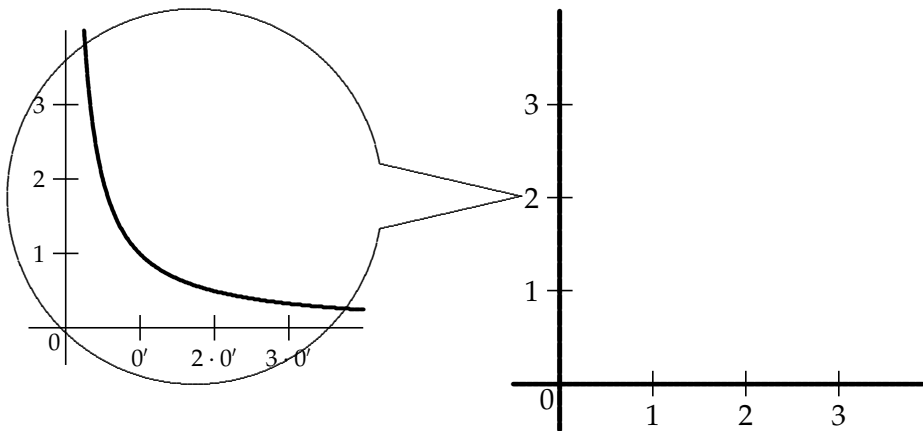
FIG. 29: Axial function $A(x)$
as proper unfolded reciprocal function

The axial function can be made semiregular at the unfolded level by choosing a proper unfolding. In this case, we use the proper unfolding $A(x) :\equiv \frac{0'}{x}$, shown in Figure 29.

Function $\sin \frac{1}{x}$
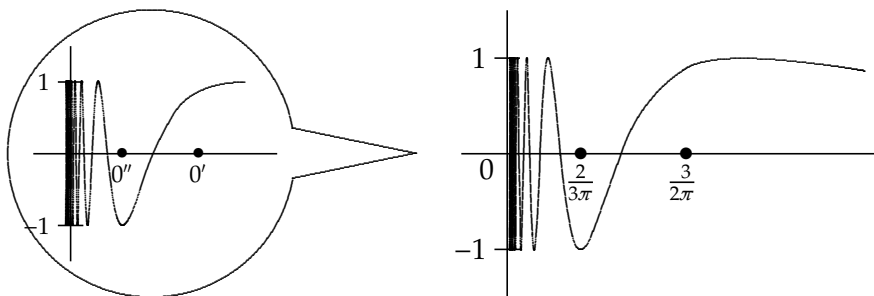
FIG. 30: Essential singularity of $S(x) := \sin \frac{1}{x}$
with microscope view of y axis infinitely expanded in x
direction and unchanged scale in y direction

The function $S(x) := \sin \dfrac{1}{x}$ is graphed in Figure 30. Like the pole in Figure 26, $S(x)$ at 0 maintains its shape from macroscope to microscope. Within

the microscope, $S(0)$ takes on every value within the interval $[-1, +1]$. Algebraically we can see this by observing that $\infty + r = \infty$ for every real perfinite r, so $\sin \infty = S(0) = [-1, +1]$.

Since $S(0)$ is not determinate, it is a singularity. An offset value $S(0')$ can be any point within $[-1, +1]$, so the offset values are not uniform or semiuniform, and the singularity is not a removable discontinuity or jump discontinuity.

The following calculation shows that the singularity is also not a pole. As defined in **Types of singularity** (p. 72), a pole of a function f is a point p such that $f(x) = \dfrac{g(x)}{h(x)}$, g and h are analytic, $h(x)$ has a root (zero) at p, and the multiplicity of the root is finite. The following converts the power series for $S(x)$ to a fraction, using the *Pochhammer symbol* $(n)_r$ to denote the *falling factorial* function $\dfrac{n!}{(n-r)!}$, for which $(n)_n = (n)_{n-1} = n!$, $(n)_1 = n$, $(n)_0 = 1$.

$$
\begin{aligned}
\sin \frac{1}{x} &= \frac{1}{x} - \frac{1}{3!x^3} + \frac{1}{5!x^5} - \cdots \\
&= \frac{3!x^2 - 1}{3!x^3} + \frac{1}{5!x^5} - \cdots \\
&= \frac{5!x^4 - (5)_2 x^2 + 1}{5!x^5} - \frac{1}{7!x^7} + \cdots \\
&= \frac{7!x^6 - (7)_4 x^4 + (7)_2 x^2 - 1}{7!x^7} + \frac{1}{9!x^9} - \cdots \\
&= \frac{\sum_{k=0}^{\infty'} (-1)^k (2\infty' + 1)_{2(\infty'-k)} x^{2(\infty'-k)}}{(2\infty' + 1)! x^{2\infty'+1}}
\end{aligned}
$$

The denominator of the final fraction has a root at $x = 0$ of infinite multiplicity. Since the singularity is not a removable discontinuity, jump discontinuity, or pole, it is an essential singularity.

The derivative $S'(x) = -\dfrac{\cos \frac{1}{x}}{x^2}$ is indeterminate at the singularity, but the antiderivative $\displaystyle\int S(x)dx = x \sin x + \int_{\frac{1}{x}}^{\infty} \frac{\cos t}{t} dt + k$ is determinate.

Weierstrass function

Weierstrass gave an example of a class of functions that are continuous everywhere but differentiable nowhere in conventional analysis. We now examine a function which is simpler but still shows the essential features of the original Weierstrass functions:

$$W(x) := \sum_{n=0}^{\infty} \frac{\sin(2^n x)}{2^n} = \sin x + \frac{\sin 2x}{2} + \frac{\sin 4x}{4} + \ldots$$

Conventional analysis cannot differentiate this function because

$$\liminf_{\delta \to 0} \frac{W(x+\delta) - W(x)}{\delta} > \limsup_{\delta \to 0} \frac{W(x+\delta) - W(x)}{\delta},$$

at every point, and thus

$$W'(x) = \lim_{\delta \to 0} \frac{W(x+\delta) - W(x)}{\delta}$$

does not exist anywhere.

Equipoint analysis does not have any such requirement. It requires only that a function be defined on an interval. Then, using an unfolded $0'$-level arithmetic, it computes

$$W'(x) = \frac{W(x+0') - W(x)}{0'}.$$

The conventional theory of infinite series allow us to use the commutative, associative, and distributive properies of addition and multiplication on infinite series only in restricted cases. ***Equipoint summation***, on the other hand, developed in [CD] for the summation of divergent series, allows unrestricted use of the these properties with no known inconsistencies. Therefore, when coupled with the algebraic definition of derivatives developed here, we feel confident that we can differentiate W term by term as we would a finite sum, and so:

$$W'(x) = \sum_{n=0}^{\infty} \cos(2^n x).$$

Another way to differentiate this function is to define it with an unfolded upper limit:

$$W(x) :\equiv \sum_{n=0}^{\infty'} \frac{\sin(2^n x)}{2^n} = \sin x + \frac{\sin 2x}{2} + \frac{\sin 4x}{4} + \ldots$$

and differentiate at an unfolded level beyond the unfolding of the upper limit, where $W(x)$ is smooth, just as $\sin x$ is smooth at an unfolded level:

$$0' :\equiv \frac{1}{\infty'^2}$$

$$W'_{0'}(x) \equiv \frac{{}^{0'}dW(x)}{{}^{0'}dx}$$

$$\sum_{n=0}^{\infty'} \cos(2^n x).$$

Figures 31 and 32 show $W(x)$ and $W'(x)$.

In the first calculation, the derivative $W'(x)$ is can be calculated at every point within an unfolding of x and is single valued everywhere, but the offset values $W(x+0')$ vary with each $0'$, so $W(x)$ is not continuous anywhere. Thus every point is a singularity, each of which is nonisolated.

In the second calculation, $W(x+0')$ matches $W(x)$, but only at levels of unfolding beyond $0'$. It can be said to be continuous at those unfoldings.
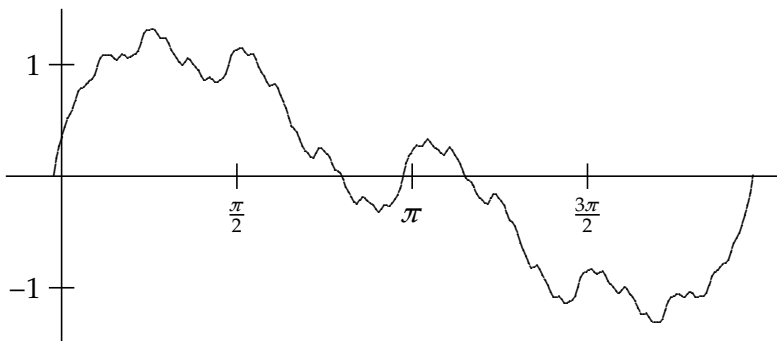
FIG. 31:
Weierstrass-like function $W(x)$

Equipoint Analysis
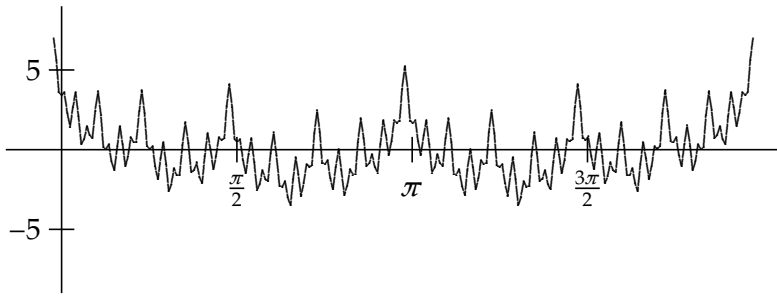
FIG. 32:
Derivative of Weierstrass-like function

Fourier transform

The Fourier transform, in the unitary asymmetric form, maps a function $f(x)$ to the transform $\widehat{f}(k)$ by

$$\widehat{f}(k) = \int_{-\infty}^{+\infty} f(x)e^{-2\pi ikx}\, dx.$$

We will not redevelop Fourier theory here but only note the Fourier transform of some proper unfolded functions. These are variations of the two elementary transforms

$\mathbf{f(x)}$	$\widehat{\mathbf{f}}(\mathbf{k})$
1	$\delta(k)$
$e^{2\pi iax}$	$\delta(k - a)$

The proper unfolded variations are

$\mathbf{f(x)}$	$\widehat{\mathbf{f}}(\mathbf{k})$
0	$0'\delta(k) = \delta_{k,0}$
$0e^{2\pi iax}$	$0'\delta(k - a) = \delta_{k,a}$

These two results assume that the width and height of $\delta(0)$ are $0'$ and $\frac{1}{0'}$. These connect the **Dirac delta** (p. 79) $\delta(x - a)$ with the **Kronecker delta** (p. 78) $\delta_{x,a}$.

Other singularities
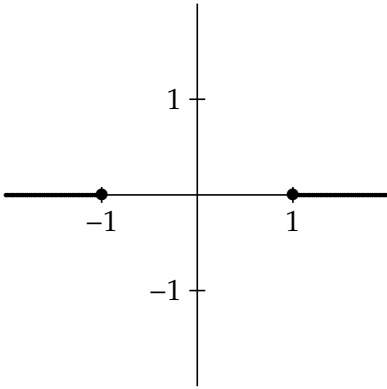
FIG. 33:
Gapped interval in $G(x)$
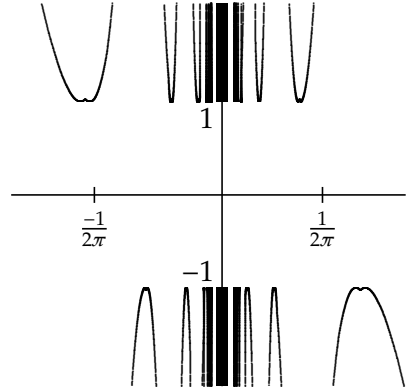
FIG. 34:
Accumulation point
of poles of $C(x) = \csc \frac{1}{x}$
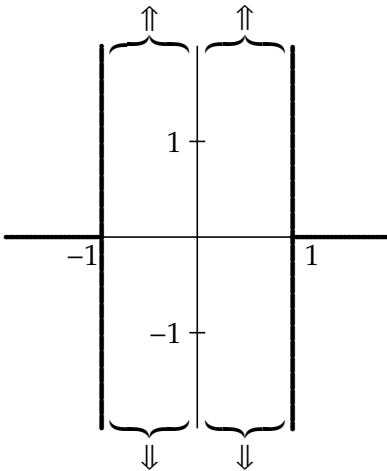
FIG. 35:
Interval of poles
in $J(x) = \frac{1}{x^{\infty}}$
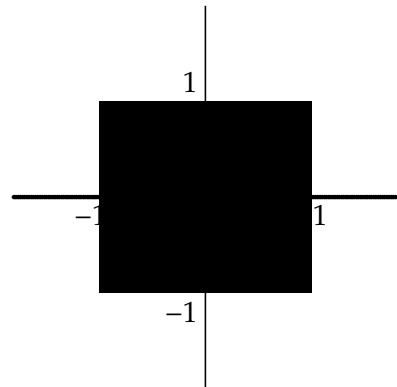
FIG. 36:
Interval of intervals
in $T(x) = \sin \frac{1}{x^{\infty}}$

Nonisolated singularities. Examples of nonisolated singularities are shown in Figures 33 through 36.

- Figure 33: A gapped interval in the function
$$G(x) := \begin{cases} 0 & \text{for } |x| \geq 1 \\ \emptyset & \text{for } |x| < 1 \end{cases}$$

- Figure 34: An accumulation point of poles at $x = 0$ in the function $C(x) := \csc \dfrac{1}{x}$. Every neighborhood around $x = 0$, and the unfolded point itself, has an infinite number of poles.

- Figures 35: An interval of poles in the function $J(x) := \dfrac{1}{x^\infty}$. For every $x \in [-1, +1]$, $J(x)$ is a pole. For $x = \pm 1$, $J(x) = \varnothing$, and elsewhere $J(x) = 0$.

- Figure 36: An interval of intervals in the function $T(X) = \sin \dfrac{1}{x^\infty}$. For every point $x \in [-1, +1]$, $T(x) = [-1, +1]$. Elsewhere, $T(x) = 0$.

- Characteristic function of the rational numbers. See **Using class counts in derivatives and integrals** (p. 130).

Complex singularities. The following are analyzed in the **Complex functions** (p. 96) chapter.

- **Complex poles** (p. 105), the complex ananlogs of **real poles** (p. 85) described above.

- The **complex axial function** (p. 107), the complex analog of the **real axial function** (p. 87) described above.

- The **complex function $e^{\frac{1}{x}}$** (p. 108), which includes the **real function $\sin \frac{1}{x}$** (p. 89) described above.

COMPLEX FUNCTIONS

Complex derivative

The complex derivative is similar to the real derivative but allows folded and unfolded complex numbers and extended complex numbers and functions.

It is single valued and finite if the real and imaginary partial derivatives are single valued, finite, and analytic. It may be multivalued otherwise. For example, at $x = 0$, for real x,

$$\frac{d|x|}{dx} = \pm 1,$$

while for complex z,

$$\frac{d|z|}{dz} = e^{i\mathbb{R}}.$$

The general complex derivative (possibly multivalued, infinite, and/or non-analytic) is as follows. For a complex function $f(z)$ we first define

$$f(z) \equiv \operatorname{Re} f(\operatorname{Re} z + i \operatorname{Im} z) + i \operatorname{Im} f(\operatorname{Re} z + i \operatorname{Im} z)$$
$$\equiv g(x, y) + ih(x, y).$$

We then have

$$
\begin{aligned}
\frac{^{0'}df(z)}{^{0'}dz} &\equiv \frac{g(\operatorname{Re}(z + 0'), \operatorname{Im}(z + 0')) - g(\operatorname{Re} z, \operatorname{Im} z)}{0'} \\
&\quad + i\frac{h(\operatorname{Re}(z + 0'), \operatorname{Im}(z + 0')) - h(\operatorname{Re} z, \operatorname{Im} z)}{0'} \\
&\equiv \frac{g(x + \operatorname{Re} 0', y + \operatorname{Im} 0') - g(x, y)}{0'} \\
&\quad + i\frac{h(x + \operatorname{Re} 0', y + \operatorname{Im} 0') - h(x, y)}{0'} \\
&\equiv \frac{g(x + \operatorname{Re} 0', y + \operatorname{Im} 0') - g(x, y + \operatorname{Im} 0')}{\operatorname{Re} 0'} \cdot \frac{\operatorname{Re} 0'}{0'}
\end{aligned}
$$

$$+ \frac{g(x, y + \operatorname{Im} 0') - g(x, y)}{\operatorname{Re} 0'} \cdot \frac{\operatorname{Re} 0'}{0'}$$

$$+ i\frac{h(x + \operatorname{Re} 0', y + \operatorname{Im} 0') - h(x, y + \operatorname{Im} 0')}{\operatorname{Im} 0'} \cdot \frac{\operatorname{Im} 0'}{0'}$$

$$+ i\frac{h(x, y + \operatorname{Im} 0') - h(x, y)}{\operatorname{Im} 0'} \cdot \frac{\operatorname{Im} 0'}{0'}$$

$$\equiv \frac{\partial g(x, y)}{\partial x} \cdot \frac{\operatorname{Re} 0'}{0'} + \frac{\partial g(x, y)}{\partial y} \cdot \frac{\operatorname{Im} 0'}{0'}$$

$$+ i\frac{\partial h(x, y)}{\partial x} \cdot \frac{\operatorname{Re} 0'}{0'} + i\frac{\partial h(x, y)}{\partial y} \cdot \frac{\operatorname{Im} 0'}{0'}$$

$$\equiv \frac{\partial g(x, y) + i\partial h(x, y)}{\partial x} \cdot \frac{\operatorname{Re} 0'}{0'} + \frac{\partial g(x, y) + i\partial h(x, y)}{\partial y} \cdot \frac{\operatorname{Im} 0'}{0'}$$

$$\equiv \frac{\partial f(z)}{\partial \operatorname{Re} z} \cdot \frac{\cos \arg 0'}{\operatorname{sgn} 0'} + \frac{\partial f(z)}{\partial \operatorname{Im} z} \cdot \frac{\sin \arg 0'}{\operatorname{sgn} 0'}.$$

If f is analytic, then this becomes

$$\frac{df(z)}{dz} \equiv \frac{\partial f(z)}{\partial \operatorname{Re} z} \cdot \frac{\cos \arg 0'}{\operatorname{sgn} 0'} + i\frac{\partial f(z)}{\partial \operatorname{Re} z} \cdot \frac{\sin \arg 0'}{\operatorname{sgn} 0'}$$

$$\equiv \frac{\partial f(z)}{\partial \operatorname{Re} z} \cdot \frac{\operatorname{Re} 0' + i\operatorname{Im} 0'}{0'} \equiv \frac{\partial f(z)}{\partial \operatorname{Re} z} \cdot \frac{0'}{0'} \equiv \frac{\partial f(z)}{\partial \operatorname{Re} z}$$

$$\equiv -i\frac{\partial f(z)}{\partial \operatorname{Im} z} \cdot \frac{\cos \arg 0'}{\operatorname{sgn} 0'} + \frac{\partial f(z)}{\partial \operatorname{Im} z} \cdot \frac{\sin \arg 0'}{\operatorname{sgn} 0'}$$

$$\equiv -i\frac{\partial f(z)}{\partial \operatorname{Im} z} \cdot \frac{\operatorname{Re} 0' + i\operatorname{Im} 0'}{0'} \equiv -i\frac{\partial f(z)}{\partial \operatorname{Im} z} \cdot \frac{0'}{0'} \equiv -i\frac{\partial f(z)}{\partial \operatorname{Im} z}.$$

As an example of the general complex derivative, let $f(z) := 3\operatorname{Re} z + 2i\operatorname{Im} z$, which is not analytic. We then have

$$\frac{{}^{0'}df(z)}{{}^{0'}dz} = \frac{3\cos \arg 0' + 2i\sin \arg 0'}{\operatorname{sgn} 0'}.$$

Letting $\theta := \arg 0'$, this becomes

$$\frac{{}^{0'}df(z)}{{}^{0'}dz} \equiv \frac{3\cos \theta + 2i\sin \theta}{e^{i\theta}}$$

$$\equiv [\cos^2 \theta + 2] - i[\cos \theta \sin \theta]$$

Letting $x := \operatorname{Re} \frac{df}{dz}$ and $y := \operatorname{Im} \frac{df}{dz}$ gives

$$x = \frac{1}{2}\cos 2\theta + \frac{5}{2}$$

$$y = -\frac{1}{2}\sin 2\theta$$
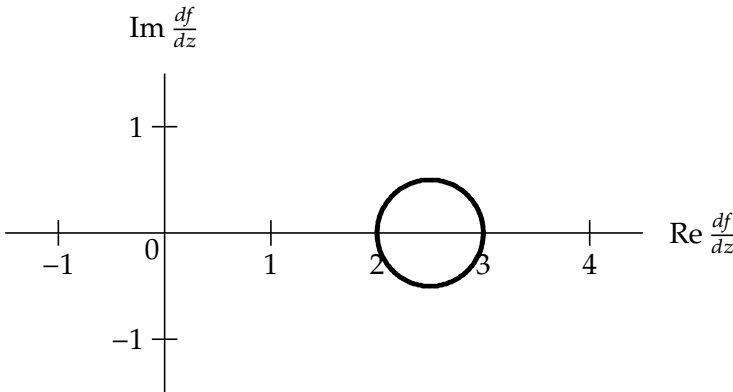
FIG. 37:
Derivative of $f(z) = 3\,\mathrm{Re}\,z + 2i\,\mathrm{Im}\,z$

The derivative is the class of all points on a circle with radius $\frac{1}{2}$ and centered on $\frac{5}{2}$, shown in Figure 37. Since the partial derivatives are constant with respect to z, so is the total derivative. While the derivative does not depend on z, it does depend on $0'$. When $\arg 0' = 0$, the derivative is 3, but when $\arg 0' = \frac{\pi}{2}$, the derivative is 2.

Generalizing this example, let a and b be real coefficients, and

$$f(z) := a\,\mathrm{Re}\,z + bi\,\mathrm{Im}\,z$$

$$\frac{{}^{0'}df(z)}{{}^{0'}dz} \equiv b + (a - b)\cos^2\theta + (b - a)i\sin^2\theta$$

$$\left(x - \frac{a+b}{2}\right)^2 + y^2 = \left(\frac{a-b}{2}\right)^2.$$

If $a = b$, this becomes

$$f(z) := az$$

$$\frac{df(x)}{dz} = a$$

and, since a is real, the circle shrinks to the point $z = a$.

The Cauchy integral formula
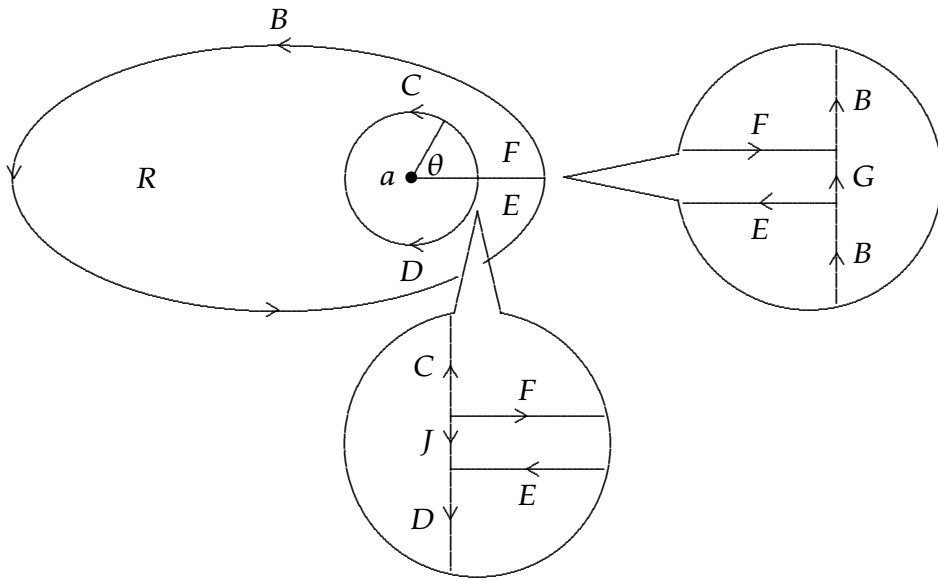
FIG. 38:
Contours for path independence
around a singularity

The Cauchy integral formula is a theorem of complex analysis that is conventionally proved with limits. Below is an equipoint proof. First we derive a preliminary theorem which is also used in the derivation of **Laurent series coefficients** (p. 103).

Given a function f that is analytic within a region R with boundary B, with the possible exception that f is not analytic at some point $a \in R$, and any closed path C within R that goes once around a, then $\int_B f(z)\, dz = \int_C f(z)\, dz$, assuming that we integrate along B and C in the same direction.

PROOF. We start by drawing the contours shown in Figure 38:

- Without loss of generality, assume *B* is directed counterclockwise.

- Draw a path *D* coincident with *C* but directed clockwise.

- Draw a directed line *E* from any point on *B* to any point on *C*, and line *F*, separated from line *E* by a distance of $0'$, directed out from *C*.

- Let *G* be the infinitesimally short portion of *B* between *E* and *F*, and let *H* be the portion of *B* with *G* removed. Then $H :\equiv B \setminus G =' B$.

- Let *J* be the infinitesimally short portion of *D* between *E* and *F*, and let *K* be the portion of *D* with *J* removed. Then $K :\equiv D \setminus J =' D$.

- Let *L* be the concatenation of, in order, *H*, *E*, *K*, and *F*. That is, start from the point where *F* meets *B*, go almost all the way around *B* to *E*, go in on *E* to *D*, go almost all the way around *D* to *F*, and go out on *F* to the starting point at *B*.

L is a closed contour which does not include *a*. By the Cauchy-Goursat integral theorem, we then have

$$
\begin{aligned}
0 &= \int_L f(z)\, dz \\
&\equiv \int_H f(z)\, dz + \int_E f(z)\, dz + \int_K f(z)\, dz + \int_F f(z)\, dz \\
&\equiv \int_H f(z)\, dz + \int_K f(z)\, dz \\
&= \int_B f(z)\, dz + \int_D f(z)\, dz \\
&\equiv \int_B f(z)\, dz - \int_C f(z)\, dz,
\end{aligned}
$$

or

$$
\int_B f(z)\, dz = \int_C f(z)\, dz. \quad \square
$$

We note that the curves *B* and *C* can be finite, infinite, or infinitesimal, with lengths of *E* and *F* to match, and the separators *G* and *J* infinitesimal compared to *B* and *C*.
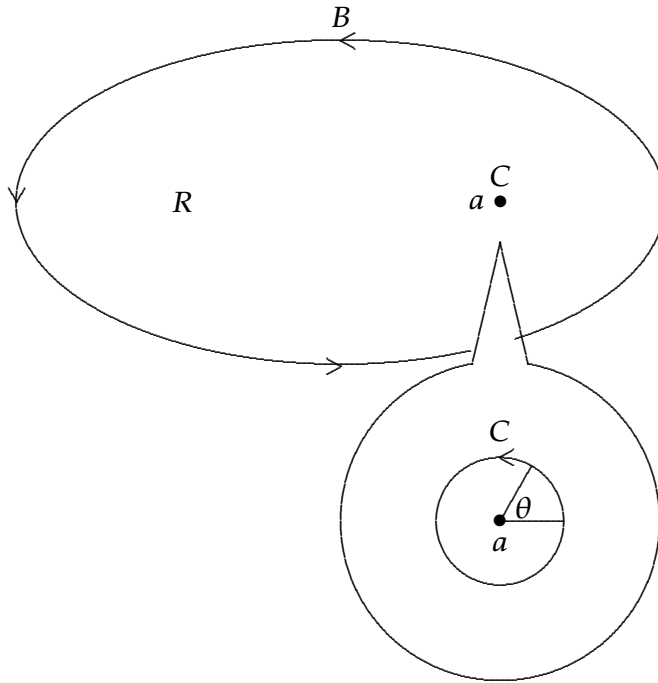
FIG. 39:
Contours for proof of the
Cauchy integral formula

CAUCHY INTEGRAL FORMULA: Given a function $f(z)$ that is analytic in a simply connected region R and on its boundary B, and given a point $a \in R$, then

$$f(a) = \frac{1}{2\pi i} \int_B \frac{f(z)\,dz}{z - a}.$$

PROOF. We draw contours as in Figure 39. Without loss of generality, we again assume B is directed counterclockwise. We draw an infinitesimal circle C, of diameter $0'$, also directed counterclockwise, around a. By the previous theorem, the integral around the boundary B equals the integral around the circle C. To integrate around C, since it is infinitesimal, we make the substitution

$$z \equiv a + 0'e^{i\theta}$$
$$dz \equiv 0'ie^{i\theta}$$

and compute

$$\int_C \frac{f(z)\,dz}{z-a} \equiv \int_C \frac{f\left(a + 0'ie^{i\theta}\right) 0'ie^{i\theta}\,d\theta}{0'e^{i\theta}}$$

$$\equiv i \int_C f\left(a + 0'ie^{i\theta}\right) d\theta$$

$$= if(a) \int_C d\theta$$

$$= 2\pi i f(a)$$

or

$$f(a) = \frac{1}{2\pi i} \int_B \frac{f(z)\,dz}{z-a}. \ \square$$

We now make the point a variable and rewrite this theorem as

$$f(z) = \frac{1}{2\pi i} \int_B \frac{f(w)\,dw}{w-z}.$$

CAUCHY INTEGRAL FORMULA FOR DERIVATIVES: Given the same conditions as in the previous theorem,

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_B \frac{f(w)\,dw}{(w-z)^{n+1}}.$$

PROOF. Since an integral is an infinite series and a derivative is a quotient difference, and since we establish in the numeristic theory of infinite series [CD] that we can handle them much as we do finite series, e.g. they commute, associate, and distribute as finite series do, we can calculate simply:

$$\frac{df(z)}{dz} = \frac{d}{dz} \frac{1}{2\pi i} \int_B \frac{f(w)\,dw}{w-z}$$

$$\equiv \frac{1}{2\pi i0'} \int_B \frac{f(w)\,dw}{w-(z-0')} - \frac{1}{2\pi i0'} \int_B \frac{f(w)\,dw}{w-z}$$

$$\equiv \frac{1}{2\pi i0'} \int_B \left[\frac{f(w)}{w-(z-0')} - \frac{f(w)}{w-z} \right] dw$$

$$\equiv \frac{1}{2\pi i} \int_B \frac{d}{dz} \frac{f(w)}{w-z}\,dw$$

$$= \frac{1}{2\pi i} \int_B \frac{f(w)\,dw}{(w-z)^2}.$$

Further differentiations yield the theorem. $\square$

Taylor and Laurent series

The numeristic theory of infinite series [CD] also establishes that certain series, when summed through extended numersitic arithmetic, are valid not only where they converge, but also where they diverge. The following is one such series, which is valid everywhere in the complex plane, even though it converges only within the unit circle:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots = \sum_{k=0}^{\infty} x^k$$

We now derive two alternate forms of this series, which we will use in the proofs of the following two theorems:

$$\frac{1}{w-z} \equiv \frac{1}{(w-a)-(z-a)}$$

$$\equiv \frac{\frac{1}{w-a}}{1 - \frac{z-a}{w-a}}$$

$$\equiv \frac{1}{w-a}\left[1 + \frac{z-a}{w-a} + \left(\frac{z-a}{w-a}\right)^2 + \left(\frac{z-a}{w-a}\right)^3 + \ldots\right]$$

$$\equiv \frac{1}{w-a} + \frac{z-a}{(w-a)^2} + \frac{(z-a)^2}{(w-a)^3} + \frac{(z-a)^3}{(w-a)^4} + \ldots,$$

$$\frac{-1}{w-z} \equiv \frac{1}{(z-a)-(w-a)}$$

$$\equiv \frac{\frac{1}{z-a}}{1 - \frac{w-a}{z-a}}$$

$$\equiv \frac{1}{z-a}\left[1 + \frac{w-a}{z-a} + \left(\frac{w-a}{z-a}\right)^2 + \left(\frac{w-a}{z-a}\right)^3 + \ldots\right]$$

$$\equiv \frac{1}{z-a} + \frac{w-a}{(z-a)^2} + \frac{(w-a)^2}{(z-a)^3} + \frac{(w-a)^3}{(z-a)^4} + \ldots.$$

> **TAYLOR SERIES COEFFICIENTS: Given a function $f(z)$ that is analytic within a simply connected region R and a point $a \in R$, then, for any $z \in R$:**
>
> $$f(z) = f(a) + (z-a)f'(a) + (z-a)^2\frac{f''(a)}{2!} + (z-a)^3\frac{f^{(3)}(a)}{3!} + \ldots$$

$$= \sum_{k=0}^{\infty} (z-a)^k \frac{f^{(k)}(a)}{k!}.$$

PROOF. Let C be any closed path around a. C could be infinitesimal. Using the first of the above identities and the n-th derivative Cauchy integral formula, we compute

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(w)\,dw}{w-z}$$

$$= \frac{1}{2\pi i} \int_C \frac{f(w)\,dw}{w-a} + \frac{z-a}{2\pi i} \int_C \frac{f(w)\,dw}{(w-a)^2} + \frac{(z-a)^2}{2\pi i} \int_C \frac{f(w)\,dw}{(w-a)^3} + \cdots$$

$$= f(a) + (z-a)f'(a) + (z-a)^2 \frac{f''(a)}{2!} + (z-a)^3 \frac{f^{(3)}(a)}{3!} + \cdots$$

$$= \sum_{k=0}^{\infty} (z-a)^k \frac{f^{(k)}(a)}{k!}. \ \square$$

LAURENT SERIES COEFFICIENTS: Given a function $f(z)$ that is analytic within a region R between an outer boundary B (which may be infinite) and an inner boundary C (which may be infinitesimal), and given a point a inside C (so that $a \notin R$), then, for any $z \in R$:

$$f(z) = \sum_{k=-\infty}^{\infty} \frac{(z-a)^k}{2\pi i} \int_B \frac{f(w)\,dw}{(w-a)^{k+1}}.$$

PROOF. Following Figure 38, let L be the concatenation of, in order, H, E, K, and F. Let z be any point in R and let $g(w) :\equiv \frac{f(w)}{w-z}$. Then

$$\int_L g(w)\,dw \equiv \int_H g(w)\,dw + \int_E g(w)\,dw + \int_K g(w)\,dw + \int_F g(w)\,dw$$

$$\equiv \int_H g(w)\,dw + \int_K g(w)\,dw$$

$$= \int_B g(w)\,dw + \int_D g(w)\,dw$$

$$\equiv \int_B g(w)\,dw - \int_C g(w)\,dw.$$

Since L encloses z, and using both of the above identies,

$$f(z) = \frac{1}{2\pi i} \int_L g(w)\, dw$$

$$= \frac{1}{2\pi i} \int_B g(w)\, dw - \frac{1}{2\pi i} \int_C g(w)\, dw$$

$$= \frac{1}{2\pi i} \int_B \frac{f(w)\, dw}{w - z} - \frac{1}{2\pi i} \int_C \frac{f(w)\, dw}{w - z}$$

$$= \frac{1}{2\pi i} \int_B \frac{f(w)\, dw}{w - a} + \frac{z - a}{2\pi i} \int_B \frac{f(w)\, dw}{(w - a)^2} + \frac{(z - a)^2}{2\pi i} \int_B \frac{f(w)\, dw}{(w - a)^3} + \dots$$

$$+ \frac{1}{2\pi i(z - a)} \int_C f(w)\, dw + \frac{1}{2\pi i(z - a)^2} \int_C f(w)(w - a)\, dw$$

$$+ \frac{1}{2\pi i(z - a)^3} \int_C f(w)(w - a)^2\, dw + \dots$$

$$= \sum_{k=0}^{\infty} \frac{(z - a)^k}{2\pi i} \int_B \frac{f(w)\, dw}{(w - a)^{k+1}} + \sum_{k=1}^{\infty} \frac{1}{2\pi i(z - a)^k} \int_C f(w)(w - a)^{k-1}\, dw.$$

The integrals in the second sum vanish since they do not enclose any singuarities. Hence

$$f(z) = \sum_{k=-\infty}^{\infty} \frac{(z - a)^k}{2\pi i} \int_B \frac{f(w)\, dw}{(w - a)^{k+1}}. \quad \square$$

Complex poles

In **Types of singularity** (p. 72), we defined a pole of a function f as a point x such that $f(x) = \dfrac{g(x)}{h(x)}$, g and h are analytic, $h(x)$ has a root (zero) at x, and the multiplicity of the root is finite.

In numeristics, every elementary function is defined over the whole complex plane, even at its singularities. Since a function may be defined at a singularity, the domain of such a function may still be simply connected.

As in conventional analysis, we transform a contour by parameterizing it into a directed real interval.
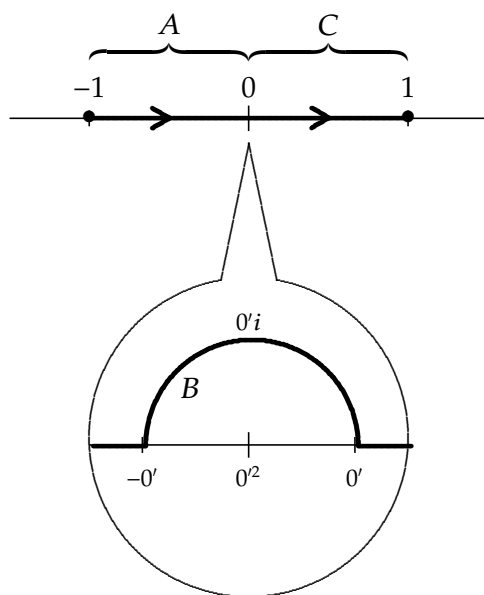
FIG. 40:
Contour G consisting of
three portions A, B, C

In real analysis, as described in **real poles** (p. 85), the effective antiderivative of $\dfrac{1}{x}$ is $\ln |x|$, which assumes that we integrate either completely on the negative side of the real axis or completely on the positive side.

In complex space, the antiderivative is $\ln x$, and the path of integration is connected. A path which includes an infinitesimal region is shown in Figure 40. This path, contour G, has three portions that link two points, -1 and $+1$:

    A: A path along the real axis from -1 to $-0'$,

    B: An infinitesimal semicircle around the origin from $-0'$ to $+0'$, and

    C: A path along the real axis from $+0'$ to $+1$.

In real space, we must omit portion B and use the effective antiderivative $\ln |x|$. Although this path includes all but one point of the real interval

$[-1, +1]$, the resulting integral of $\dfrac{1}{x}$ differs from the complex version:

$$\int_{-1}^{+1} \frac{dx}{x} \equiv \int_{A+C} \frac{dx}{x} \equiv \ln|x|\Big|_{-1}^{-0'} + \ln|x|\Big|_{+0'}^{+1} \equiv [-\infty' - 0] + [0 + \infty'] = 0.$$

In complex space, we can include portion B and use the actual antiderivative $\ln x$:

$$\int_G \frac{dz}{z} \equiv \int_{A+B+C} \frac{dz}{z} \equiv \int_{-1}^{-0'} \frac{dz}{z} + \int_{\pi}^{0} d\theta + \int_{1}^{0'} \frac{dz}{z} \equiv \ln z\Big|_{-1}^{-0'} + \theta\Big|_{\pi}^{0} + \ln z\Big|_{+0'}^{+1}$$
$$\equiv \big[\ln(-0') - \ln(-1)\big] + [0 - \pi] + \big[\ln 1 - \ln 0'\big]$$
$$\equiv \ln 0' - \pi - \ln 0' = -\pi.$$

Additional windings around the pole, inside this same infinitesimal complex space hidden within the real line, give the class of values $(2\mathbb{Z} + 1)\pi i$. This agrees with the Fundamental Theorems of Calculus:

$$\int_G \frac{dz}{z} = \ln z\Big|_{-1}^{+1} = \ln 1 - \ln(-1) = 2\mathbb{Z}\pi - (2\mathbb{Z} + 1)\pi = (2\mathbb{Z} + 1)\pi.$$

With other poles, the discrepancy between real and complex integrals happens whenever the integral of portion B is nonzero. Some examples of this integral:

$$\int_B \frac{dx}{x} = \frac{2\mathbb{Z} + 1}{2}$$

$$\int_B \frac{dx}{x^2} = \infty$$

$$\int_B \frac{dx}{x^3} = 0.$$

Complex axial function

The axial function is defined:

$$A(z) := \begin{cases} 0 & \text{for } z \neq 0 \\ \cancel{\infty} & \text{for } z = 0 \end{cases}$$

Its name derives from the real version of this function, discussed in **Axial function** (p. 87). The graph of the real function coincides the coordinate

axes. The complex version of this function coincides with the plane that contains the two axes of the domain, and the plane containing the two axes of the range.

In real space, an integral through the origin yields an indeterminacy, as it does with a pole. To integrate from one side of the origin to the other, we have to integrate piecewise on each side, which yields two independent constants of integration.

In complex space, we can integrate on a path around the origin, as with did with a **complex pole** (p. 105). This yields a single constant of integration.

Function $e^{\frac{1}{x}}$

$$\mathrm{Re}\, E(\mathrm{Re}\, z) = e^{\frac{1}{\mathrm{Re}\, z}}$$

$$\mathrm{Im}\, E(\mathrm{Re}\, z) = \mathrm{Im}\, e^{\frac{1}{\mathrm{Re}\, z}} = 0$$

$$\mathrm{Re}\, E(i\,\mathrm{Im}\, z) = \mathrm{Re}\, e^{\frac{i}{\mathrm{Im}\, z}} = \cos\frac{1}{\mathrm{Im}\, z}$$

$$\mathrm{Im}\, E(i\,\mathrm{Im}\, z) = -\,\mathrm{Im}\, e^{\frac{i}{\mathrm{Im}\, z}} = -\sin\frac{1}{\mathrm{Im}\, z}$$
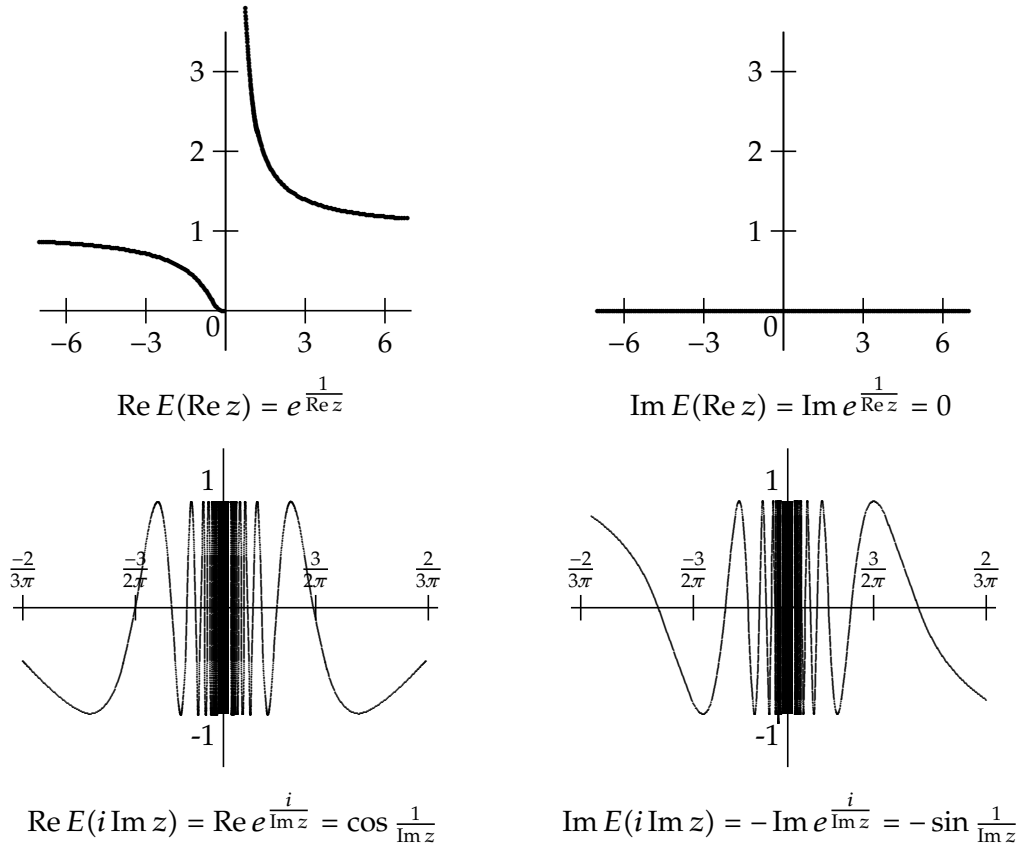
FIG. 41: Essential singularity of $E(z) := e^{\frac{1}{z}}$

     The function $E(z) := e^{\frac{1}{z}}$ is graphed via real and imaginary parts in Figure 41. $E(x)$ is a complex version of the function $S(x)$ investigated in **Function $\sin\frac{1}{x}$** (p. 89).

     As a real function, $E(z)$ has a jump discontinuity at $z = 0$, but as a complex function, this is not the case. The offset values $E(0')$ for imaginary $0'$ can be any value within $[-1, +1]$, so the offset values are not semiuniform, and the singularity is not a removable discontinuity or jump discontinuity. The

following shows that it is not a pole.

$$e^{\frac{1}{z}} = 1 + \frac{1}{z} + \frac{1}{2!x^2} + \dots$$

$$= \frac{x+1}{x} + \frac{1}{2!x^2} + \dots$$

$$= \frac{2!x^2 + (2)_1 x + 1}{2!x^2} + \frac{1}{x^3} + \dots$$

$$= \frac{3!x^3 + (3)_2 x^2 + (3)_1 x + 1}{3!x^3} + \frac{1}{x^4} + \dots$$

$$= \sum_{k=0}^{\infty'} (\infty')_{\infty'-k} x^{\infty'-k} \infty'! x^{\infty'}$$

The denominator of the final fraction has a root at $z = 0$ of infinite multiplicity. Thus the singularity is an essential singularity.

PICARD'S THEOREM: If a complex function f has an essential singularity at x, then $f(x) = \infty\!\!\!/$.

We will not prove the general case of this theorem, but we will show that it holds for $E(z)$, i.e. that $E(0) = e^{\frac{1}{0}} = \infty\!\!\!/$. To do this, we will show that for given any z, we can find $0'$ such that $z = e^{\frac{1}{0'}}$. If we write $z = re^{i\theta} = e^{\frac{1}{0'}}$, then we are looking for r and θ such that $\ln r + i\theta$ is infinite.

For infinite or zero r, $\ln r$ is infinite, and θ can be any value. For perfinite r, θ must be infinite. Since, as we saw in **Function $\sin \frac{1}{x}$** (p. 89), $\sin \infty = [-1, +1]$, then for an unfolded infinite θ, $e^{i\theta}$ is on the unit circle as it is for finite θ. Thus, for any z, $\ln r + i\theta = \infty'$ for some complex infinite value ∞', and $0' =' \frac{1}{\infty'}$. □

CALCULUS OF VARIATIONS

Definition of functional

The terminology, definitions, notation, and approaches of the calculus of variations are not yet fully standardized. The calculus of variations is also known as *variational calculus*. Older mathematical literature also called it *functional calculus*, but currently this term has a much different meaning.

Here we use terms, notation, and definitions which are common but not universal. The approach that we use for the main topic of the calculus of variations, the functional, is to define it as a type of infinite dimensional vector space. For definitions and theorems, first we state the finite dimensional vector case, and then the functional case.

For an entertaining account of the conventional vector space approach to functionals, see [W16 p. 341-370].

We define an ***n-dimensional vector function*** to be of the form
$$f(\{x_j\}) = h,$$
where $\{x_j\}$ is a sequence (not a class) of real or complex scalars, h is a real or complex scalar, and the index j ranges from 1 to n. This could be stated as $f(x_1, x_2, x_3, \ldots, x_n) = h$, but we are choosing the first form to emphasize its connection to functionals.

Example: Let
$$f(\{x_j\}) := \sum_{j=1}^{n} x_j^2$$
then for $n = 3$, $f(\{1, 2, 3\}) = 14$.

A *functional*
$$F[x(s)] = h$$
maps the real or complex function $x(s)$ to h, where $s, x(s), h$ are real or complex scalars.

Example: Let

$$F[x(s)] := \int_0^1 x(s)^2 ds$$

then $F[2s] = \frac{4}{3}$.

Iteration

Some definitions in this chapter use iterations of operators which we denote with *iteration notation*. This notation defines an indexed iteration, a simple type of recursion. Two or more relations are required to define a function recursively, but iteration notation can define an iterated function with a single expression.

Discrete iteration denotes a finite number of iterations at unit intervals of a function on an initial value, a *seed*:

$$\overset{n}{\underset{k=m}{\text{I}}} h(\langle a \rangle, k) := r_n, \text{ the last term of the following sequence:}$$

$$r_m = h(a, m)$$
$$r_k = h(r_{k-1}, k)$$
$$r_n = h(r_{n-1}, n)$$

Continuous iteration denotes an infinite number of iterations at infinitesimal intervals of a function on a seed:

$$\overset{0'}{\mathscr{I}}\overset{v}{\underset{t=u}{}} h(\langle a \rangle, t) := s(t), \text{ the last term of the following sequence:}$$

$$s(u) = h(a, u)$$
$$s(t) = h(s(t - 0'(v - u)), t)$$
$$s(v) = h(s(v - 0'(v - u)), v)$$

Sums, products, integrals, and prodegrals are easily translated to iteration notation:

$$\sum_{k=m}^{n} f(k) = \overset{n}{\underset{k=m}{\text{I}}} \langle 0 \rangle + f(k)$$

$$\prod_{k=m}^{n} f(k) = \overset{n}{\underset{k=m}{\text{I}}} \langle 1 \rangle f(k)$$

$$\int_a^b f(x) dx = \overset{dx}{\mathscr{I}}\overset{b}{\underset{x=a}{}} \langle 0 \rangle + f(x)$$

$$\int_a^b f(x)^{dx} = {}^{dx}\mathscr{G}_{x=a}^{b}\langle 1\rangle \cdot f(x)$$

Functional differentials

For a vector function, applying the definition in **Differentials and integrants** (p. 55) gives the following definition of a *partial differential* of a vector function:

$$\mathop{}^{0'}\partial_{x_i} f(\{x_j\}) :\equiv f(\{x_j + 0'\delta_{i,j}\}) - f(\{x_j\}).$$

Adapting this definition for a functional yields the following definition for the *(partial) differential of a functional*:

$$\mathop{}^{0'}\delta_{t,x(s)} F[x(s)] :\equiv F[x(s) + 0'\delta(s-t)] - F[x(s)]$$

The functional differential is sometimes called the *variation* of F, but the **corresponding derivative** (p. 113), defined in the next section, is also sometimes called the variation.

The *total differential* of a vector function is the sum of the partial differentials:

$$\mathop{}^{0'}d_{\{x_j\}} f(\{x_j\}) :\equiv \sum_{i=1}^{n} {}^{0'}\partial_{x_i} f(\{x_j\}).$$

Similarly, the total differential of a functional is the integral of the partial differentials:

$$\mathop{}^{0'}\delta_{*,x(s)} F[x(s)] :\equiv \int_{-\infty}^{+\infty} {}^{0'}\delta_{t,x(s)} F[x(s)] dt.$$

The *omnivariate differential* of a vector function is the repeated partial differential for all variables:

$$\partial_{x_1,\dots,x_n}^{n} f(\{x_j\}) \equiv \prod_{i=1}^{n} \partial_{x_i}\langle f(\{x_j\})\rangle.$$

Analogously, the *omnivariate differential* of a functional is the partial differential repeated for each point in the function space:

$$\mathop{}^{0'}\mathfrak{D}_{t,x(s)} F[x(s)] :\equiv {}^{0'}\mathscr{G}_{t=-\infty}^{+\infty} {}^{0'}\delta_{t,x(s)}\langle F[x(s)]\rangle.$$

In the independent function space, for both vector and functional cases, the differentials are:

$$\partial x_1 \ldots \partial x_n \equiv \prod_{i=1}^{n} \partial x_i$$

$${}^{0'}\delta_{x(t)} x(t) :\equiv {}^{0'}d_{x(t)} x(t)$$

$${}^{0'}\mathfrak{D}_{x(t)} x(t) :\equiv {}^{0'}\!\int_{t=-\infty}^{+\infty} {}^{0'}\delta_{x(t)} x(t)$$

Functional derivatives

Applying the **definition of derivative** (p. 31) and the **definition of partial differential** (p. 55) to a vector function, the *partial derivative* of a vector function is:

$$\frac{\partial f(\{x_j\})}{\partial x_i} :\equiv \frac{{}^{0'}\partial_{x_i} f(\{x_j\})}{{}^{0'}d_{x_i} x_i} \equiv \frac{f(\{x_j + 0'\delta_{i,j}\}) - f(\{x_j\})}{0'}.$$

Analogously, the *functional derivative* is defined as:

$$\frac{\delta F[x(s)]}{\delta x(t)} :\equiv \frac{{}^{0'}\delta_{t,x(s)} F[x(s)]}{{}^{0'}\delta_{x(t)} x(t)} \equiv \frac{F[x(s) + 0'\delta(s - t)] - F[x(s)]}{0'}.$$

As mentioned above, both the functional derivative and the functional differential are called the *variation*.

Quadratic example, vector function case: Let

$$f(\{x_j\}) := \sum_{j=1}^{n} x_j^2.$$

Then

$$\frac{\partial f(\{x_j\})}{\partial x_i} \equiv \frac{\partial \sum_{j=1}^{n} x_j^2}{\partial x_i} \equiv \sum_{j=1}^{n} 2x_j \cdot \delta_{i,j} = 2x_i.$$

Quadratic example, functional case: Let

$$F[x(s)] := \int_a^b [x(s)]^2 ds.$$

Then

$$\frac{\delta F[x(s)]}{\delta x(t)} \equiv \frac{\int_a^b [x(s) + 0'\delta(s-t)]^2 \, ds - \int_a^b x(s)^2 ds}{0'}$$

$$\equiv \frac{\int_a^b [x(s)^2 + 2x(s)\, 0'\delta(s-t) + 0'^2\delta(s-t)^2] \, ds - \int_a^b x(s)^2 ds}{0'}$$

$$\equiv \int_a^b [2x(s)\, \delta(s-t) + 0'\delta(s-t)^2] \, ds$$

$$\equiv 2x(t) + \int_a^b 0'\delta(s-t) \, ds = 2x(t).$$

For further reading on the conventional theory and applications of the functional derivative, the author suggests [WC].

Higher functional derivatives

Higher derivatives of vector functions are, of course, iterations of the first derivative. The following is the form using binomial coefficients that was derived in **Higher order derivatives and integrals** (p. 60).

$$\frac{\partial^2 f(\{x_j\})}{\partial x_i^2} \equiv \frac{f(\{x_j + 2 \cdot 0'\delta_{j,i}\}) - 2f(\{x_j + 0'\delta_{j,i}\}) + f(\{x_j\})}{0'^2}$$

$$\frac{\partial^n f(\{x_j\})}{\partial x_i^n} \equiv \frac{\sum_{k=0}^m (-1)^{n-k} \binom{n}{k} f(\{x_j + 0'k\delta_{j,i}\})}{0'^n}$$

Higher functional derivatives are also iterations of the first derivative, which are derived analogously to the higher derivatives of vector and ordinary functions.

$$\frac{\delta^2 F[x(s)]}{\delta F[x(t)]^2} :\equiv \frac{F[x(s) + 2 \cdot 0'\delta(s-t)] - 2F[x(s) + 0'\delta(s-t)] + F[x(s)]}{0'^2}$$

$$\frac{\delta^n F[x(s)]}{\delta F[x(t)]^n} :\equiv \frac{\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}F[x(s)+0'k\delta(s-t)]}{0'^n}$$

The second functional derivative also called the *second variation*, etc.

Functional product rule

PRODUCT RULE FOR VECTOR FUNCTIONS:
$$\frac{\partial}{\partial x_i}f(\{x_j\})g(\{x_j\}) = f(\{x_j\})\frac{\partial}{\partial x_i}g(\{x_j\}) + g(\{x_j\})\frac{\partial}{\partial x_i}f(\{x_j\}).$$

PROOF. The proof is essentially the same as for the **ordinary product rule** (p. 40).

$$\frac{\partial}{\partial x_i}f(\{x_j\})g(\{x_j\})$$

$$\equiv \frac{f(\{x_j + 0'\delta_{j,i}\})g(f(\{x_j + 0'\delta_{j,i}\})) - f(\{x_j\})g(\{x_j\})}{0'}$$

$$\equiv \frac{1}{0'}\Big[f(\{x_j\})g(\{x_j\})$$
$$+ f(\{x_j\})\left[g(\{x_j + 0'\delta_{j,i}\}) - g(\{x_j\})\right]$$
$$+ \left[f(\{x_j + 0'\delta_{j,i}\}) - f(\{x_j\})\right]g(\{x_j\})$$
$$+ \left[f(\{x_j + 0'\delta_{j,i}\}) - f(\{x_j\})\right]\left[g(\{x_j + 0'\delta_{j,i}\}) - g(\{x_j\})\right]$$
$$- f(\{x_j\})g(\{x_j\})\Big]$$

$$= \frac{1}{0'}\Big[f(\{x_j\})g(\{x_j\})$$
$$+ f(\{x_j\})\left[g(\{x_j + 0'\delta_{j,i}\}) - g(\{x_j\})\right]$$
$$+ \left[f(\{x_j + 0'\delta_{j,i}\}) - f(\{x_j\})\right]g(\{x_j\})$$
$$- f(\{x_j\})g(\{x_j\})\Big]$$

$$\equiv \frac{f(\{x_j\})\left[g(\{x_j + 0'\delta_{j,i}\}) - g(\{x_j\})\right] + g(\{x_j\})\left[f(\{x_j + 0'\delta_{j,i}\}) - f(\{x_j\})\right]}{0'}$$

$$\equiv f(\{x_j\})\frac{\partial}{\partial x_i}g(\{x_j\}) + g(\{x_j\})\frac{\partial}{\partial x_i}f(\{x_j\}). \ \square$$

PRODUCT RULE FOR FUNCTIONALS:

$$\frac{\delta}{\delta x(t)}F[x(s)]G[x(s)] = F[x(s)]\frac{\delta}{\delta x(t)}G[x(s)] + G[x(s)]\frac{\delta}{\delta x(t)}F[x(s)].$$

PROOF.

$$\frac{\delta}{\delta x(t)}F[x(s)]G[x(s)]$$

$$\equiv \frac{F[x(s) + 0'\delta(s-t)]G[x(s) + 0'\delta(s-t)] - F[x(s)]G[x(s)]}{0'}$$

$$\equiv \frac{1}{0'}\Big[F[x(s)]G[x(s)]$$

$$+ F[x(s)][G[x(s) + 0'\delta(s-t)] - G[x(s)]]$$

$$+ [F[x(s) + 0'\delta(s-t)] - F[x(s)]]\, G[x(s)]$$

$$+ [F[x(s) + 0'\delta(s-t)] - F[x(s)]]\, [G[x(s) + 0'\delta(s-t)] - G[x(s)]]$$

$$- F[x(s)]G[x(s)]\Big]$$

$$= \frac{1}{0'}\Big[F[x(s)]G[x(s)]$$

$$+ F[x(s)]\, [G[x(s) + 0'\delta(s-t)] - G[x(s)]]$$

$$+ [F[x(s) + 0'\delta(s-t)] - F[x(s)]]\, G[x(s)]$$

$$- F[x(s)]G[x(s)]\Big]$$

$$\equiv \frac{1}{0'}\Big[F[x(s)]\, [G[x(s) + 0'\delta(s-t)] - G[x(s)]]$$

$$+ G[x(s)]\, [F[x(s) + 0'\delta(s-t)] - F[x(s)]]\Big]$$

$$\equiv F[x(s)]\frac{\delta}{\delta x(t)}G[x(s)] + G[x(s)]\frac{\delta}{\delta x(t)}F[x(s)]. \ \square$$

Functional power rule

The power rule for a vector function follows easily from the **product rule** (p. 115).

POWER RULE FOR VECTOR FUNCTIONS:
$$\frac{\partial}{\partial x_i} \sum_j x_j^n = nx_i^{n-1}.$$

PROOF.
$$\frac{\partial}{\partial x_i} \sum_j x_j^n \equiv \frac{\sum_j (x_j + 0'\delta_{j,i})^n - \sum_j x_j^n}{0'}$$

$$\equiv \frac{\sum_j \sum_{k=1}^n \binom{n}{k} x_j^{n-k} 0'^k \delta_{j,i}^k}{0'}$$

$$\equiv \sum_j nx_j^{n-1}\delta_{j,i} + \sum_j \sum_{k=2}^n \binom{n}{k} x_j^{n-k} 0'^{k-1} \delta_{j,i}^k$$

$$= \sum_j nx_j^{n-1}\delta_{j,i} \equiv nx_i^{n-1}. \ \square$$

POWER RULE FOR FUNCTIONALS:
$$\frac{\delta}{\delta x(t)} \int x(s)^n ds = nx(t)^{n-1}.$$

PROOF.
$$\frac{\delta}{\delta x(t)} \int x(s)^n ds \equiv \frac{\int [x(s) + 0'\delta(s-t)^n]ds - \int x(s)^n ds}{0'}$$

$$\equiv \frac{\int \sum_{k=1}^n \binom{n}{k} x(s)^{n-k} 0'^k \delta(s-t)^k}{0'}$$

$$\equiv \int nx(s)^{n-1}\delta(s-t)ds + \int \sum_{k=2}^{n} \binom{n}{k} x_j^{n-k} 0'^{k-1}\delta(s-t)^k ds$$

$$= \int nx(s)^{n-1}\delta(s-t)ds \equiv nx(t)^{n-1}. \;\square$$

This proof applies only when n is a positive integer and is analogous to the proof of the ordinary **power rule** (p. 43) for n a positive integer. The other cases of n in that proof also have analogs for functional derivatives, but they are not presented here.

Functional transfer rule

The transfer rules show how the derivatives of certain vector functions and functionals simplify to an ordinary derivative. They make use of the **power rules** (p. 117).

TRANSFER RULE FOR VECTOR FUNCTIONS: If f is analytic,

$$\frac{\partial}{\partial x_i} \sum_j f(x_j) \equiv f'(x_j).$$

PROOF.

$$\frac{\partial}{\partial x_i} \sum_j f(x_j) \equiv \frac{\partial}{\partial x_i} \sum_j \sum_{k=0}^{\infty} a_k x_j^k \equiv \frac{\partial}{\partial x_i} \sum_{k=0}^{\infty} \sum_j a_k x_j^k$$

$$\equiv \sum_{k=0}^{\infty} a_k \frac{\partial}{\partial x_i} \sum_j x_j^k \equiv \sum_{k=1}^{\infty} a_k k x_j^{k-1}$$

$$\equiv \frac{d}{dx_j} f(x_j) \equiv f'(x_j). \;\square$$

TRANSFER RULE FOR FUNCTIONALS: If f is analytic,

$$\frac{\delta}{\delta x(t)} \int f(x(s))ds \equiv f'(x(t)).$$

PROOF.

$$\frac{\delta}{\delta x(t)} \int f(x(s))ds \equiv \frac{\delta}{\delta x(t)} \int \sum_{k=0}^{\infty} a_k x(s)^k ds \equiv \frac{\delta}{\delta x(t)} \sum_{k=0}^{\infty} \int a_k x(s)^k ds$$

$$\equiv \sum_{k=0}^{\infty} a_k \frac{\delta}{\delta x(t)} \int x(s)^k ds \equiv \sum_{k=1}^{\infty} a_k k x(t)^{k-1}$$

$$\equiv \frac{d}{dx(t)} f(x(t)) \equiv f'(x(t)). \ \square$$

Functional chain rule

The **chain rule** (p. 39) for ordinary functions has to be somewhat modified for vector functions and functionals. Since vector functions and functionals reduce the dimensions of their domains, chain rules involving them must start with domains of increased dimensions.

An *indexed vector function* maps a two-dimensional vector (a matrix) to a one-dimensional vector. The indexed vector function $f(\{x_{i,j}\})_j$ is evaluated on the matrix $\{x_{i,j}\}$ over all j, yielding a one-dimensional vector $\{y_i\}$. The indexed function $f(\{x_i\})_i$ is the same as the simple vector function $f(\{x_i\})$, which maps to a scalar.

An *indexed functional* maps a two-argument function to a one-argument function. The indexed functional $F[x(s,t)]_t$ is evaluated over all t, yielding a one-argument function $y(s)$. The indexed functional $F[x(s)]_s$ is the same as the simple functional $F[x(s)]$, which maps to a scalar.

The following are examples of indexed vector functions and functionals and how they can be composed with simple vector functions and functionals:

$$g(\{x_{j,k}\})_j := \left\{ \sum_{j=1}^{m} x_j^2 + y_k^2 \right\}_{k=1}^{n}$$

$$f(\{z_k\}) := \sum_{k=1}^{n} z_k$$

$$f(g(\{x_{j,k}\})_j) = \sum_{k=1}^{n} \sum_{j=1}^{m} x_j^2 + y_k^2$$

$$G[x(s,t)]_s := \int_{1}^{2} [x(s)^2 + y(t)^2]ds$$

$$F[z(t)] := \int_0^1 z(t)\,dt$$

$$F[G[x(s,t)]_s] = \int_0^1 \int_1^2 [x(s)^2 + y(t)^2]\,ds\,dt$$

CHAIN RULE FOR VECTOR FUNCTIONS: Given a simple vector function f and an indexed vector function g,

$$\frac{\partial f(g(\{x_{j,k}\})_j)}{\partial x_i} \equiv \sum_l \frac{\partial f(g(\{x_{j,k}\}_j)}{\partial g(\{x_{j,l}\})_j} \cdot \frac{\partial g(\{x_{j,l}\})}{\partial x_i}.$$

PROOF. Define the following:

$$\{y_n\} := g(\{x_{j,n}\})_j$$

$$\{0_n\} :\equiv g(\{x_{j,n} + 0'\delta_{i,j})_j - g(\{x_{j,n}\})_j.$$

Then

$$\frac{\partial f(g(\{x_{j,k}\})_j)}{\partial x_i} \equiv \frac{f(g(\{x_{j,k} + 0'\delta_{i,j}\})_j) - f(g(\{x_{j,k}\})_j)}{0'}$$

$$\equiv \frac{f(\{y_k + 0_k\}) - f(\{y_k\})}{0'}$$

$$\equiv \frac{\sum_n [f(\{y_k + 0_k\delta_{k,n}\}) - f(\{y_k\})]}{0'}$$

$$\equiv \sum_n \frac{f(\{y_k + 0_k\delta_{k,n}\}) - f(\{y_k\})}{0_n} \cdot \frac{0_n}{0'}$$

$$\equiv \sum_n \frac{\partial f(\{y_k\})}{\partial y_n} \cdot \frac{\partial y_n}{\partial x_i} \equiv \sum_n \frac{\partial f(g(\{x_{j,k}\}_j)}{\partial g(\{x_{j,n}\})_j} \cdot \frac{\partial g(\{x_{j,n}\})}{\partial x_i}. \ \square$$

CHAIN RULE FOR FUNCTIONALS: Given a simple functional F and an indexed functional G,

$$\frac{\delta F[G[x(s,u)]_s]}{\delta x(t)} \equiv \int \frac{\delta F[G[x(s,v)]_s]}{\delta G[x(s,v)]_s} \cdot \frac{\delta G[x(s,v)]_s}{\delta x(t)}\,dv.$$

PROOF. Define the following:

$$y(v) := G[x(s,v)]_s$$

$$0_v :\equiv G[x(s,v) + 0'\delta(s-t)]_s - G[x(s,v)]_s.$$

Then

$$\frac{\delta F[G[x(s,u)]_s]}{\delta x(t)} \equiv \frac{F[G[x(s,u) + 0'\delta(s-t)]_s] - F[G[x(s,u)]_s]}{0'}$$

$$\equiv \frac{F[y(u) + 0_u] - F[y(u)]}{0'}$$

$$\equiv \frac{\int (F[y(u) + 0_u\delta(u-v)] - F[y(u)])\, dv}{0'}$$

$$\equiv \int \frac{F[y(u) + 0_u\delta(u-v)] - F[y(u)]}{0_v} \cdot \frac{0_v}{0'}\, dv$$

$$\equiv \int \frac{\delta F[y(u)]}{\delta y(v)} \cdot \frac{\delta y(v)}{\delta x(t)}\, dv$$

$$\equiv \int \frac{\delta F[G[x(s,v)]_s]}{\delta G[x(s,v)]_s} \cdot \frac{\delta G[x(s,v)]_s}{\delta x(t)}\, dv. \ \square$$

Straight line theorem

A classic example of the application of functional derivatives is a proof that the shortest path between two points is a straight line. The equipoint proof of this theorem is not only simpler than conventional proofs but, since a numeristic function is unrestricted and can be multivalued, covers all possible paths, including the case of a vertical line.

> **STRAIGHT LINE THEOREM: In a plane, the shortest path between two points is a straight line.**

PROOF. We define an arc length functional L on the space of functions $f(X)$ and minimize L using its functional derivative.

$$L[f(X)] := \int_a^b \sqrt{1 + f'(X)^2}\, dX$$

$$\frac{\delta L[f(X)]}{\delta f(x)} = \frac{\delta}{\delta f(x)} \int_a^b \sqrt{1 + f'(X)^2}\, dX$$

$$= \frac{d}{df(x)} \sqrt{1 + f'(x)^2} \quad \text{by \textbf{transfer rule} (p. 118)}$$

$$= \frac{f'(x)}{\sqrt{1 + f'(x)^2}} \frac{d^2 f(x)}{df(x)\, dx}$$

$$= \frac{f''(x)}{\sqrt{1 + f'(x)^2}}$$

The minimum occurs when this last expression is 0. This can happen in two ways:

- When $f''(x) = 0$. In this case, $f(x)$ is a horizontal or oblique straight line of the form $f(x) = mx + b$, where m is finite.

- When $f'(x)$ is infinite and $f''(x)$ is finite. In this case, $f(x)$ is a vertical straight line at a point $x = a$ and infinite valued elsewhere: $f(x) = \infty(x - a)$. □

Functional integration

Like vector and functional derivatives, vector and functional integrals map a vector or functional (respectively) to a scalar. Unlike an ordinary integral, vector and functional integrals are not inverses of their respective derivatives.

To define these integrals, we first look at the special case of a two-dimensional vector. Since functional integrals are definite integrals that integrate over the entire real range, from $-\infty$ to $+\infty$, we examine only vector function integrals of this type. From the **definition of definite integral** (p. 33) and **infinite bounds on integrals and path integral** (p. 34), we have

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x, y) \, dx \, dy := \sum_{m=1}^{\infty'^2} \sum_{k=1}^{\infty'^2} f\left(-\infty' + \frac{2\infty'k}{\infty'^2}, -\infty' + \frac{2\infty'm}{\infty'^2}\right) \frac{4}{\infty'^2}$$

$$\equiv \sum_{k_2=1}^{\infty'^2} \sum_{k_1=1}^{\infty'^2} f\left(\frac{2k_1}{\infty'} - \infty', \frac{2k_2}{\infty'} - \infty'\right) \frac{4}{\infty'^2}.$$

The general *vector function integral*, using **iteration notation** (p. 111), is given by:

$$\int \ldots \int_{-\infty}^{+\infty} f(\{x_j\}) \, dx_1 \ldots dx_n := \underbrace{\int_{-\infty}^{+\infty} \ldots \int_{-\infty}^{+\infty}}_{n} f(\{x_j\}) \, dx_n \ldots dx_2 dx_1$$

$$\equiv \prod_{i=1}^{n} \int_{-\infty}^{+\infty} \langle f(\{x_j\}) \rangle \, dx_i$$

$$\equiv \prod_{i=1}^{n} \sum_{k_i=1}^{\infty'^2} \left\langle f\left(\left\{\frac{2k_i}{\infty'} - \infty'\right\}\right)\right\rangle \frac{2}{\infty'}.$$

Now we adapt this definition to functionals. Conventional analysis usually denotes and defines the *functional integral* as follows:

$$\int F[x(s)]\,\mathfrak{D}x(t) := \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} F[x(s)] \prod_{t} dx(t),$$

which uses an ellipsis for iterated integrals and and the product symbol $\prod$ for a continuous product. The equipoint definition instead uses continuous iteration notation:

$$\int F[x(s)]\,\mathfrak{D}x(t) \equiv \mathscr{J}_{t=-\infty}^{+\infty} \int \langle F[x(s)]\rangle\,\delta x(t)$$

$$\equiv \mathscr{J}_{t=-\infty}^{+\infty} \sum_{k_t=1}^{\infty'^2} \left\langle F\left[x(t) + \frac{2k_t}{\infty'}\right]\right\rangle \frac{2}{\infty'}.$$

For further reading on the conventional theory and applications of the functional integral, the author suggests [K16].

Functional delta function

The functional delta function is the functional analog of the **Dirac delta function** (p. 79). The functional delta function is a functional, not an ordinary function. The integration property is expressed in terms of **functional integration** (p. 122).

First we examine the vector function case of $n = 2$.

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x,y)\delta(x-a)\,\delta(y-b)\,dx\,dy = \int_{-\infty}^{+\infty} f(a,y)\,\delta(y-b)\,dy = f(a,b).$$

This enables us to easily define the general vector delta function in terms of ordinary delta function:

$$\delta(\{x_j\}) := \prod_{j=1}^{n} \delta(x_j) \equiv \prod_{j=1}^{n} \int_{-\infty}^{+\infty} e^{2\pi i x_j y_j}\,dy_j \equiv \underbrace{\int \cdots \int_{-\infty}^{+\infty}}_{n} e^{2\pi i \sum_{k=1}^{n} x_k y_k} \prod_{i=1}^{n} dx_i,$$

which has the property

$$\underbrace{\int \cdots \int_{-\infty}^{+\infty}}_{n} f(\{x_j\})\,\delta(\{x_j\} - \{a_j\}) \prod_{i=1}^{n} dx_i = f(\{a_j\}).$$

The functional delta function is thus defined:

$$\delta[x(s)] := \oint_{s=-\infty}^{+\infty} \delta(x(s)) \equiv \oint_{s=-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{2\pi i s t} \, dt \equiv \int e^{2\pi i \int_{-\infty}^{+\infty} x(s) y(s) ds} \, \mathcal{D}y(s),$$

and has the property

$$\int F[x(s)] \, \delta[x(s) - a(s)] \, \mathcal{D}x(s) = F[a(s)].$$

CLASS COUNTS

Class count comparisons

In set theory, the notation *#C* means the cardinality of the set *C*. Since numeristics does not use the concept of cardinality, we use this notation to mean simply the number of elements in the class *C*, which we call a ***class count***.

We will need to address finite and infinite counts separately. We will avoid saying that a class "is finite" or "is infinite," because such characterizations do not distinguish between the elements and the count. A class could have a finite number of infinite elements, or an infinite number of finite elements, or both finite, or both infinite.

Finite counts

A finite count can obviously be established through a bijection: $\# \pm 1 = 2$, for example, can be established through a bijection from $\mathbb{Z}_2$ to ± 1, such as $n \mapsto (-1)^n$ or $n \mapsto 2n - 1$. For all finite counts, the count is independent of the bijection.

Infinite counts

Of course, the counts of all infinite classes are the same value:
$$\#\mathbb{N} = \#\mathbb{Q} = \#\mathbb{R} = \#\mathbb{C} complex number = \infty.$$

On the other hand, as with other infinite quantities, ratios and other operations between two infinite values may be finite. Much as the value of $\dfrac{dy}{dx}$ depends on the relation between x and y in unfolded space, $\dfrac{\#C}{\#D}$, so $\#C - \#D$, and other such expressions may depend on the relation between the counts of C and D in unfolded space, which in turn may depend on the way that C and D are mapped to each other.

Comparing $\mathbb{N}^*$ and $\mathbb{Z}^*$

As an example, consider $\mathbb{N}$ and $\mathbb{Z}^*$. If we map $\mathbb{Z}^*$ to $\mathbb{N}^*$ with a function f that takes n to $|n|$, then f maps two elements of $\mathbb{Z}^*$ to each element of $\mathbb{N}^*$. If $\infty' :\equiv \#\mathbb{N}^*$, then $\#\mathbb{Z}^*$ established through this map is $2\infty'$, and $\dfrac{\#\mathbb{Z}^*}{\#\mathbb{N}^*} = 2$.

But if we map $\mathbb{Z}^*$ to $\mathbb{N}^*$ with a function g that takes n to $2n$ for positive n and $-2n-1$ for negative n, which maps $\ldots -3, -2, -1, 1, 2, 3 \ldots$ to $5, 3, 1, 2, 4, 6 \ldots$, then $\#\mathbb{Z}^*$ established through this map is ∞', and $\dfrac{\#\mathbb{Z}^*}{\#\mathbb{N}^*} = 1$.

Principles of infinite counts

As the above example shows, if the count of *folded* elements in a class is infinite, then comparing the count to other class counts in *unfolded* space may depend on the map or maps that are used to connect the classes.

Given $f : A \to B$, then $\#A$ is the number of mappings in f, and the relationship of $\#A$ to $\#B$ is determined from the way f maps A to B. We will use the notation $(\# : f)$ before an expression containing such a comparison to show that the classes are connected with the map f. In the above example, $(\# : f)\,\dfrac{\#\mathbb{Z}^*}{\#\mathbb{N}^*} = 2$, and $(\# : g)\,\dfrac{\#\mathbb{Z}^*}{\#\mathbb{N}^*} = 1$.

General rules for comparison maps

In general, for a class count comparison map f:

- If $f : A \to \mathbb{Z}_n$ is bijective, then $(\# : f)\ \#A \equiv n$. In this case, A has a finite count, which is independent of f.

- If $f : A \to B$ is bijective, then $(\# : f)\ \#A =' \#B$.

- If $f : A \to B \cup C$ is bijective and $B \cap C = \emptyset$, then $(\# : f)\ \#A =' \#B + \#C$.

- If $f : A \to (B, C)$ is bijective, then $(\# : f)\ \#A =' \#B\#C$.

The notation (B, C) means the Cartesian product of B and C, $\{(b, c) \mid b \in B \land c \in C\}$. Introduced in [CN], this notation is used to avoid confusion with $B \times C$, which means a product distributed over the elements of B and C, $\{bc \mid b \in B \land c \in C\}$.

- If $f : A \to B^{\swarrow C}$ is bijective, then $(\# : f)\ \#A =' \#B^{\#C}$.

  The notation $B^{\swarrow C}$ means the class of functions from C to B, $\{f \mid f : C \to B\}$. Introduced in [CN], this notation is used to avoid confusion with B^C, which means the elements of C distributing powers over the elements of B, $\{b^c \mid b \in B \land c \in C\}$.

From these, we can derive the following:

- If $f : A \to B$ is surjective and maps n elements of A to one element of B, i.e. $\#f(a) \equiv n$ for each $a \in A$, then $\#A =' n\#B$.

- If $f : A \to B^{\times n}$ is bijective, then $(\# : f)\#A =' \#B^n$.

  The notation $A^{\times n}$ means A extended to n dimensions: $(\underset{1}{A}, \underset{2}{A}, \ldots, \underset{n}{A})$. Introduced in [CN], this notation is used to avoid confusion with A^n, which means a power distributed over the elements of A, $\{a^n \mid a \in A\}$.

- If $f : A \to \mathbb{Z}_2^{\swarrow C}$ is bijective, then $(\# : f)\#A =' 2^{\#C}$. This is the number of subclasses of C, as each subclass corresponds to a selection function that takes C to a class of two logical elements, true and false, which decides whether each element is in the subclass.

Comparing $\mathbb{Z}$ and $\mathbb{Q}$
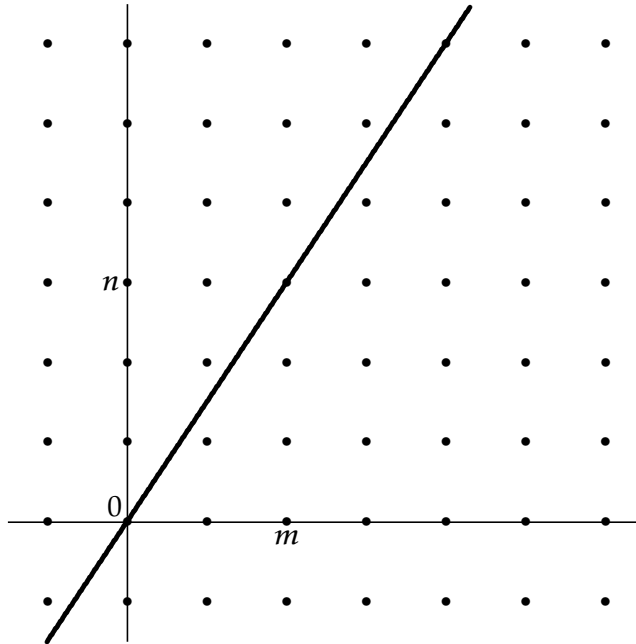
FIG. 42:
Map s from (m, n) to line of slope $\frac{m}{n}$

A class x is ***integrous*** if there is a bijection between the elements of x and some subset of the integers. Classes with a finite number of elements, $\mathbb{N}$, and $\mathbb{Z}$, are all integrous. In set theory, such classes are called ***countable*** or ***denumerable***, but from a numeristic point of view, these terms are misleading, since we can count and compare the number of elements of any class, including the nonintegrous classes $\mathbb{R}$ and $\mathbb{C}$.

Through a well known diagonal technique, it is possible to construct a bijection d^+ between $\mathbb{N}$ and $\mathbb{Q}^+$, and a similar bijection d between $\mathbb{Z}$ and $\mathbb{Q}$. This means that

$$(\# : d^+) \ \#\mathbb{N} =' \mathbb{Q}^+$$
$$(\# : d) \ \#\mathbb{Z} =' \mathbb{Q}$$

and that $\mathbb{Q}^+$ and $\mathbb{Q}$ are integrous. But d is not a very natural map, since among other things, it does not preserve order.

For a more natural map, we define $s : \mathbb{N}^{\times 2} \to \mathbb{Q}^+$. Figure 42 shows this map geometrically. s takes $(m, n) \in \mathbb{N}^{\times 2}$ to the line through (m, n) and the origin, which has slope $\dfrac{n}{m} \in \mathbb{Q}^+$. This is a many-to-one map with duplicates whenever m and n are not coprime (relatively prime).

Cesàro and others (see [H75, thm. 332, p. 269]) have shown that the probability of two random integers being coprime is $\dfrac{1}{\zeta(2)} = \dfrac{6}{\pi^2}$, where ζ is the Riemann zeta function. Therefore

$$(\# : s) \quad \frac{\#\mathbb{Q}^+}{\#\mathbb{N}^{\times 2}} = \frac{6}{\pi^2}$$

$$(\# : s_2) \quad \frac{\#\mathbb{Q}}{\#\mathbb{N}^{\times 2}} = \frac{3}{\pi^2}$$

$$\text{where } s_2 : \mathbb{N}^{\times 2} \to \mathbb{Q}$$

$$(\# : s_3) \quad \frac{\#\mathbb{Q}}{\#\mathbb{Z}^{\times 2}} = \frac{3}{2\pi^2}$$

$$\text{where } s_3 : \mathbb{Z}^{\times 2} \to \mathbb{Q}.$$

Comparing $\mathbb{N}$ and $\mathbb{R}$

We first map $\mathbb{N}$ to the half-open unit interval $I = [0, 1)$ through base two radix representations (base two decimals). The expansion of $r \in I$ consists of a radix (decimal) point followed by an infinite string of binary digits. Such a string can be considered a map $k : \mathbb{N} \to \mathbb{Z}_2$. We define a class K of all possible such k, and then we define a map $j : K \to I, k \mapsto r$.

K has $2^{\#\mathbb{N}}$ elements, each of which maps to a unique r, except for duplicates of the form $0.(digits)0111\ldots = 0.(digits)1000\ldots$, which appear at $\#\mathbb{N}$ unique positions. Hence

$$(\# : j) \quad \#I =' 2^{\#\mathbb{N}} - \#\mathbb{N}.$$

In order to cover the real line, we make $\#\mathbb{Z}$ copies of I, using the map $y : \mathbb{R} \to I, r \mapsto r - \lfloor r \rfloor$. Hence

$$(\# : k, y) \quad \#\mathbb{R} =' \#\mathbb{Z}\#I$$

$$(\# : k, j, y) \quad \#\mathbb{R} =' \#\mathbb{Z} \left[2^{\#\mathbb{N}} - \#\mathbb{N} \right].$$

Letting $\infty' :\equiv \#\mathbb{N}$, we have

$$(\# : k, j, y) \quad \sqrt[\#\mathbb{N}]{\#\mathbb{R}} =' \sqrt[\#\mathbb{N}]{\#\mathbb{Z}} \sqrt[\#\mathbb{N}]{2^{\#\mathbb{N}} - \#\mathbb{N}}$$

$$\equiv (2\infty')^{\frac{1}{\infty'}} \left(2^{\infty'} - \infty'\right)^{\frac{1}{\infty'}}$$

$$\equiv e^{\frac{\ln 2\infty'}{\infty'}} e^{\frac{\ln 2^{\infty'} - \infty'}{\infty'}} \equiv e^{\frac{2}{2\infty'}} e^{\frac{(\ln 2)2^{\infty'} - 1}{2^{\infty'} - \infty'}}$$

$$=' e^{0'} e^{\frac{(\ln 2)^2 2^{\infty'}}{(\ln 2)2^{\infty'} - 1}} =' e^{\frac{(\ln 2)^3 2^{\infty'}}{(\ln 2)^2 2^{\infty'}}} = 2.$$

A similar result holds for any radix. For a general j_b which uses a radix b instead of 2, we obtain

$$(\# : k, j_b, y) \ \sqrt[\#\mathbb{N}]{\#\mathbb{R}} = b.$$

Comparing $\mathbb{R}$ and $\mathbb{C}$

If we define $p : \mathbb{C} \to \mathbb{R}^{\times 2}, a + bi \mapsto (a, b)$, then

$$(\# : p) \ \frac{\#\mathbb{C}}{\#\mathbb{R}^{\times 2}} = 1.$$

Using class counts in derivatives and integrals

We now use class counts and other equipoint arguments to calculate the derivative and integral of the *indicator* or *characteristic function of the rational numbers*:

$$[\mathbb{Q}](x) := \begin{cases} 1 & \text{for } x \in \mathbb{Q} \\ 0 & \text{for } x \notin \mathbb{Q}. \end{cases}$$

First we calculate $\dfrac{\#\mathbb{Q}}{\#\mathbb{R}}$, using the maps s, k, j, and y from the previous section:

$$(\# : s, k, j, y) \ \frac{\#\mathbb{Q}}{\#\mathbb{R}} =' \frac{\frac{3}{\pi^2}\#\mathbb{N}^{\times 2}}{2^{\#\mathbb{N}}}$$

$$\equiv \frac{\frac{3}{\pi^2}\#\mathbb{N}^{\times 2}}{1 + \#\mathbb{N}\ln 2 + \frac{(\#\mathbb{N}\ln 2)^2}{2!} + \frac{(\#\mathbb{N}\ln 2)^3}{3!} + \frac{(\#\mathbb{N}\ln 2)^4}{4!} + \ldots}$$

$$\equiv \frac{\frac{3}{\pi^2}}{\frac{1}{\#\mathbb{N}^{\times 2}} + \frac{\ln 2}{\#\mathbb{N}} + \frac{(\ln 2)^2}{2!} + \frac{\#\mathbb{N}(\ln 2)^3}{3!} + \frac{\#\mathbb{N}^{\times 2}(\ln 2)^4}{4!} + \ldots}$$

$$\equiv \frac{\frac{3}{\pi^2}}{0'^2 + 0'\ln 2 + \frac{(\ln 2)^2}{2!} + \frac{\infty'(\ln 2)^3}{3!} + \frac{\infty'^2(\ln 2)^4}{4!} + \ldots} = 0$$

Next we investigate the continuity of $[\mathbb{Q}](x)$ using the definition above, namely that $[\mathbb{Q}](x)$ is continuous at x when $[\mathbb{Q}](x + 0') = [\mathbb{Q}](x)$.

To compute $[\mathbb{Q}](x + 0')$, pick an unfolded integer ∞' and let $0' :\equiv 10^{-\infty'}$. The ∞'-th digit of the decimal representation of x is the origin of the space unfolded with $0'$. Call this digit d. The unit in this place has the value $0'$.

If x is rational, the decimal preserves the repetend of x, even in the ∞'-th place. Adding the unit $0'$ to x changes d to $d + 1$ for $d < 9$, and 9 to 0 with a finite number of carries, with one exception noted later. With this change of at least one digit, the repetend is broken, and the number is no longer rational. Hence $[\mathbb{Q}](x + 0') = 0 \neq [\mathbb{Q}](x)$, and $[\mathbb{Q}](x)$ is discontinuous at x.

The exception to the above process occurs when the repetend is 9, in which case there are an infinite number of carries. The 9s to the left of the ∞'-th place change to 0, but the digits to the right of this place remain 9. In this case also, the repetend is broken, and $[\mathbb{Q}](x)$ is discontinuous at x.

If x is irrational, the same thing occurs, except that there is never a repetend of 9, because there is never any repetend. So there are at most a finite number of carries, the folded digits are never affected, and $[\mathbb{Q}](x + 0')$ is also irrational at the folded level. Hence $[\mathbb{Q}](x + 0') = 0 = [\mathbb{Q}](x)$, and $[\mathbb{Q}](x)$ is continuous at x.

This differs from the conclusion of conventional analysis, which says that the function is discontinuous everywhere because the limit $\lim_{x \to a}[\mathbb{Q}](x)$ does not exist at any point. In equipoint, while there are an infinite number of discontinuities in any finite interval, the function is continuous at most points, since

$$(\# : s, k, j, y) \ \frac{\#\mathbb{Q}}{\#\mathbb{R}} = 0$$

$$(\# : s, k, j, y) \ \frac{\#(\mathbb{R} \setminus \mathbb{Q})}{\#\mathbb{R}} = 1.$$

We can now compute the derivative of $[\mathbb{Q}](x)$:

$$(\# : s, k, j, y) \ \frac{{}^{0'}d[\mathbb{Q}](x)}{{}^{0'}dx} \equiv \begin{cases} \delta'_{x+0',x} & \text{for } x \in \mathbb{Q} \\ 0 & \text{for } x \notin \mathbb{Q}. \end{cases}$$

$$\equiv \sum_{x \in \mathbb{Q}} \delta'_{\frac{1}{0'^2}}(x).$$

$\int_0^a [\mathbb{Q}](x)\,dx$ is simply the ratio $\frac{\#\mathbb{Q}}{\#\mathbb{R}}$, which we have already seen is 0.

STRUCTURE OF UNFOLDED
REAL NUMBERS

Decimal expansions

Folded real numbers

A folded real number r, identified with a decimal expansion, has the form

$$r = \sum_{m=-\infty}^{+\infty} d_m 10^m, \text{where } d_m \in \mathbb{Z}_{10} = 0, 1, \ldots, 9.$$

For example, for

$$\sqrt{2} = 1.414\ldots$$

we have

$$d_n = 0 \text{ for } n > 0.$$
$$d_0 = 1$$
$$d_{-1} = 4$$
$$d_{-2} = 1$$
$$d_{-3} = 4$$
$$\ldots$$

Conventional analysis allows only a finite number of nonzero digits to the left of the decimal point, i.e. it requires that there be an n such that $d_m = 0$ for all $m > n$. But here we allow an infinite number of such digits and call such a digital representation an *infinite left decimal*. An example of a repeating infinite left decimal is $\ldots 333 = \overline{3}$, which means an infinite number of 3's to the left of the decimal point, just as a repeating right decimal $0.333\ldots = 0.\overline{3}$ means an infinite number of 3's to the right.

Both left and right infinite decimals lead to duplicates, but we do not need uniqueness for this discussion, and allowing both makes the discussion easier.

$\overline{9} = \ldots 999$ and $0.\overline{9} = 0.999\ldots$ are examples of such duplicates, as we will now see. We start by recognizing them as infinite geometric series:

$$0.\overline{9} = 0.999\ldots = \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \ldots$$

$$\overline{9} = \ldots 999 = 9 + 90 + 900 + \ldots.$$

SUM OF AN INFINITE GEOMETRIC SERIES:
$$\sum_{k=n}^{\infty} a^k = a^n + a^{n+1} + a^{n+2} + \ldots = \frac{a^n}{1-a}.$$

PROOF. Let

$$x := \sum_{k=n}^{\infty} a^k = a^n + a^{n+1} + a^{n+2} + \ldots$$

then

$$ax = \sum_{k=n+1}^{\infty} a^k = a^{n+1} + a^{n+2} + a^{n+3} + \ldots$$
$$x - ax = a^n$$
$$x = \frac{a^n}{1-a} \; \square$$

Unlike conventional theories of convergent and divergent series, the numeristic theory of infinite series, *equipoint summation*, developed in [CD], allows the use of ordinary commutative, associative, and distributive properties of addition and multiplication of both convergent and divergent infinite series without any known inconsistencies. Equipoint summation allows the above result for both the convergent ($|a| < 1$) and divergent ($|a| \geq 1$) cases.

Using this result for the above examples, we have:

$$0.\overline{9} = 0.999\ldots = 9\sum_{k=1}^{\infty} 10^{-k} = \frac{\frac{9}{10}}{1 - \frac{1}{10}} = 1$$

$$\overline{9} = \ldots 999 = 9\sum_{k=0}^{\infty} 10^{k} = \frac{9}{1 - 10} = -1$$

$$0.\overline{3} = 0.333\ldots = 3\sum_{k=1}^{\infty} 10^{-k} = \frac{\frac{3}{10}}{1 - \frac{1}{10}} = \frac{1}{3}$$

$$\overline{3} = \ldots 333 = 3\sum_{k=0}^{\infty} 10^{k} = \frac{3}{1 - 10} = -\frac{1}{3}.$$

The numeristic theory of repeating decimals, including infinite left decimals, is developed in detail in [CR].

Unfolded real numbers

In general, an unfolded real number is the sum of one or more of: (1) an infinite number, (2) a perfinite number, and (3) an infinitesimal number.

In the first unfolding, an arbitrary real number r takes this form:

$$r \equiv r_{+1}\,\infty' + r_0 + r_{-1}\,0'$$

$$\equiv \sum_{m=-\infty''}^{+\infty''-1} d_{+1,m}10^m\,\infty' + \sum_{m=-\infty''}^{+\infty''-1} d_{0,m}10^m + \sum_{m=-\infty''}^{+\infty''-1} d_{-1,m}10^m\,0',$$

where $\infty' := \frac{1}{0'}$ and r_{+1}, r_0, r_{-1} are folded real numbers, and we choose ∞'' so that this becomes a single sequence from smallest infinitesimal to largest infinite:

$$10^{-\infty''}\,\infty' \equiv 10^{+\infty''} \quad \text{and} \quad 10^{-\infty''} \equiv 10^{+\infty''}\,0' \equiv 10^{+\infty''}\,\infty'^{-1}$$

$$\infty' \equiv 10^{2\infty''}$$

$$\infty'' \equiv \frac{1}{2}\log_{10}\infty' \equiv -\frac{1}{2}\log_{10}0'.$$

Then

$$r \equiv \sum_{m=-\infty''}^{+\infty''-1} d_{+1,m}10^{2\infty''+m} + d_{0,m}10^m + d_{-1,m}10^{-2\infty''+m}$$

$$\equiv \sum_{m=-3\infty''}^{3\infty''-1} d_m 10^m$$

where $d_m = \begin{cases} d_{+1,m-2\infty''} & \text{for } +\infty'' \le m \le +3\infty''-1 \\ d_{0,m} & \text{for } -\infty'' \le m \le +\infty''-1 \\ d_{-1,m+2\infty''} & \text{for } -3\infty'' \le m \le -\infty''-1 \end{cases}$

or $d_m = d_{k,m-2k\infty''}$ for $(2k-1)\infty'' \le m \le (2k+1)\infty''-1$ and $k = +1, 0, -1$.

In the second unfolding,

$$r \equiv r_{+2}\,\infty'^2 + r_{+1}\,\infty' + r_0 + r_{-1}\,0' + r_{-2}\,0'^2$$

$$\equiv \sum_{m=-\infty''}^{+\infty''-1} d_{+2,m}10^m\,\infty'^2 + d_{+1,m}10^m\,\infty' + d_{0,m}10^m + d_{-1,m}10^m\,0' + d_{-2,m}10^m\,0'^2$$

$$\equiv \sum_{m=-\infty''}^{+\infty''-1} d_{+2,m}10^{4\infty''+m} + d_{+1,m}10^{2\infty''+m} + d_{0,m}10^m + d_{-1,m}10^{-2\infty''+m} + d_{-2,m}10^{-4\infty''+m}$$

$$\equiv \sum_{m=-\infty''}^{+\infty''-1} \sum_{k=-2}^{+2} d_{k,m} 10^{2k\infty''+m}$$

$$\equiv \sum_{m=-5\infty''}^{+5\infty''-1} d_m 10^m.$$

In the *n*-th unfolding,

$$r \equiv \sum_{m=-\infty''}^{+\infty''-1} \sum_{k=-n}^{+n} d_{k,m} 10^{2k\infty''+m}$$

$$\equiv \sum_{m=-(2n+1)\infty''}^{+(2n+1)\infty''-1} d_m 10^m.$$

In the ultimate unfolding, $n = \infty$.

Successor operation

The folded real numbers $\mathbb{R}$ are defined from the rational numbers $\mathbb{Q}$, which in turn are defined from the integers $\mathbb{Z}$, and these are defined from the natural numbers $\mathbb{N}$. The natural numbers are often defined with the *successor function* $S(n) := n + 1$. The whole structure of the real numbers can thus be built on the foundation of the successor function.

The unfolded real numbers can be built on nearly the same foundation, namely an unfolded successor function that distinguishes n from $n+1$ for both finite and infinite n. This leads immediately to the first unfolding, but, as we have seen above, all unfoldings following from the first.
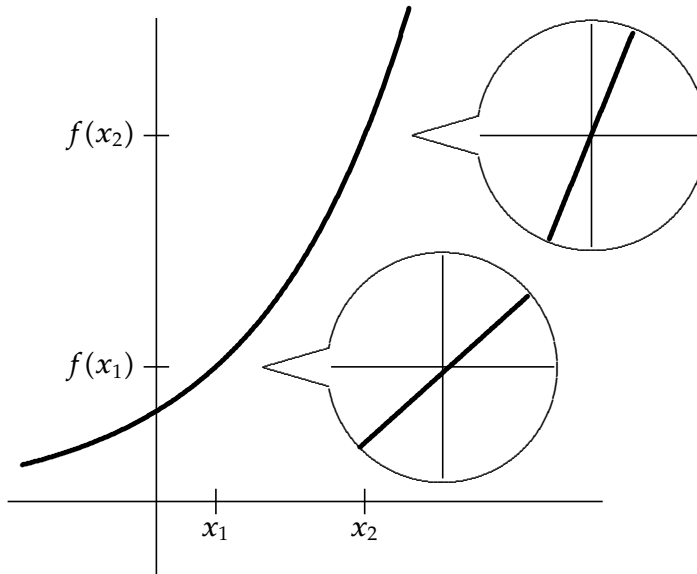
Limitation of microscopes

FIG. 43:
Microscopes of unfolded space
at two points in the graph of $f(x)$

Figure 43 shows typical microscope views of unfolded space within two points in the graph of $f(x)$. We have used this type of microscope to **calculate the first derivative** (p. 31). As long as we are not looking at some kind of singularity, a function $f(x)$ appears curved outside the microscope but straight within it. We used the fact of straighness within an unfolded space to draw an analogy to the slope of a straight line in folded space and thus calculate the derivative.

The straightness of this line can be misleading. In folded space, the derivative of a straight line is constant and the second derivative is zero. This could lead us to think that the second derivative of a curve such as the one in Figure 43 is constant and the third derivative is zero. In fact, the slope of this curve deviates infinitesimally from a straight line, but this is not visible at infinite magnification. By the time the curve moves from x_1 to x_2, for instance, this infinitesmial deviation adds up to a clearly nonzero change in slope.

CONVERGENT SERIES PARADOX

In equipoint analysis, infinite series and definite integrals are both simple sums with an infinite number of terms. In a definite integral, all the terms are zero, while in an infinte series, an infinite number of terms are nonzero. When the terms can be directly compared, this may lead to a paradoxical condition wherein both a series and an integral yield a finite result. For example, consider that

$$\sum_{n=1}^{\infty} 2^{-n} = 1 < \int_0^1 2\,dx = \sum_{n=1}^{\infty'} \frac{2}{\infty'} = 2,$$

even though, if we look at individual terms,

$$2^{-n} \geq \frac{2}{\infty}$$

for all n, with equality holding only for infinite n.

In the numerisitic theory of infinite series, we find that convergent series such as the one above actually have *two* values, one finite and one infinite. The infinite value arises when we consider the series to be an infinite sum of strictly positive values, and the finite value comes from the identification of $+\infty$ and $-\infty$ in the projectively extended real numbers. This is explained in detail in [CD] and [CR].

QUANTUM RENORMALIZATION

Renormalization is a procedure used in quantum physics to "tame" infinities that occur in quantum formulas. The correctness of values derived through renormalization is well verified experimentally, but the mathematics of this procedure is poorly understood, and therefore its theoretical validity is controversial.

Equipoint analysis should improve the understanding of quantum renormalization. The following example may show this. Although realistic examples of quantum renormalization usually involve very difficult formulas, we use here a very simplified example given by Klauber [K14, p. 305]. The problem is to evaluate

$$\int_{-\infty}^{\infty} x^2\,dx.$$

In conventional analysis, this evaluation is done in two main steps, regularization and renormalization. In this example, the regularized form is

$$\lim_{\Lambda \to \infty} \int_{-\Lambda}^{\Lambda} x^2\,dx.$$

This still diverges, so we renormalize by multiplying by the factor $\frac{1}{\Lambda^3}$. Then we have

$$\lim_{\Lambda \to \infty} \frac{1}{\Lambda^3} \int_{-\Lambda}^{\Lambda} x^2\,dx = \frac{2}{3}.$$

In equipoint analysis, we avoid limits and directly write

$$\frac{1}{\infty'^3} \int_{-\infty'}^{\infty'} x^2\,dx = \frac{2}{3}.$$

We can also separate the factor $\frac{1}{\infty'^3}$ from the integral and evaluate these expressions separately as infinite quantities.

See also [T99, ch. 18] and [D04] for elementary introductions to quantum renormalization.

APPENDIX: COMPARISON
OF EQUIPOINT WITH OTHER
THEORIES OF ANALYSIS

Comparison of equipoint and conventional analysis

Differences

The differences between equipoint analysis and conventional limit-based analysis have been observed throughout this document. Here we summarize these differences.

Conventional analysis	Equipoint analysis
• There are no infinite values.	• There are one or more infinite numeric values at the folded level.
• Every expression is single valued.	• An expression can represent a single value or a multivalued numeristic class.
• As a result of the previous two points, some operations are left undefined or regarded as meaningless.	• As a result of the previous two points, all operations are defined. An expression which is syntactically correct has a value and is never regarded as meaningless.
• Infinite and infinitesimal quantities are handled indirectly through limits.	• Infinite and infinitesimal quantities are handled directly through an extended multiple-level number system.
• Many proofs of simple results are difficult.	• Many proofs are short and easy.

- The Leibnitz derivative and Riemann integral cannot be used in some cases, giving rise to the need for constructs such as the Lebesgue integral.

- The Leibnitz derivative and Riemann integral suffice for any real or complex function.

Examples

Below are definitions and examples of the derivative and definite integral in coventional limit-based analysis. For equipoint equivalents, see **Definition of derivative** (p. 31) and **Definition of definite integral** (p. 33) above.

Conventional definition of derivative:
$$\frac{df(x)}{dx} = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}.$$

Sample application of this definition:
$$\begin{aligned}
\frac{dx^2}{dx} &= \lim_{h \to 0} \frac{(x + h)^2 - x^2}{h} \\
&= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} \\
&= \lim_{h \to 0} \frac{2xh + h^2}{h} \\
&= \lim_{h \to 0} (2x + h) \\
&= 2x.
\end{aligned}$$

Conventional definition of definite integral:
$$\int_a^b f(x)\, dx = \lim_{N \to \infty} \sum_{k=1}^{N} f\left(a + \frac{k(b - a)}{N}\right) \frac{b - a}{N}.$$

Sample application of this definition:
$$\begin{aligned}
\int_0^u 2x\, dx &= \lim_{N \to \infty} \sum_{k=1}^{N} 2\frac{ku}{N}\frac{u}{N} \\
&= \lim_{N \to \infty} \frac{2u^2}{N^2} \sum_{k=1}^{N} k \\
&= \lim_{N \to \infty} \frac{2u^2}{N^2} N \frac{N + 1}{2}
\end{aligned}$$

$$= \lim_{N \to \infty} u^2 \left(1 + \frac{1}{N} \right)$$

$$= u^2.$$

Comparison of equipoint and nonstandard analysis

Similarities

Equipoint analysis has much in common with nonstandard analysis and has borrowed many of its concepts.

Nonstandard analysis	Equipoint analysis
• Nonstandard values around a standard point.	• Unfolded values within a folded point.
• Nonstandard infinite values which are reciprocals of nonstandard infinitesimal values.	• Unfolded infinite values which are reciprocals of unfolded infinitesimal values.
• Microscope diagram of nonstandard values.	• Microscope diagram of unfolded values.
• Equality relation =.	• Equivalence relation ≡.
• Approximate equality relation ≈.	• Folded equality relation =.

Differences

Nonstandard analysis	Equipoint analysis
• Set theoretic foundation.	• Number based foundation.
• Every expression is single valued.	• An expression can represent a single value or a multivalued numeristic class.

- Two levels of sensitivity: standard and nonstandard.

- There are no standard infinite values.

- Division by zero is not allowed.

- Some operations are left undefined or regarded as meaningless.

- Conversion from nonstandard to standard is handled by the standard part function.

- The standard part of nonstandard infinite values is undefined.

- Two infinite values are approximately equal only if their difference is infinitesimal. Thus for an infinite H, $H \not\approx H + 1$.

- The function $\dfrac{1}{x}$ is not continuous at 0.

- The function $2x$ is not continuous at infinite values.

- Infinite number of levels of sensitivity: one folded, and the rest unfolded.

- There are one or more infinite numeric values at the folded level.

- One or more folded infinite values and multivalued expressions allow division by zero.

- All operations are defined.

- Conversion from unfolded to folded is handled by the equality relation.

- Unfolded infinite values are folded into an infinite value.

- Two infinite values are equal if they fold to the same infinite element. Thus, for an infinite a, $a = a + 1$.

- The function $\dfrac{1}{x}$ is continuous at 0 on at least one side.

- The function $2x$ is continuous at infinite values.

For source material on nonstandard analysis, see [R74], [KE] and [KF].

Examples

Below are definitions and examples of the derivative and definite integral in nonstandard analysis. For equipoint equivalents, see **Definition of derivative** (p. 31) and **Definition of definite integral** (p. 33) above.

Nonstandard definition of derivative:

$$\frac{df(x)}{dx} = \text{st}\left(\frac{f(x+\varepsilon) - f(x)}{\varepsilon}\right),$$

where ε is an infinitesimal, which in nonstandard analysis is nonzero but smaller than all nonzero reals, and st() is the standard part function, which maps a number of the form $a + \varepsilon$ to a, where a is real.

Sample application of this definition:

$$\frac{dx^2}{dx} = \text{st}\left(\frac{(x+\varepsilon)^2 - x^2}{\varepsilon}\right)$$

$$= \text{st}\left(\frac{x^2 + 2x\varepsilon + \varepsilon^2 - x^2}{\varepsilon}\right)$$

$$= \text{st}\left(\frac{2x\varepsilon + \varepsilon^2}{\varepsilon}\right)$$

$$= \text{st}(2x + \varepsilon)$$

$$= 2x.$$

Nonstandard definition of definite integral:

$$\int_a^b f(x)\,dx = \text{st}\left(\sum_{k=1}^{H} f\left(a + \frac{k(b-a)}{H}\right)\frac{b-a}{H}\right),$$

where H is an infinite number, the reciprocal of an infinitesimal.

Sample application of this definition:

$$\int_0^u 2x\,dx = \text{st}\left(\sum_{k=1}^{H} 2\frac{ku}{H}\frac{u}{H}\right)$$

$$= \text{st}\left(\frac{2u^2}{H^2}\sum_{k=1}^{H} k\right)$$

$$= \text{st}\left(\frac{2u^2}{H^2}H\frac{H+1}{2}\right)$$

$$= \text{st}\left(u^2\left(1 + \frac{1}{H}\right)\right)$$

$$= \text{st}\left(u^2(1 + \varepsilon)\right)$$

$$= u^2.$$

Stroyan's uniform derivative

Stroyan's system of analysis is a variant of nonstandard analysis. One of its main features is the *uniform derivative*, which is defined over an interval and contrasts with the usual nonstandard derivative, which Stroyan calls the *pointwise derivative*.

Stroyan gives several equivalent definitions of the uniform derivative, the first of which is as follows. A real function $f(x)$ has a derivative $f'(x)$ on the interval (a, b) iff for every hyperreal x such that $a < x < b$, $x \not\approx a$, $x \not\approx b$, and $\delta x \approx 0$,

$$f(x + \delta x) - f(x) = f'(x)\delta x + \varepsilon \cdot \delta x$$

for some $\varepsilon \approx 0$ [S97, p. 54].

Comparison of equipoint and Fermat's adequality

Fermat's adequality is one of several 17th century antecedents to calculus. Other systems were developed by Cavalieri and Wallis, but Fermat's was the first known general method for determining extrema.

Fermat described his method in a manuscript written about 1636 and published in 1679 [Fe79]. Both Newton and Leibnitz acknowledged Fermat's adequality as an antecedent of their own work.

Fermat said that adequality derived from a technique used by the Greek mathematician Diophantus, who called it παρισότης *parisotēs,* but Diophantus used this word only to mean approximation.

Similarities

Fermat adequality	Equipoint analysis
• Uses the adequality relation to convert a finite difference into an instantaneous difference.	• Uses the equality relation to convert an unfolded quantity into a folded quantity.

• Uses a term e which the adequality relation discards when added to another quantity.	• Uses an unfolded element $0'$ which the equality relation discards when added to another quantity.

Differences

Fermat adequality	Equipoint analysis
• Used only for computing maxima and minima.	• Used for a wide variety of applications.
• Suitable only for polynomial functions.	• Suitable for any type of function.
• Does not define a derivative or integral but uses a procedure which is algebraically equivalent to computing a derivative and setting it to zero.	• Defines a derivative and an integral which are used in many different ways.
• Adquality is stated in formal terms with little justification. The nature of adequality and the reason it discards the special term e are not made clear.	• Sensitivity and multilevel equality are defined and explained in both formal and informal terms which make its nature and application clear.

Outline and example

Fermat's method for finding a maximum or minimum of $f(x)$:

1. Ad-equate $f(x)$ and $f(x + e)$.

2. Simplify as you would with equality, including dividing both sides by e, until there is at least one term that does not contain e.

3. Discard terms containing e.

4. Convert the adequation to an equation.

Fermat gave the following example to find the maximum of $bx - x^2$. The adequality relation is denoted $\backsim$.

$$bx - x^2 \backsim b(x + e) - (x + e)^2 = bx + be - x^2 - 2xe + e^2$$
$$0 \backsim be - 2xe + e^2$$
$$be \backsim 2xe - e^2$$
$$b \backsim 2x - e$$
$$b = 2x.$$

Comparison of equipoint and ultrasmall/relative analysis

Similarities

Relative analysis	Equipoint analysis
• Multiple layers of numbers.	• Multiple levels of sensitivity.
• Ultrasmall and ultralarge numbers.	• Unfolded infinitesimal and infinite numbers.
• Equalityis relative to a given level, a proper class often denoted V, and the equality denoted $\simeq_V$.	• Equality is relative to a given level, an infinitesimal often denoted $0'$, and the equality denoted $='$.
• Variables "appear" at certain levels.	• Values become distinguishable at certain levels.

Differences

	Relative analysis	**Equipoint analysis**

- Set theoretic foundation, to which is added a new set theoretic relation which is used to build set theoretic proper classes of numbers, which are formed into layers.

- Number based foundation, to which is added the principle of unfolding a point into layers of sensitivity.

- Every expression is single valued.

- An expression can represent a single value or a multivalued numeristic class.

- There are no real infinite values.

- There are one or more infinite real values at the folded level.

- Division by zerois not allowed.

- One or more folded infinite values and multivalued expressions allow division by zero.

- Some operations are left undefined or regarded as meaningless.

- All operations are defined.

For source material on relative analysis, see [H10], [OD09], and [OD11].

Examples

Relative analysis definition of derivative:

$$f'(x) := n\left(\frac{f(x+h) - f(x)}{h}\right),$$

where $n(x)$ or $n_V(x)$ denotes the ***neighbor*** of x, the unique real number ultraclose to x at level V.

Relative analysis definition of definite integral:

$$\int_a^b f(x)\,dx := n\left(\sum_{i=0}^{N-1} f(x_i)h\right),$$

for h ultrasmall and N ultralarge, $h := \frac{b-a}{N}$ and $x_i := a + ih$.

For equipoint equivalents, see **Equipoint definition of derivative** (p. 31) and **Equipoint definition of definite integral** (p. 33).

Comparison of equipoint and smooth infinitesimal analysis

Similarities

Smooth infinitesimal analysis	Equipoint analysis
• Simple algebra of infinitesmial operations.	• Simple algebra of infinitesmial operations.

Differences

Smooth infinitesimal analysis	Equipoint analysis
• Intuitionistic logic.	• Classical logic.
• A single level of equality.	• Multiple levels of equality.
• Every expression is single valued.	• An expression can represent a single value or a multivalued numeristic class.
• There are no infinite values.	• There are one or more infinite values at folded and unfolded levels.
• Division by zero or infinitesimals is not allowed.	• Division by zero and infinitesimals is allowed.
• Some operations are left undefined or regarded as meaningless.	• All operations are defined.
• Indirect definition of derivative and integral.	• Direct definition of derivative and integral.

- The square of an infinitesimal ε is 0.

- An infinitesimal $0'$ has an infinite number of powers which are distinguishable at various levels of sensitivity.

For source material on smooth infinitesimal analysis, see [BI], [BP], and [La].

Smooth infinitesimal definitions

Smooth infinitesimal analysis has two postulates for infinitesimals:

$$(\exists! D)(\forall \varepsilon) f(\varepsilon) = f(0) + D$$
$$[(\forall \varepsilon) \varepsilon a = \varepsilon b] \Rightarrow a = b$$

Using intuitionistic logic, these postulates imply $\varepsilon \neq 0$ and $\neg\, (\varepsilon \neq 0)$, but the second is not equivalent to $\varepsilon = 0$, i.e. ε equality does not obey the law of excluded middle. These postulates also imply $\varepsilon^2 = 0$, i.e. infinitesimals are nilpotent.

This approach leads to formulae such as

$$f(x + \varepsilon) - f(x) = \varepsilon f'(x)$$

as the definition of derivative, which is only implicit and has to be proved to exist. Since there is no division by infinitesimals, we cannot use notation such as $\dfrac{dy}{dx}$ or $f'(x) = \dfrac{f(x + \varepsilon) - f(x)}{\varepsilon}$.

Integration is even less direct, being given not by a formula but by an **Integration Principle**: Given a smooth function $f : [0, 1] \to \mathbb{R}$, there exists a unique smooth function $g : [0, 1] \to \mathbb{R}$, such that $g' = f$ and $g(0) = 0$.

The equipoint equivalents of these definitions are explicit formulas given in **Definition of derivative** (p. 31) and **Definition of definite integral** (p. 33).

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