EQUIPOINT ANALYSIS

A Numeristic Approach to Calculus

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Non-standardanalysis frequently simplifies substantially the proofs, not only of elementary theorems, but also of deep results. This is true, e.g., also for the proof of the existence of invariant subspaces for compact operators, disregarding the improvement of the result; and it is true in an even higher degree in other cases. This state of affairs should prevent a rather common misinterpretation of non-standard analysis, namely the idea that it is some kind of extravagance or fad of mathematical logicians. Nothing could be farther from the truth. Rather, there are good reasons to believe that non-standard analysis, in some version or other, will be the analysis of the future.—Kurt Gödel, [G74]

And there is every reason to believe that the codification of intuitive concepts and the reinterpretation of accepted principles will continue also in future and will bring new advances, into territory still uncharted.—Abraham Robinson, [R68]

[Srinivasa Ramanujan] sometimes spoke of 'zero' as the symbol of the Absolute (Nirguna Brahman) of the extreme monistic school of Hindu Philosophy, that is, the reality to which no qualities can be attributed, which cannot be defined or described by words and is completely beyond the reach of the human mind; according to Ramanujan, the appropriate symbol was the number 'zero', which is the absolute negation of all attributes. He looked on the number 'infinity' as the totality of all possibilities which was capable of becoming manifest in reality and which was inexhaustible. According to Ramanujan, the product of infinity and zero would supply the whole set of finite numbers. Each act of creation, as far as I could understand, could be symbolized as a particular product of infinity and zero, and from each such product would emerge a particular individual of which the appropriate symbol was a particular finite number. . . . He spoke with such enthusiasm about the philosophical questions that sometimes I felt he would have been better pleased to have succeeded in establishing his philosophical theories than in supplying rigorous proofs of his mathematical conjectures.—P. C. Mahalanobis, [Mn]

Unlike my colleagues, I think that an attempt to reconsider the idea of an infinitesimal as a variable finite quantity is fully scientific, and that the proposal to replace variable infinitesimals by fixed ones, far from having purely pedagogical significance, has in its favor something immeasurably deeper, and that this idea is growing roots in modern analysis. ...

I have a clear recollection of my ideas on infinitesimal analysis. I was a second-year student. When the professors announced that $\frac{dy}{dx}$ is the limit of a ratio, I thought: "What a bore! Strange and incomprehensible. No! They won't fool me: it's simply the ratio of infinitesimals, nothing else." ...

Imagine what would happen in physics if physicists held on to earlier view on atoms, i.e. if they imagined a small sphere, a little ball of matter covered with a shell. ... It is a sobering thought that if we had adhered to tradition we would not have modern quanta! *It is much the same in mathematics.* ... I cannot but see a stark contradiction between the intuitively clear fundamental formulas of the integral calculus and the incomparably artificial and complex work of their "justification" and their "proofs". —N. N. Luzin, [Lu] (emphasis in the original)

I still remember the sight of [my high school caclculus teacher] standing in front of the blackboard w[h]ere she had drawn a wonderfully smooth parabola, inserting a secant and telling us that $\Delta y/\Delta x$ is its slope, until finally she convinced us that the slope of the tangent is dy/dx where dx is infinitesimally small and dy accordingly. ... This, I admit, impressed me deeply. Until then our school Math had consisted largely of Euclidean geometry, with so many problems of constructing triangles from some given data. This was o.k. but in the long run that stuff did not strike me as more than boring exercises. But now, with those infinitesimals, Math seemed to have more interesting things in stock than I had met so far. ... [However, at the university,] we were told to my disappointment that my Math teacher had not been up to date after all. We were warned to beware of infinitesimals since they do not exist, and in any case they lead to contradictions. Instead, although one writes dy/dx ..., this does not really mean a quotient of two entities, but it should be interpreted as a symbolic notation only, namely the limit of the quotient $\Delta y/\Delta x$. I survived this disappointment too. ... [Later,] when I learned about Robinson's infinitesimals, my early school day experiences came to my mind again and I wondered whether that lady teacher had not been so wrong after all. The discussion with Abraham Robinson kindled my interest and I wished to know more about it. Some time later there arose the opportunity to invite him to visit us in Germany where he have lectures on his ideas, first in Tübingen and later in Heidelberg, after I had moved there.—P. Roquette, [R10]

अणोरणीयान् महतो महीयान् आत्मास्य जन्तोर्निहितो गुहायाम् ॥
Aṇoraṇīyān mahato mahīyān ātmāsya jantornihito guhāyām.
The Self is smaller than the smallest, bigger than the biggest, and is hidden in a secret place of all creatures.—Katha Upanishad 2.20

यथा पिण्डे तथा ब्रह्माण्डे ॥
Yathā piṇḍe tathā brahmāṇḍe.
As is the point, so is the infinite.—Charaka Samhita

SUMMARY

This part of the book extends the concepts of **numeristics** (p. 31–125) to analysis. Here a theory of analysis is developed, based on infinitesimals which are all equal to zero, and infinite values that are their reciprocals.

Fundamental concepts derive from Maharishi Mahesh Yogi's Vedic Mathematics, Charles Musès's analysis of zero and infinity, and Abraham Robinson's nonstandard analysis. This theory uses multiple levels of unfolding to extend real and complex arithmetic and evaluate equality. It then defines derivatives and integrals solely in terms of elementary arithmetic operations in this extended arithmetic.

Topics include:

- **Unfolding** (p. 138), including multilevel numbers, functions, and relations.
- **The fundamental theorems of calculus** (p. 168).
- **Chain rule** (p. 171), **product rule** (p. 172), derivatives of **trigonometric** (p. 177) and **exponential** (p. 179) functions.
- **Limit** (p. 185) defined in terms of unfolding levels, and **continuity** (p. 187) in terms of these limits.
- **The natural logarithm developed as a polynomial** (p. 180) in the extended arithmetic.
- **Singularities** (p. 203): jump singularities, removable singularities, poles, essential singularities.
- **Complex analysis** (p. 234): complex derivative, Cauchy integral formula, Taylor and Laurent series, complex poles, complex essential singularities.
- **Calculus of variations** (p. 249): functional derivative, product rule, chain rule, transfer rule, application to straight line theorem.

An **appendix** (p. 288) compares equipoint analysis to other theories of analysis: conventional analysis, nonstandard analysis, relative analysis, and smooth infinitesimal analysis.

HOW TO USE THIS PART

This is not a textbook. This part of the book describes a new system of calculus and analysis, equipoint analysis, and shows the differences between it and other systems of calculus and analysis. This part should therefore be used as a supplement to other mathematical texts at that level.

At a minimum, this text assumes familiarity with calculus. Some material is aimed at a more advanced level, such as complex analysis and functionals. Those who are not familiar with these areas can skip these sections.

To understand equipoint analysis, it is essential to understand its total, unrestricted arithmetic and the refinement of this arithmetic by expanding a point to a space.

- The unrestricted arithmetic of total elementary functions is developed in the chapters on **Classes** (p. 53) and **Infinity and infinite element extensions** (p. 64).

- The refinement of this arithmetic is covered in the chapter on **unfolding** (p. 138).

- The refined arithmetic is then used to **redefine the derivative and integral** (p. 163).

Equipoint analysis is also used in an alternative theory of divergent series, described in the third part of this book, **Divergent Series** (p. 301–409), and this alternative theory is applied to repeating decimals in the fourth part, **Repeating Decimals** (p. 409–457).

NON-CONVENTIONAL THEORIES OF ANALYSIS

The conventional theory of analysis, based on set theory and limits, was first developed in the 19th century. Since 1960, the following theories of analysis have emerged as alternatives to classical analysis. Here we briefly describe the history of these theories. See the **appendix** (p. 288) for a more detailed description and comparison to equipoint analysis.

Nonstandard Analysis

Nonstandard analysis has its roots in the original development of calculus in terms of infinitesimals by Leibnitz in the 17th century. In the intervening centuries, calculus was found to be very useful, but the explanation of it in terms of infinitesimals did not satisfy very many mathematicians. With the increasing demand for rigor in the 19th century, the theory of infinitesimals was replaced by classical limits-based analysis.

In 1960, Abraham Robinson resurrected the theory of infinitesimals by developing it as a modern set theoretic system he called ***nonstandard analysis*** [R74]. Jerome Keisler used the principles of nonstandard analysis in his elementary calculus textbook [KE] and undergraduate analysis text [KF]. Nonstandard analysis is widely considered to be a significantly simpler and more elegant system than classical analysis, yet in the more than 50 years since its introduction, it has not achieved widespread use, either in teaching or in research.

Relative Analysis

More recently, ***relative analysis*** was developed by Karel Hrbáček, Oliver Lessman, and Richard O'Donovan [H10], and used by O'Donovan in high school instruction [OD09]. This theory uses the terms *ultrasmall* and *ultralarge*, whereas *infinitesimal* and *infinite* are used in nonstandard analysis. Like nonstandard analysis, relative analysis has not achieved widespread use.

Smooth Infinitesimal Analysis

Smooth infinitesimal analysis was developed by John L. Bell, in [BI] and [BP], as a branch of synthetic differential geometry. It was originally developed by F. William Lawvere from category theory starting in 1967, but it remained obscure until Lawvere's 1998 article [La]. Like other alternatives, smooth infinitesimal analysis has not achieved widespread use.

ORIGIN OF EQUIPOINT ARITHMETIC

As explained in **Foundations of numeristics** (p. 48), numeristics is based on the infinite and the experience of the silent, unmanifest point of infinity, *samādhi* or zero. In numeristics this is conceptualized to give an arithmetic of 0 and ∞, including total, unrestricted multiplication and division by these quanities, such as $\frac{1}{0}$, $\frac{0}{0}$, and $\infty + 1$.

Some of these unrestricted operations, including $\frac{0}{0}$, $\infty - \infty$, and $\infty \cdot 0$, give rise to indeterminate expressions. Numeristics gives a value to these expressions, the ***full class*** ($\not{\infty}$), a class which includes all numeric values.

In some cases, numeristic arithmetic yields an inderminate expression in response to a question which clearly has a determinate result. One example is the calculation of the slope of the tangent to a curve $y = f(x)$ at a point a. Numeristic arithmetic alone yields the result $\frac{f(x) - f(x)}{0} = \frac{0}{0} = \not{\infty}$. In such cases, numeristic arithmetic needs to be refined to yield a determinate result. This need is called the ***principle of determinacy***, and it is implemented through equipoint arithmetic.

In this and similar works, the principle of determinacy is used in the following:

- **Derivatives** (p. 163).
- **Integrals** (p. 165).
- **Offset derivatives** (p. 203).
- **Class count comparisons** (p. 274).
- Divergent series in **Divergent Series** (p. 301–409).
- Infinite left decimals in **Infinite left decimals** (p. 434).

In conventional analysis, determinacy is handled through limits, but limits have limitations which equipoint analysis overcomes, chiefly because a limit considers only at values *around* a point, whereas equipoint arithmetic considers what happens *at* and even *within* a point. Moreover, a limit at infinity looks only at finite values, since the infinite is not even accorded the status of a number in conventional analysis, whereas equipoint arithmetic considers infinity to consist of one or more numeric values.

The existence of infinitesimals is suggested by dual decimal representations of some real numbers, such as 1.000... and 0.999.... In numeristic arithmetic, each of these pairs has an identical value, yet they each suggest an infinitesimal difference.

Numeristics starts with the experience of infinity and zero as the point of infinity. Equipoint arithmetic extends with the experience of the point opening up into a vast inner space distinct from and much richer than ordinary space, and the contraction of this space back into ordinary space.

Equipoint analysis conceptualizes some aspects of the expanded space by considering it as a space of points which all have the ordinary single value of zero from the perspective of ordinary space, but which form a class of distinct zeros from the perspective of the expanded space.

This allows us to do calculus with infinitesimals that are exactly zero, and with numeric infinites that are reciprocals of these zeros. In the next chapter, we formalize these two perspectives with ***unfolding***.

UNFOLDING

Unfolding zero

In equipoint analysis, every number can be ***unfolded*** into a space of distinct numbers, and this space can be ***folded*** back to a single number.

Ordinary real numbers, integers, rational numbers, etc. are completely folded and are called ***folded numbers***. Our first example of unfolding with be the unfolding of zero.

When, as described in the previous chapter, zero opens up into a space of zeros, these zeros are all exactly equal to zero at the folded level, but when zero is unfolded, zero is multivalued, and the individual zeros are distinct elements. This is shown pictorially below.

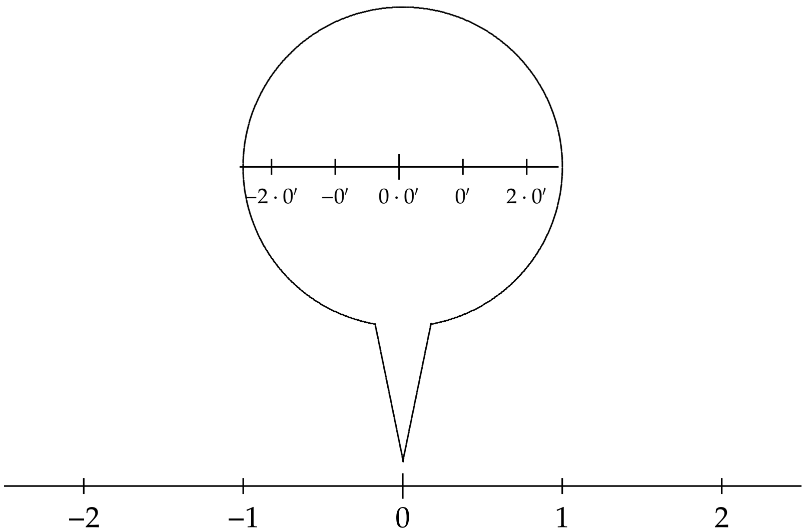

FIG. 39:
Real number line with
microscope view of unfolded 0

In Figure 39, we have the ordinary real number line with the real number 0 expanded into a space. When 0 is expanded into a space, we find multiple distinct zeros in that space. The ordinary number line is the ***folded space*** and the expanded space the ***unfolded space*** around 0. The bubble showing the unfolded space is called a ***microscope***, and the original graph of folded space is called a ***macroscope***.

The figure shows one of the unfolded zeros denoted as $0'$, and it also shows some multiples of $0'$. At the folded level, $0' = 0 \cdot 0' = 0$, but at the unfolded level, $0' \neq 0 \cdot 0'$, and $0' \in 0$.

In the unfolded space of 0, each of the individual values of 0 in that space has a well defined ratio with every other point in that space. For instance, if $0'' \equiv 3 \cdot 0'$, then $\dfrac{0''}{0'} = 3$. This also means that the unfolded space is ordered analogously to real space, e.g. $0'' > 0'$ at the unfolded level.

Finite multiples of each value in the unfolded space are distinct, but squares and higher powers of any value in this space end up at the origin of the unfolded space: $0'^2 = 0'^n = 0 \cdot 0'$ for any $0'$ in the space and any $n > 1$.

We use the notation **=′** for equality at the folded level, and **<′** and **>′** for order at the folded level. For example, $0' \neq 0$ and $0' < 0''$, but $0' =' 0$. Since $0'$ is contained within 0, we can write $0' \subset 0$.

The unfolded space of 0 is called the ***body*** of 0, and the origin of the body is called the ***core***. If it is necessary to distinguish these two symbolically, the body can be denoted $0_{[1]}$ ("zero unit one") and the core can be denoted $0_{[0']}$ ("zero unit zero prime") or, as in Figure 39, $0 \cdot 0'$. The bracketed subscript denotes the unit of the space.

If an unfolded number is multiplied by an unmarked number, the unmarked factor is assumed to be folded. For example, $3 \cdot 0' = 3_{[1]} \cdot 0'$.

Unfolded zeros are the infinitesimals of this system of analysis. The name ***equipoint*** reflects the fact that these infinitesimals are all equal in folded space.

To summarize, at the folded level, there is only one 0:

$$0' =' \mathbb{R} \cdot 0 =' 0'^2 =' 0,$$

while at the unfolded level, there are multiple distinct values of 0:

$$\begin{aligned} &0' =' 0 \\ &0' \neq 2 \cdot 0' \neq 0 \cdot 0' \\ &0' \subset 0 \\ &0'^2 \subset 0 \cdot 0' \end{aligned}$$

Unfolding perfinite numbers

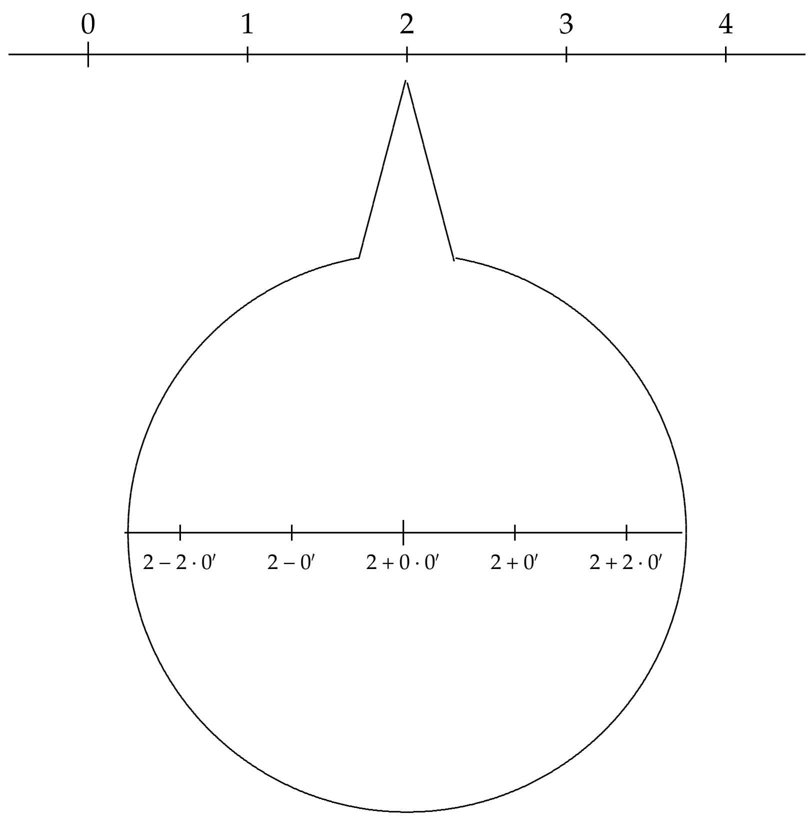

FIG. 40:
Real number line with
microscope view of unfolded +2

Since $a + 0 = a$ for any real a, every real number a can open up into a space consisting of a plus zeros. Figure 40 shows this for $a = 2$. Just as we locate $0'$ within 0 and use it to perform unfolded arithmetic within 0, we can also locate $a' \equiv a + 0'$ within a and use it as the base of unfolded arithmetic within the body of a.

If an unfolded number is added to an unmarked number, the unmarked term is assumed to be unfolded, and thus their sum is unfolded. So $a + 0' = a_{[0']} + 0'$.

Following are examples of arithmetic within a finite real a.

$$a - a = 0$$
$$a + 0' \neq a + 0 \cdot 0' \neq a$$
$$a + 0' =' a + 0 \cdot 0' =' a$$
$$(a + 0') - (a + 0 \cdot 0') = (a_{[0']} + 0') - a_{[0']} = 0'$$
$$a + 0' = a_{[0']} + 0' \subset a = a_{[1]}$$

Unfolding infinite numbers

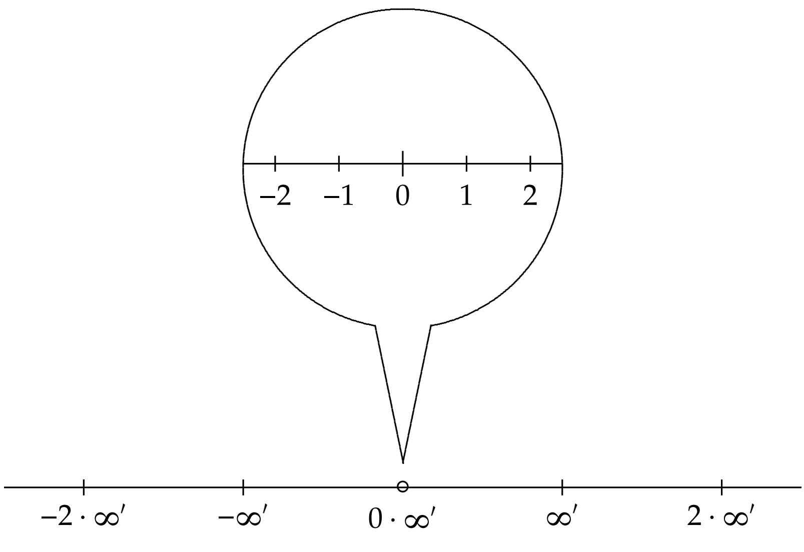

FIG. 41:
Line of infinities with microscope view of
real number line within $0 \cdot \infty'$

Inversely to the expansion of a point of real space, the real number line can collapse into a point within a space of infinities. This is shown in Figure 41.

When an infinite number is unfolded, the roles of microscope and macroscope are reversed: The microscope shows folded finite space, and the macroscope shows unfolded infinite space. The symbol $\circ$ at the center of the microscope shows that it is not part of the unfolded infinites. The origin of the unfolded infinities is at the periphery of the macroscope.

The unfolded number ∞' is the unit in this unfolded space, so we can use bracket subscripts as we did with unfolded zeros, e.g. $\infty_{[1]}$ for folded infinity, and $\infty_{[\infty']}$ for the core of the unfolded space.

The class of folded infinite values, and the unfoldings derived from them, depend on the type of infinite element extension. In the projectively extended real numbers, we defined $\infty \equiv \frac{1}{0}$, while in the affinely extended reals, we defined $\infty \equiv \left|\frac{1}{0}\right|$.

$$\begin{aligned} \infty' &=' \infty \\ \infty' &\neq 2 \cdot \infty' \neq \infty \cdot \infty' \\ \infty' + \infty' &= 2\infty' \\ \infty' &\subset \infty \\ \infty'^2 &\subset \infty \cdot \infty' \end{aligned}$$

Although the folded real numbers in the microscope may appear to have been embedded within the unfolded infinite numbers in the macroscope of Figure 41, in fact the folded real numbers contain the unfolded infinite numbers, since the infinites in the macroscope have been unfolded from the folded ∞, which is on the remote periphery of the figure, infinitely far removed from the center. Instead of a point expanding into a line of unfolded finite numbers, the folded ∞ *contracts* into a line of unfolded infinite numbers.

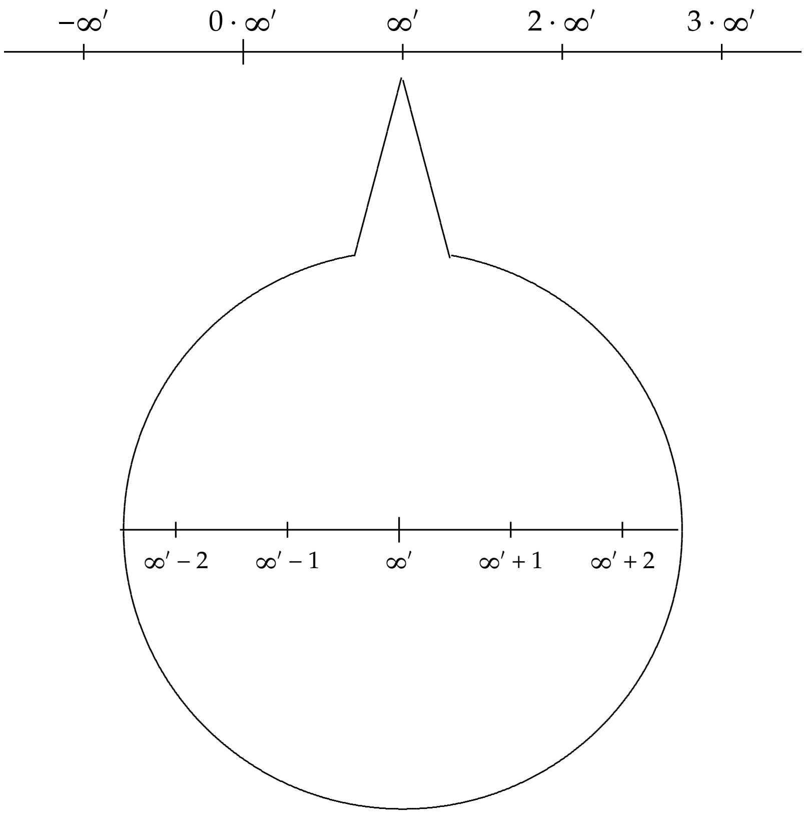

FIG. 42:
Line of infinities with
perfinite unfolding of ∞'

Figure 42 shows that each infinite element in the space of infinities unfolds into a space in which finite numbers added to the infinite element are distinct points.

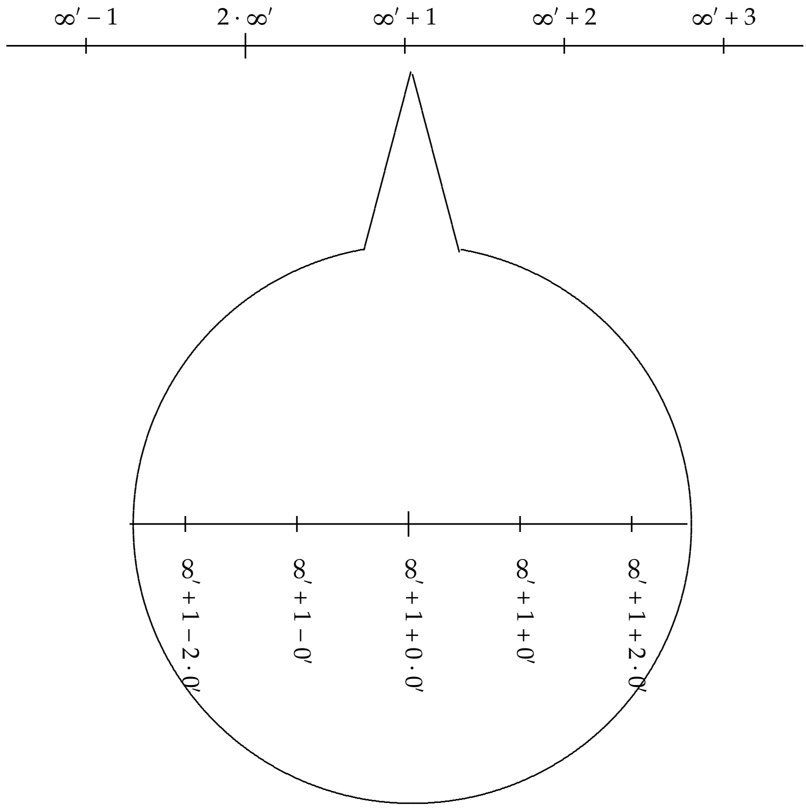

FIG. 43:
Line of infinities with
infinitesimal unfolding of $\infty' + 1$

Figure 43 shows that each point of infinite plus finite unfolds into a space with infinitesimals added.

Unfolding infinity

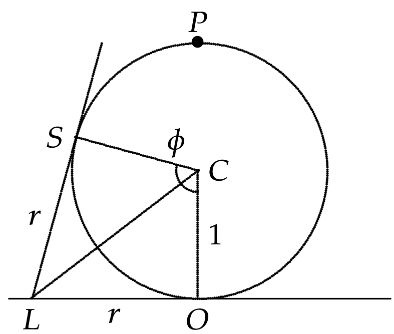

FIG. 44:
Geometric mapping of projectively extended real numbers,
$r = \tan \frac{\phi}{2}$

The previous section unfolded infinite values that are centered on finite numbers. In such unfoldings, the whole class of finite numbers become a single point, and the folded ∞ contracts into a line of unfolded infinite numbers.

To pictorialize this more clearly, we use methods of mapping an infinite value to a point of a finite geometric figure, and then we develop an unfolded space around these points.

We first examine the unfolding of the single folded infinity of the projectively extended real numbers. Figure 44 shows the basic method we used in **Real infinite element extensions** (p. 68) of mapping this infinity to a point on a circle. In this figure, the real axis is line OL, and an arbitrary point L on this line is mapped via a tangent line to the point S on the circle CO, which has a radius of 1. A real number r is mapped to an angle $\phi = \angle OCS$, with $r = \tan \dfrac{\phi}{2}$.

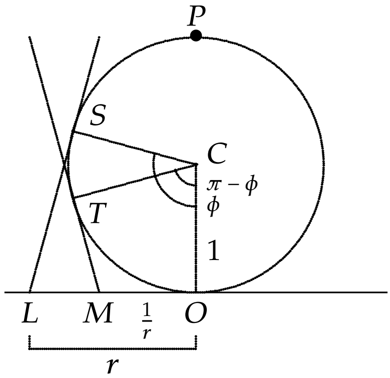

FIG. 45:
Geometric relation
of reciprocals in
projectively extended
real numbers

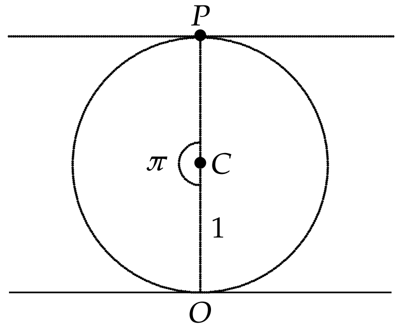

FIG. 46:
Geometric relation
of 0 and ∞ in
projectively extended
real numbers

Since $\frac{1}{r} = \cot\frac{\phi}{2} = \tan\frac{\pi - \phi}{2}$, a pair of reciprocals map to a pair of supplementary angles. Figure 45 shows that if S is a point on the upper half of the circle that is directly above a point T on the lower half of the circle, and if $L \mapsto S$ and $M \mapsto T$, then $M = \frac{1}{L}$. S and T are directly above and below each other in the top and bottom halves of the circle.

In particular, the real number 0 maps to the point O, and the extended real number ∞ maps to the point P, as shown in Figure 46.

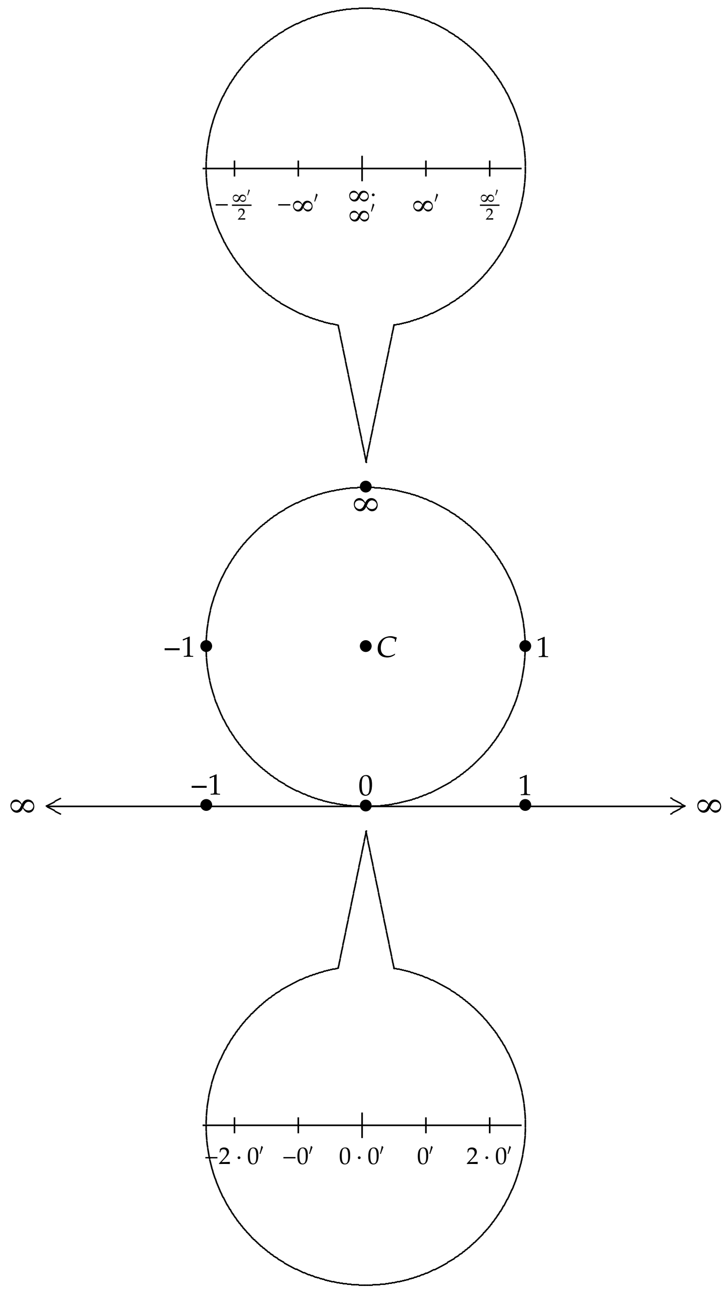

FIG. 48:
Unfolding infinity in
projectively extended real numbers

Figure 48 shows the unfoldings of points O and P. In the lower microscope, we see the unfolding of 0 at point O, containing infinitesimals. In the upper microscope, we see the unfolding of ∞ at point P, containing infinite values. Each point within the upper microscope is the reciprocal of the point directly below it in the lower microscope.

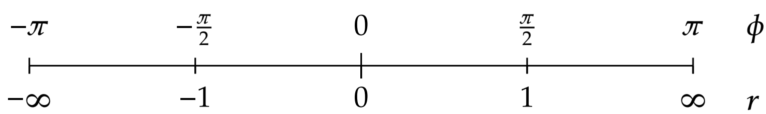

FIG. 47:
Geometric mapping of
affinely extended
real numbers

To pictorialize the affine infinities, we use the method of **tangent scale plots** (p. 82). In Figure 47, the circle in Figure 44 has been cut at P and unrolled into a line segment. An affinely extended real number r is mapped to the now linear parameter ϕ via the mapping $\tan\dfrac{\phi}{2}$ for $\phi \in [-\pi, +\pi]$, with $-\infty$ and $+\infty$ mapped to the endpoints.

Again we have $\dfrac{1}{r} = \cot\dfrac{\phi}{2} = \tan\dfrac{\pi-\phi}{2}$. Hence, in the right half of the line segment, two points that are equidistant from and on opposite sides of $+1$ are reciprocals, and similarly for the left side and -1.

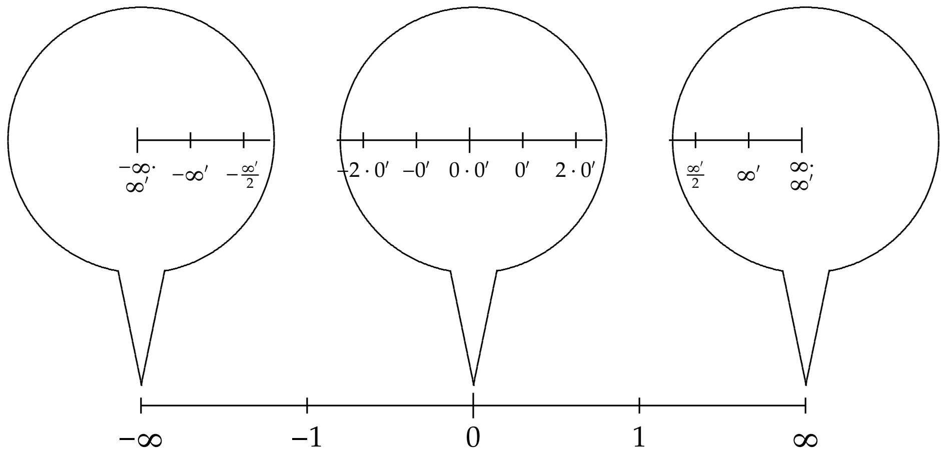

FIG. 49:
Unfolding infinity in
affinely extended real numbers

In Figure 49, we see the line segment with microscopes at 8, 0, and $+\infty$. The right microscope at $+\infty$ ends at $\infty \cdot \infty'$ because there is no value greater than $+\infty$, and similarly the left microscope ends at $-\infty \cdot \infty'$. Each point in the two infinite microscopes is the reciprocal of the corresponding point in the microscope at 0.

In the unfoldings of both projective and affine infinite points, we use the same unfolded infinite values that are reciprocals of unfolded infinitesimals that we used in previous sections. The only additional numbers are the center points of the microscopes, which are simply the folded infinity multiplied by an unfolded infinity.

Higher powers of unfoldings

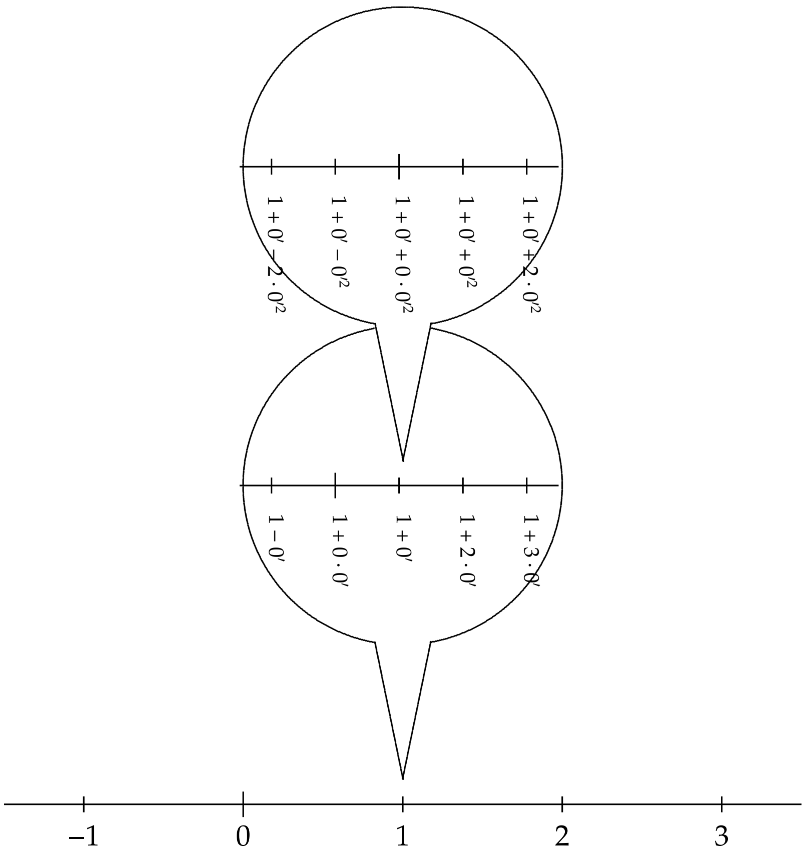

FIG. 50:
Unfolding of $1 + 0'$,
second unfolding of 1

Within an unfolded space, any point can be unfolded again. This ***second unfolding*** uses a unit of $0'^2$ instead of $0'$. We can also call this an unfolding with respect to $0'^2$.

Figure 50 shows the second unfolding of $1+0'$, which is itself an element of the unfolded 1.

For each positive integral power $0'^n$, we can make an ***n-th unfolding*** or unfolding with respect to $0'^n$. Any unfolding beyond the first is called a ***higher unfolding***.

$$\begin{aligned}
0' &\subset 0 \\
0'^2 &\subset 0 \cdot 0' \\
0'^n &\subset 0 \cdot 0'^{n-1} \\
\infty' &\subset \infty \\
\infty'^2 &\subset \infty \cdot \infty' \\
\infty'^n &\subset \infty \cdot \infty'^{n-1}
\end{aligned}$$

Power series enable us to evaluate various unfoldings of non-polynomial functions:

$$\begin{aligned}
e^{0'x} &=' 1 \\
&=' 1 + 0'x \\
&=' 1 + 0'x + \frac{0'^2 x^2}{2} \\
&= \sum_{k=1}^{\infty} \frac{0'^k x^k}{k!}.
\end{aligned}$$

Nonintegral powers of unfoldings

A **higher power of unfolding** (p. 151) normally uses an unfolding unit of $0'^p$, where p is an integer. The microscope of such an unfolded space magnifies space by a factor of ∞'^p, where $\infty' \equiv \dfrac{1}{0'}$.

Any positive perfinite p, including noninteger p, results in a magnification by an infinite amount, since ∞'^p is infinite. For any positive perfinite $q < p$, the unfolding with $0'^p$ is an unfolding of the unfolding with $0'q$, since $p - q$ is also positive perfinite and ∞'^{p-q} is infinite. Thus there is a continuum of unfoldings corresponding to the real continuum, each with its own unfolding level.

However, by the general binomial theorem, for any real r,

$$(1+\infty')^r = \sum_{n=0}^{\infty} \frac{r^{\underline{n}}}{n!} \infty'^n,$$

where

$$r^{\underline{n}} \equiv \prod_{k=0}^{n-1} (r-k),$$

is the *falling factorial* of n. In this way, a nonintegral power of ∞' can be expressed in terms of integral powers.

Unfolded real numbers

Starting with the projectively extended real numbers $\widehat{\mathbb{R}}$ and the affinely extended real numbers $\overline{\mathbb{R}}$, the above sections have described the unfolding of these classes at various $0'$ unfolding levels.

When each of these extended real number classes is extended again to unfoldings, we call them ***unfolded real number classes*** and denote them $\widehat{\mathbb{R}}'$ and $\overline{\mathbb{R}}'$.

In this section, $\infty' \equiv \dfrac{1}{0'}$.

Elementary unfoldings can be stated through identities from a seed unfolded number $0'$:

$$\begin{aligned}
0 &= \mathbb{R}\,0' && \text{(first unfolding)}\\
&= \mathbb{R}\,(1+0)\,0' && \text{(second unfolding)}\\
&= \underset{1}{\mathbb{R}} \left(1 + \underset{2}{\mathbb{R}}\,0'\right) 0' \\
&= \underset{1}{\mathbb{R}}\,0' + \underset{2}{\mathbb{R}}\,0'^2 \\
1 &= 1 + 0 \\
&= 1 + \mathbb{R}\,0' && \text{(first unfolding)}\\
&= 1 + \mathbb{R}\,(1+0)\,0' && \text{(second unfolding)}\\
\infty &= \infty + \mathbb{R} \\
\infty &= \frac{1}{\mathbb{R}} \infty',
\end{aligned}$$

where an expression such as $1 + 0'$ is interpreted as $1_{[0']} + 0'$, and $1_{[0']}$ means the core of 1 in the unfolded $0'$ space.

These unfoldings give the unfolded real elements as

$$\widehat{\mathbb{R}}' \equiv \sum_{n=-\infty}^{\infty} \widehat{\mathbb{R}}_n \infty'^n$$

$$\overline{\mathbb{R}}' \equiv \sum_{n=-\infty}^{\infty} \overline{\mathbb{R}}_n \infty'^n$$

which include the folded real numbers at $n = 0$.

The above expressions describe the elements of $\widehat{\mathbb{R}}'$ and $\overline{\mathbb{R}}'$. These elements underlie all unfoldings, which are hierarchically embedded.

Unfolded infinitesimals are clearly contained within folded 0, and each unfolding of $0'^n$ is contained within the origin of $0'^{n-1}$, i.e. $0'^n \subset 0'^{n-1}$. This is diagrammed for $n = 1$ in Figure 39.

Unfolded infinite numbers are contained within the folded class ∞, and each unfolding of ∞'^n is contained within the infinitely removed *periphery* of ∞^{n-1}, i.e. $\infty'^n \subset \infty'^{n-1}$. Although diagrams may make the folded real numbers appear to be embedded within unfolded infinites, the reverse is actually true, since the unfolded infinites are unfoldings of folded ∞. This is diagrammed for $n = 1$ in Figure 41.

If $m > n$, then $r\infty'^m + s\infty'^n$ means $(r\infty'^m)_{[\infty'^n]} + (s\infty'^n)_{[\infty'^n]}$, i.e. in a sum, each term is unfolded at the level of the lowest power of ∞' (highest power of $0'$) in the terms. For example, see the expressions in Figure 40, where $m = 0$ and $n = -1$, and in Figure 42, where $m = 1$ and $n = 0$.

Nonzero coefficients with exponent n in the above sums yields the following classes.

$n = -1$	**first unfolding of 0** (p. 138)
$n = +1$	**first unfolding of ∞** (p. 146)
$n \in \mathbb{Z}$	**higher power of unfoldings** (p. 151)
$n \notin \mathbb{Z}$	**nonintegral power of unfoldings** (p. 152)
$n < 0$	infinitesimals
$n > 0$	infinites
$n = 0$	folded numbers

See **Decimal expansions** (p. 281) for a discussion of the structure of the unfolded projectively extended real number class in terms of decimal expansions.

The above definitions are not complete enumerations of all possible unfoldings. For example, the summation index and exponent n of the above summations have not been unfolded. If they were, what are called elements above would unfold into multivalued classes. But the above definitions will be more than adequate for all the proofs in this book, especially since they form power series in $0'$ and ∞'.

Unfolded integers

In **Extended integers and rational numbers** (p. 76), integers and natural numbers were extended with infinite elements:

$$\widehat{\mathbb{N}} \equiv \mathbb{N} \cup \overline{\infty}$$
$$\overline{\mathbb{N}} \equiv \mathbb{N} \cup \infty^{+}$$
$$\widehat{\mathbb{Z}} \equiv \mathbb{Z} \cup \overline{\infty}$$
$$\overline{\mathbb{Z}} \equiv \mathbb{Z} \cup \pm\infty^{+}$$

Since $\dfrac{2}{0'} - \dfrac{1}{0'} = \dfrac{1}{0'} > 1$, there must be at least one positive integer ∞' between $\dfrac{1}{0'}$ and $\dfrac{2}{0'}$.

Analogous to the real numbers, we might theorize the unfolded integers to be the following:

$$\widehat{\mathbb{N}}' \stackrel{?}{=} \sum_{n=-\infty}^{\infty} \widehat{\mathbb{N}}_n \, \overline{\infty}'^n$$

$$\overline{\mathbb{N}}' \stackrel{?}{=} \sum_{n=-\infty}^{\infty} \overline{\mathbb{N}}_n (\infty'^{+})^n$$

$$\widehat{\mathbb{Z}}' \stackrel{?}{=} \sum_{n=-\infty}^{\infty} \widehat{\mathbb{Z}}_n \, \overline{\infty}'^n$$

$$\overline{\mathbb{Z}}' \stackrel{?}{=} \sum_{n=-\infty}^{\infty} \overline{\mathbb{Z}}_n (\pm\infty'^{+})^n$$

However, there are other unfolded integers if ∞' is composite. For instance, if ∞' is a perfect square ∞''^2, then ∞'' does not have any of the above forms. Our notation should therefore indicate that the above classes of unfolded integers are only those generated from the seed ∞':

$$\widehat{\mathbb{N}}[\infty'] \equiv \sum_{n=-\infty}^{\infty} \widehat{\mathbb{N}}_n \, \overline{\infty}'^n$$

$$\overline{\mathbb{N}}[\infty'] \equiv \sum_{n=-\infty}^{\infty} \overline{\mathbb{N}}_n (\infty'^{+})^n$$

$$\widehat{\mathbb{Z}}[\infty'] \equiv \sum_{n=-\infty}^{\infty} \widehat{\mathbb{Z}}_n \overline{\infty}'^n$$

$$\overline{\mathbb{Z}}[\infty'] \equiv \sum_{n=-\infty}^{\infty} \overline{\mathbb{Z}}_n (\pm\infty'^{+})^n$$

Comprehensive classes of unfolded integers can be defined as unfolded real numbers with zero fractional part:

$$\widehat{\mathbb{N}}' \equiv \left\{ r \in |\widehat{\mathbb{R}}'| \;\middle|\; r = \lfloor r \rfloor \right\}$$

$$\overline{\mathbb{N}}' \equiv \left\{ r \in |\overline{\mathbb{R}}'| \;\middle|\; r = \lfloor r \rfloor \right\}$$

$$\widehat{\mathbb{Z}}' \equiv \left\{ r \in \widehat{\mathbb{R}}' \;\middle|\; r = \lfloor r \rfloor \right\}$$

$$\overline{\mathbb{Z}}' \equiv \left\{ r \in \overline{\mathbb{R}}' \;\middle|\; r = \lfloor r \rfloor \right\}$$

Unfolded integers are used as summation limits in **Equipoint summation** (p. 351–351).

Unfolded rational numbers

In **Extended integers and rational numbers** (p. 76), rational numbers were extended with infinite elements:

$$\widehat{\mathbb{Q}} \equiv \mathbb{Q} \cup \overline{\infty}$$

$$\overline{\mathbb{Q}} \equiv \mathbb{Q} \cup \pm\infty^{+}$$

The unfolding of these classes is analogous to that of real numbers:

$$\widehat{\mathbb{Q}}' \equiv \sum_{n=-\infty}^{\infty} \widehat{\mathbb{Q}}_n \infty'^n$$

$$\overline{\mathbb{Q}}' \equiv \sum_{n=-\infty}^{\infty} \overline{\mathbb{Q}}_n \infty'^n$$

These classes of unfolded rational numbers are actually identical to the classes of unfolded real numbers, i.e.

$$\widehat{\mathbb{Q}}' = \widehat{\mathbb{R}}'$$

$$\overline{\mathbb{Q}}' = \overline{\mathbb{R}}'$$

This is because each element of $\widehat{\mathbb{Q}}'$ and $\overline{\mathbb{Q}}'$ is a ratio of two elements of $\widehat{\mathbb{Z}}'$ and $\overline{\mathbb{Z}}'$, respectively, which are unfolded infinite integers, while an element of $\widehat{\mathbb{R}}'$ or $\overline{\mathbb{R}}'$ in its decimal form is the same thing, a ratio of two infinite integers. For instance, the folded number π includes unfolded numbers of the form

$$\frac{314159\ldots d_{\infty'}}{10^{\infty'}},$$

where d_n is the digit of π in the n-th position after the decimal point.

Decimal representations of unfolded rational and irrational numbers are further examined in **Repeating Decimals** (p. 409–457).

Unfolded complex numbers

In **Complex infinite element extensions** (p. 77), complex numbers were extended with infinite elements:

$$\widehat{\mathbb{C}} \equiv \mathbb{C} \cup \widehat{\infty}$$
$$\widetilde{\mathbb{C}} \equiv \mathbb{C} \cup e^{i[0,\pi)}\,\overline{\infty}$$
$$\overline{\mathbb{C}} \equiv \mathbb{C} \cup e^{i[0,2\pi)}\,\infty^{+}$$

The unfolding of these classes is analogous to that of real numbers:

$$\widehat{\mathbb{C}}' \equiv \sum_{n=-\infty}^{\infty} \widehat{\mathbb{C}}_n \infty'^n$$

$$\widetilde{\mathbb{C}}' \equiv \sum_{n=-\infty}^{\infty} \widetilde{\mathbb{C}}_n \infty'^n$$

$$\overline{\mathbb{C}}' \equiv \sum_{n=-\infty}^{\infty} \overline{\mathbb{C}}_n \infty'^n$$

Unfolding functions and relations

In much the same way as natural numbers are extended to integers, rational numbers, real numbers, and complex numbers, unfolded numbers are an extension of folded numbers. Whenever a class of numbers is extended, the functions on those classes must also be extended, where applicable. We have already used versions of addition and multiplication that were extended to unfolded numbers.

An ***unfolded function*** is a function defined in an unfolded space. We postulate that unfolded functions follow the ***transfer principle***: Any function originally defined in a folded space can be uniquely extended to the corresponding unfolded space. We call such an extension the ***unfolding*** or ***canonical unfolding*** or the ***canonical extension*** of the folded function.

A function may be defined in an unfolded space but not be the unfolding of any folded function or relation; in other words, it cannnot be defined solely at the folded level. In this case, we call it a ***noncanonical unfolded***; function.

A noncanonical unfolded function can be folded, but some information will be lost:

$$f(x)_{[1]} = \{f(x + 0') \mid 0' \subset 0_{[1]}\},$$

that is, take all the values in the unfolding $f(x+\mathbb{R}0')'$ and put them into a single class that is assigned to the folded $f(x)$. If f is single valued at the unfolded level, and there is any $0'$ such that $f(x) \neq' f(x+0')$, then f will be multivalued at the folded level.

A canonical unfolded function does not lose any information in this way when folded. Unfolded polynomial functions are canonical unfolded, but the **Dirac delta function** (p. 217) is a noncanonical unfolded function.

A relation can be considered as a function that maps to a logical value. A relation that is defined in a folded space thus has a canonical extension to an unfolded space, while a relation that is defined in an unfolded space may be noncanonical.

Unfolding equality and order

Unprimed equality = in unfolded numbers is the canonical extension of unprimed equality in folded numbers. In unfolded numbers, it is reflexive, symmetric, and transitive and hence an equivalence relation.

Primed equality in $\widehat{\mathbb{R}}'$ is defined as follows:

$$a='b \quad \Leftrightarrow \quad (\exists c, d \in \widehat{\mathbb{R}})\, c \supseteq a \wedge d \supseteq b \wedge c = d$$

with the obvious analog for $\overline{\mathbb{R}}$. Note that c and d must be in $\widehat{\mathbb{R}}$ or $\overline{\mathbb{R}}$, i.e. are folded numbers.

Primed equality is noncanonical. It is reflexive, symmetric, and transitive and thus an equivalence relation, with the folded numbers as its equivalence classes.

As we have seen, a folded number a can be interpreted as an element of the class of folded numbers, or as a class of unfolded numbers. Each unfolded number can similarly be considered an element at its level of unfolding, or as a class of further unfolded numbers. Therefore an equality of two numbers $a = b$ can be interpreted as the identity of two elements, or as class equality of unfolded elements of a and b, or as a conjuctively distributed equality over unfolded elements of a and b. Example:

$$1{+}2 = 3 \quad \Leftrightarrow \quad (\forall a \in' 0)(1{+}a){+}(2{+}a) = 3{+}a \quad \Leftrightarrow \quad (\forall a \subset 0)(1{+}a){+}(2{+}a) = 3{+}a.$$

Unprimed order $<$ in unfolded numbers is the canonical extension of unprimed order in folded numbers.

Trichotomy of $<$ holds in $\overline{\mathbb{R}}$, and in $\widehat{\mathbb{R}}$ except that $\infty < a$ and $\infty > a$ for finite a. Hence, with that one exception, $<$ is a strict total order in $\overline{\mathbb{R}}$ and $\widehat{\mathbb{R}}$. But in unfolded $\overline{\mathbb{R}}'$ and $\widehat{\mathbb{R}}'$, trichotomy of $<$ does not hold, since, for example

$$0' \subset 0$$
$$0' \not< 0$$
$$0' \not> 0$$
$$0' \neq 0.$$

Unprimed order $<$ is however irreflexive, antisymmetric, and transitive, so with the sole exception for ∞ in $\widehat{\mathbb{R}}$, unprimed order $<$ is a strict partial order in $\overline{\mathbb{R}}'$ and $\widehat{\mathbb{R}}'$. Unprimed $\leq$ is a (non-strict) partial order.

Primed order in $\widehat{\mathbb{R}}'$ is defined as follows:

$$a <' b \quad \Leftrightarrow \quad (\exists c, d \in \widehat{\mathbb{R}})\,(c \supseteq a) \wedge (d \supseteq b) \wedge (c < d)$$

again with the obvious analog for $\overline{\mathbb{R}}$.

$a \leq' b$ means $a <' b \vee a =' b$, or equivalently,

$$a \leq' b \quad \Leftrightarrow \quad (\exists c, d \in \widehat{\mathbb{R}})\,(c \supseteq a) \wedge (d \supseteq b) \wedge (c \leq d).$$

Primed order $<'$ is noncanonical. It is irreflexive, antisymmetric, and transitive, but trichotomy also holds with respect to primed equality (for example, $0' =' 0$), so primed inequality $<'$ is a strict total order. Primed $\leq'$ is a non-strict total order.

As with equality, inequality of two numbers can be interpreted as a statement about two elements or as a conjuctive distribution of inequality over unfolded elements:

$$1 < 2 \quad \Leftrightarrow \quad (\forall a \in' 0)\,(1 + a) < (2 + a) \quad \Leftrightarrow \quad (\forall a \subset 0)\,(1 + a) < (2 + a).$$

Unfolding membership and inclusion

We have seen that a folded or unfolded number can be regarded either as an element of an unfolded real number class or as a subclass consisting of further unfolded numbers. Hence the status of a number as an element an unfolded class is relative to a level of unfolding rather than absolute.

Furthermore, levels of unfolding are continuous, not discrete. For instance, $0'n \subset 0{\cdot}0'm$ for any $m < n$, so there is a continuous hierarchy of inclusion over all powers of $0'$, i.e. all levels of unfolding.

To indicate this multilevel membership, we use the same subscript notation that we used in **Unfolding zero** (p. 138) and **Higher powers of unfoldings** (p. 151) to distinguish $0_{[1]}$, $0_{[0']}$, and $0_{[0'^n]}$.

The ***multilevel membership relation*** $\in_{[0'^n]}$ is defined as follows. $a \in_{[0'^n]} b$ means $a \subset b$ and a is unfolded at the level $0'^n$ but at no greater level of unfolding. The notation $a \in_{[\infty'^n]} b$ has a similar meaning when unfolding the unfolding unit is infinite.

The unmodified notation $\in$ means folded membership, i. e. $\in_{[1]}$, while the notation $\in'$ can be used to mean $\in_{[0']}$ or $\in_{[\infty']}$, depending on the context.

Some examples, assuming finite a:

$$0 \in_{[1]} \mathbb{R}, \quad \text{or equivalently} \quad 0 \in \mathbb{R}$$

$$a0' \in_{[0']} 0, \quad \text{or equivalently} \quad a0' \in' 0$$

$$a + 0' \in_{[0']} a, \quad \text{or equivalently} \quad a + 0' \in' a$$

$$a0'^2 \in_{[0'^2]} 0$$

$$a0'^2 \in_{[0'^2]} 0 \cdot 0'$$

$$a0'^n \in_{[0'^n]} 0 \cdot 0'^m, \quad \text{where } m < n$$

$$1 + a0' \in_{[0']} 1$$

$$0' + a0'^2 \in_{[0'^2]} 0'$$

$$1 + 0' + a0'^2 \in_{[0'^2]} 1 + 0'$$

Unlike membership, the inclusion relations $\subset$, $\subseteq$, $\supset$, and $\supseteq$ do not need unfolding levels, since inclusion remains the same whether the terms are single valued or multivalued. Symbolically, $a \in_{[0'^n]} b$ and $a \in_{[\infty'^n]} b$ implies $a \subset b$, regardless of $0'$, ∞', or n.

Implied unfolding

In a sum, the unfolding of one summand may imply the unfolding of other summands if no unfolding is explicitly indicated for the other summands. The default unfolding for all summands is the highest level of unfolding which is explicity indicated in the sum.

The above rule for sums does not apply to products or powers.

EXAMPLE 1. In the sum $2 + 0'$, the highest level of unfolding is in the $0'$ summand. The unfolding for the 2 summand defaults to the $0'$ level, so the sum is interpreted as $2_{[0']} + 0'$, a point in the first unfolding of the folded number 2.

EXAMPLE 2. In $2_{[1]} + 0'$, the folded number 2 is unfolded as a class at the $0'$ level. Adding $0'$ to each member of this class increases each member, so $2_{[1]} + 0' \overset{\wedge}{\geq} 2_{[1]}$, but it leaves the class as a whole unchanged, so $2_{[1]} + 0' \overset{\{\}}{=} 2_{[1]}$.

EXAMPLE 3. The sum $2 + 0' + 3 \cdot 0'^2$ is interpreted as $2_{[0'^2]} + 0'_{[0'^2]} + 3_{[1]} \cdot 0'^2_{[0'^2]}$.

EXAMPLE 4. $\dfrac{0}{0} = \not\phi$, since $\dfrac{a}{b} \equiv \{x \mid ax = b\}$, and $\{x \mid 0x = 0\} = \not\phi$. Alternatively, if 0 is regarded as a class of infinitesimals $\mathbb{R}0'$, then $\dfrac{0}{0} = \dfrac{\underset{1}{\mathbb{R}}0'}{\underset{2}{\mathbb{R}}0'} = \dfrac{\underset{1}{\mathbb{R}}}{\underset{2}{\mathbb{R}}} = \not\phi$.

EXAMPLE 5. $\dfrac{0'}{0'} = \{x \mid 0'x = 0'\} = 1$.

EXAMPLE 6. $\infty - \infty = \not\phi$, since $a - b \equiv \{x \mid a + x = b\}$, and $\{x \mid \infty + x = \infty\} = \not\phi$. Alternatively, if ∞ is regarded as a class of infinites $\mathbb{R}\infty'$, then $\infty - \infty = \underset{1}{\mathbb{R}}\infty' - \underset{2}{\mathbb{R}}\infty' = \left(\underset{1}{\mathbb{R}} - \underset{2}{\mathbb{R}}\right)\infty'$. Now $\underset{1}{\mathbb{R}} - \underset{2}{\mathbb{R}}$ could be perfinite or infinitesimal. If it is perfinite, then $\left(\underset{1}{\mathbb{R}} - \underset{2}{\mathbb{R}}\right) \infty'$ is infinite. If it is infinitesimal, then $\left(\underset{1}{\mathbb{R}} - \underset{2}{\mathbb{R}}\right) \infty'$ could be infinite, perfinite, or infinitesimal, i.e. any real value. Hence $\infty - \infty = \not\infty$.

EXAMPLE 7. $\infty' - \infty' = \{x \mid \infty' + x = \infty'\} = 0$.

EXAMPLE 8. For finite folded r,

$$\begin{aligned} \frac{0'r + 0'^2}{0'} &= \frac{0'r}{0'} + \frac{0'^2}{0'} \\ &= r + 0' \\ &=' r. \end{aligned}$$

DEFINITIONS OF DERIVATIVE AND INTEGRAL

Definition of derivative

The equipoint derivative directly calculates the rate of change at a point using unfolding levels and the transfer principle.

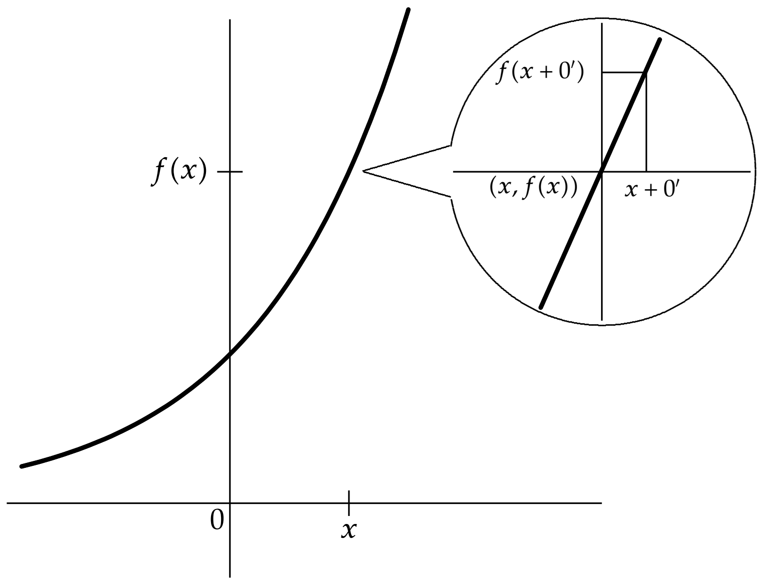

FIG. 51:
Calculation of derivative
as slope within a point

Figure 51 shows a curve $y = f(x)$ and a microscope view of the point $(x, f(x))$. Within the point, the curve is infinitely magnified and becomes a straight line. The Δx of this line is an infinitesimal $0'$, the unfolding level of the microscope, and the Δy of this line is $f(x+0') - f(x)$. In unfolded space, we denote Δx and Δy as dx and dy. The slope of the line, and the ***derivative*** of $f(x)$ at x, is

$$f'(x) \equiv \frac{df(x)}{dx} \equiv \frac{f(x+0') - f(x)}{0'}.$$

As an example of this calculation:

$$\begin{aligned}
f(x) &\equiv x^2 \\
\frac{df(x)}{dx} &= \frac{(x+0')^2 - x^2}{0'} \\
&= \frac{x^2 + 2 \cdot 0'x + 0'^2 - x^2}{0'} \\
&= \frac{2 \cdot 0'x + 0'^2}{0'} \\
&= 2x + 0' \\
&=' 2x.
\end{aligned}$$

For a comparison of this definition of the derivative with that in other systems of analysis, see the **Appendix** (p. 288).

Definition of definite integral

The equipoint integral directly calculates an area as an infinite sum of zero width rectangles.

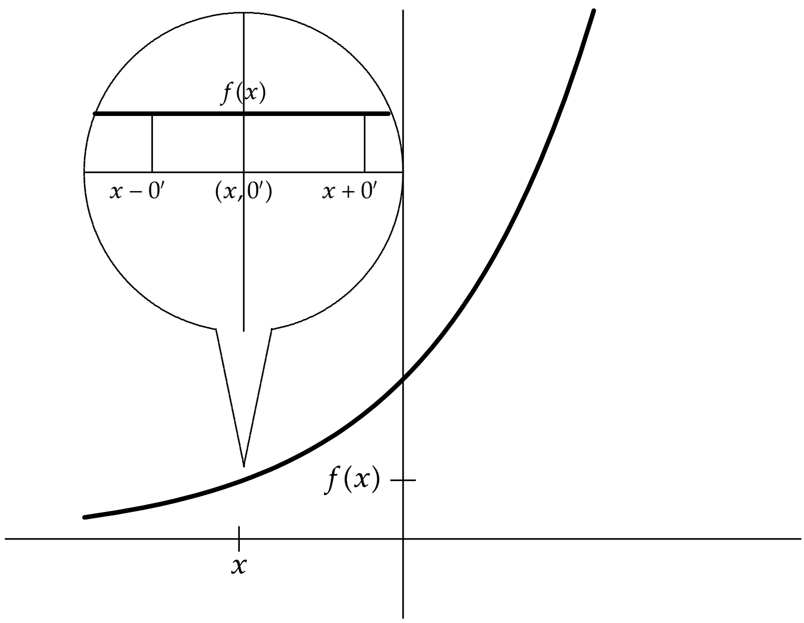

FIG. 52:
Calculation of integral
as sum of zero width rectangles

Figure 52 shows a curve $y = f(x)$ and a microscope view of the sliver at x an infinitely thin area under the curve $f()$ at the point x. The microscope expands the sliver in the x direction but not in the y direction. Within the sliver, the curve becomes a flat line, and sliver is a rectangle with height $f(x)$ and width $x + 0' - x = 0'$.

The total area under the curve from $x = a$ to $x = b$ is the sum of the areas of these slivers, the number of these slivers is $\infty' \equiv \dfrac{b-a}{0'}$, and the width of each sliver is $0' = \frac{b-a}{\infty'}$. The total area from a to b, and the ***definite integral*** of $f(x)$ from a to b, is

$$\int_a^b f(x)\,dx \equiv \sum_{k=1}^{\infty'} f\left(a + \frac{k(b-a)}{\infty'}\right)\frac{b-a}{\infty'}.$$

As an example of this calculation:

$$\begin{aligned}\int_0^u 2x\,dx &= \sum_{k=1}^{\infty'} 2\frac{ku}{\infty'}\frac{u}{\infty'}\\ &= \frac{2u^2}{\infty'^2}\sum_{k=1}^{\infty'} k\\ &= \frac{2u^2}{\infty'^2}\infty'\frac{\infty'+1}{2}\\ &= u^2\left(1+\frac{1}{\infty'}\right)\\ &= u^2(1+0')\\ &=' u^2.\end{aligned}$$

For a comparison of this definition of the definite integral with that in other systems of analysis, see the **Appendix** (p. 288).

For a definition of the indefinite integral, see **Antiderivatives** (p. 182).

Infinite bounds on integrals and path integral

In the equipoint definition of integral, $b-a$ may be infinite if either of the limits a or b is infinite. In this case, we simply choose $0'$ so that $\dfrac{b-a}{0'}$ is infinitesimal, e.g. $\dfrac{1}{(b-a)^2}$.

Equipoint analysis can be used with any infinite element extension discussed in **Infinity and infinite element extensions** (p. 64). With a projectively extended system, bounds of integration may appear ambigous, since $+\infty$ and $-\infty$ are identical. In this case, it is helpful to remember that bounds of integration implicitly establish a *path* of integration: integrating from $-\infty$ to $+\infty$ integrates through 0 and all the finite values, integrating from 0 to $+\infty$ integrates through all the positive finite values, etc. The equipoint integral along a path $x = P(t)$, where t runs from a to b, is given by

$$\int_P f(x)\,dx = \int_{t=a}^{b} f(P(t))\,dP(t) = \int_a^b f(P(t))\frac{dP(t)}{dt}\,dt.$$

Differentiability and integrability

The **Singularities** (p. 203) chapter discusses several **types of singularity** (p. 207) which may present difficulties using the above definitions of derivative and integral.

Briefly, at a jump discontinuity, the derivative is infinite, and the integral can be calculated straighforwardly through the singularity. See the discussions of the **absolute value function** (p. 208), the **Kronecker delta function** (p. 216), and the **Dirac delta function** (p. 217).

At punctured functions, poles, and essential singularity singularities, it is necessary to use an **offset derivative** (p. 203), and attempts to integrate through these singularities may be incorrect. See the discussions of the **punctured constant function** (p. 210), the **axial function** (p. 225), **poles** (p. 223), and the **function sin $\frac{1}{x}$.** (p. 227)

THE FUNDAMENTAL THEOREMS OF CALCULUS

For the following equipoint proofs of the first and second fundamental theorems of calculus, we assume the following:

1. The splitting property $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ for all c, which is easily proved from the definition of the definite integral.

2. A corollary, the zero property $\int_a^a f(x)\,dx = 0$ for all a.

3. Another corollary, the reversal property $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$.

4. The function f is **continuous** (p. 187): $f(x+0') = f(x)$ for the endpoints $x = a$ and $x = b$ in the first theorem, and for all x in the second theorem. Cases where this condition does not hold are discussed at the end of this seciton.

We do *not* assume the mean value theorem.

We recall that the definite integral is defined as

$$\int_a^b f(x)\,dx \equiv \sum_{k=1}^{\infty'} f\left(a + \frac{k(b-a)}{\infty'}\right)\frac{b-a}{\infty'},$$

and the derivative as

$$\frac{df(x)}{dx} \equiv \frac{f(x+0') - f(x)}{0'}.$$

THE FIRST FUNDAMENTAL THEOREM OF CALCULUS:

$$\boldsymbol{\int_a^b \frac{df(x)}{dx}\,dx = f(b) - f(a).}$$

PROOF.

$$\begin{aligned}
\int_a^b \frac{df(x)}{dx}\,dx &= \int_a^b \frac{f\left(x-\frac{b-a}{\infty'}\right)-f(x)}{\frac{b-a}{\infty'}}\,dx \\
&= \frac{b-a}{\infty'}\frac{\infty'}{b-a}\sum_{k=1}^{\infty'} f\left(a+\frac{k(b-a)}{\infty'}+\frac{b-a}{\infty'}\right)-f\left(a+\frac{k(b-a)}{\infty'}\right) \\
&= \sum_{k=1}^{\infty'} f\left(a+(k+1)\frac{b-a}{\infty'}\right)-f\left(a+k\frac{b-a}{\infty'}\right) \\
&= f\left(a+(\infty'+1)\frac{b-a}{\infty'}\right)-f\left(a+\frac{b-a}{\infty'}\right) \\
&= f\left(a+(b-a)+0'\right)-f\left(a+0'\right) \\
&=' f(b)-f(a). \blacksquare
\end{aligned}$$

THE SECOND FUNDAMENTAL THEOREM OF CALCULUS:

$$\boldsymbol{\frac{d}{dx}\int_c^x f(u)\,du = f(x).}$$

PROOF.

$$\begin{aligned}
\frac{d}{dx}\int_c^x f(u)\,du &= \frac{\int_c^{x+0'} f(u)\,du - \int_c^x f(u)\,du}{0'} \\
&= \frac{\int_c^x f(u)\,du + \int_x^{x+0'} f(u)\,du - \int_c^x f(u)\,du}{0'} \\
&= \frac{\int_x^{x+0'} f(u)\,du}{0'} \\
&= \frac{f(x+0')0'}{0'} \\
&= f(x+0') \\
&=' f(x). \blacksquare
\end{aligned}$$

We then have

$$\int_c^x f(u)\,du = F(x)+k,$$

where $F(x)$ is any function such that

$$\frac{dF(x)}{dx} = f(x),$$

and k is a constant that depends on c. Then we have

$$\begin{aligned}\int_a^b f(x)\,dx &= \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \\ &= \int_c^b f(x)\,dx - \int_c^a f(x)\,dx \\ &= F(b) - F(a).\end{aligned}$$

Numeristics and equipoint analysis allow us to apply these definitions and theorems to a wide range of functions. A function that is conventionally considered **discontinuous** (p. 187) may have an infinite equipoint derivative at the point of discontinuity. A similarly wide net is cast for integration. Abscissas and ordinates may be finite or infinite, single valued or multivalued.

There are a few types of singularity where these theorems do not apply completely, since the type of derivative used here at such points is not determinate. In these cases, it may be necessary to use an **offset derivative** (p. 203) and restrict the range of integration. Singularities where this occurs include **poles** (p. 223) and **essential singularities** (p. 227). This consideration is discussed in detail in the **Singularities** (p. 203) chapter.

DERIVATIVE THEOREMS

Chain rule

THE CHAIN RULE:

$$\boldsymbol{\frac{d}{dx} f(g(x)) = \left[\frac{d\, f(g(x))}{d\, g(x)}\right] \left[\frac{d}{dx} g(x)\right]}$$

It might appear that $d\, g(x)$ can simply be cancelled, but since the differentials on the left and right sides have slightly different interpretations, we must proceed more carefully.

PROOF. Define

$$\begin{aligned} y &\equiv g(x) \\ 0'' &\equiv g(x+0') - g(x) \end{aligned}$$

Then

$$\begin{aligned} \frac{df(g(x))}{dx} &= \frac{f(g(x+0')) - f(g(x))}{0'} \\ &= \frac{f(y+0'') - f(y)}{0'} \\ &= \frac{f(y+0'') - f(y)}{0''} \cdot \frac{0''}{0'} \\ &= \frac{f(y+0'') - f(y)}{0''} \cdot \frac{g(x+0') - g(x)}{0'} \\ &= \frac{df(y)}{dy} \cdot \frac{dg(x)}{dx} \\ &= \frac{df(g(x))}{dg(x)} \cdot \frac{dg(x)}{dx}. \ \blacksquare \end{aligned}$$

Product rule

THE BASIC PRODUCT RULE:

$$\frac{d}{dx}f(x)g(x) = f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x)$$

PROOF.

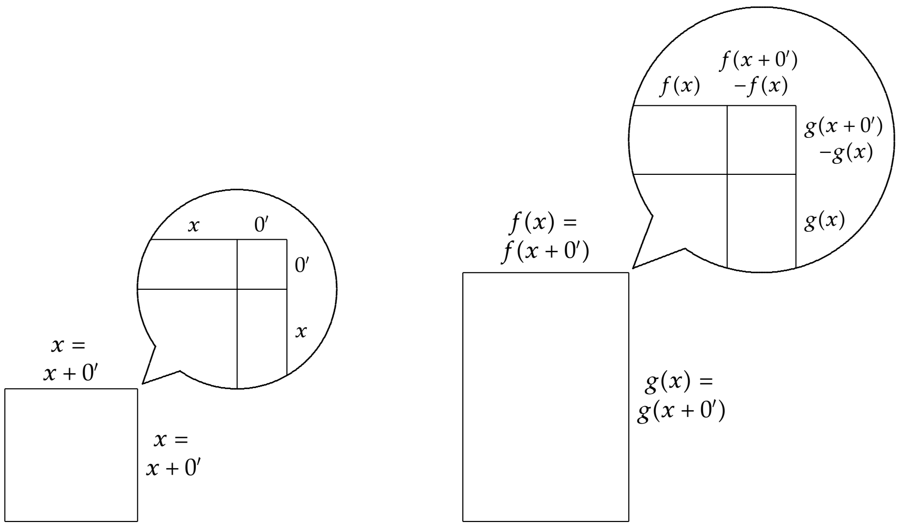

FIG. 53:
Calculation of derivative of
product of $f(x)$ and $g(x)$

Figure 53 shows, on the left, a square with sides of length $x + 0'$ and, on the right, a rectangle which is the transform of this square by $f(x)$ in the horizontal direction and $g(x)$ in the vertical direction. The rectangle has sides

$$f(x + 0') = f(x) + [f(x + 0') - f(x)]$$
$$g(x + 0') = g(x) + [g(x + 0') - g(x)].$$

The two strips on the sides of the left figure, with area $x \cdot 0'$, are infinitesimally small compared to the large portion of the left figure, with area

x^2. These strips are transformed to the two strips on the sides of the right figure, with areas $f(x) \cdot [f(x+0') - f(x)]$ and $g(x) \cdot [g(x+0') - g(x)]$, which are infinitesimally small compared to the large portion of the right figure, with area $f(x) \cdot g(x)$.

The small square in the upper right corner of the left figure, with area $0'^2$, is transformed to a small rectangle in the upper right corner of the right figure, with area $[f(x+0') - f(x)] \cdot [g(x+0') - g(x)]$. Both are infinitesimally small compared to the strips on the sides:

$$[f(x+0') - f(x)] \cdot [g(x+0') - g(x)] =' 0$$

or

$$\frac{[f(x+0') - f(x)] \cdot [g(x+0') - g(x)]}{0'} =' 0.$$

Then

$$\begin{aligned}
\frac{d}{dx} f(x)g(x) &= \frac{f(x+0')g(x+0') - f(x)g(x)}{0'} \\
&= \frac{1}{0'}\Big[f(x)g(x) \\
&\qquad + f(x)\,[g(x+0') - g(x)] \\
&\qquad + [f(x+0') - f(x)]\,g(x) \\
&\qquad + [f(x+0') - f(x)]\,[g(x+0') - g(x)] \\
&\qquad - f(x)g(x)\Big] \\
&=' \frac{1}{0'}\Big[f(x)g(x) \\
&\qquad + f(x)\,[g(x+0') - g(x)] \\
&\qquad + [f(x+0') - f(x)]\,g(x) \\
&\qquad - f(x)g(x)\Big] \\
&= \frac{f(x)\,[g(x+0') - g(x)] + g(x)\,[f(x+0') - f(x)]}{0'} \\
&= f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x). \blacksquare
\end{aligned}$$

THE MULTIPRODUCT RULE:

$$\boldsymbol{\frac{d}{dx}\prod_{k=1}^{n} f_k(x) = \sum_{j=1}^{n}\left(\prod_{\substack{k=1\\k\neq n}}^{n} f_k(x)\right)\left(\frac{d}{dx}f_j(x)\right)}$$

PROOF. By induction. The case $n = 2$ is the basic product rule proved above. Assuming the multiproduct rule for n, then:

$$\begin{aligned}
\frac{d}{dx}\prod_{k=1}^{n+1} f_n(x) &= \frac{d}{dx}\left[\left(\prod_{k=1}^{n} f_n(x)\right) f_{n+1}(x)\right] \\
&=' f_{n+1}(x)\left(\frac{d}{dx}\prod_{k=1}^{n} f_k(x)\right) + \left(\prod_{k=1}^{n} f_k(x)\right)\frac{d}{dx} f_{n+1}(x) \\
&=' f_{n+1}(x)\sum_{j=1}^{n}\left(\prod_{\substack{k=1\\k\neq n}}^{n} f_k(x)\right)\left(\frac{d}{dx} f_j(x)\right) + \frac{d}{dx} f_{n+1}(x)\left(\prod_{k=1}^{n} f_k(x)\right) \\
&= \sum_{j=1}^{n}\left(\frac{d}{dx} f_j(x)\right)\left(\prod_{\substack{k=1\\k\neq j}}^{n} f_k(x)\right) + \left(\frac{d}{dx} f_{n+1}(x)\right)\left(\prod_{\substack{k=1\\k\neq n+1}}^{n} f_k(x)\right) \\
&= \sum_{j=1}^{n+1}\left(\prod_{\substack{k=1\\k\neq n}}^{n+1} f_k(x)\right)\left(\frac{d}{dx} f_j(x)\right)
\end{aligned}$$

which is the rule for $n + 1$. ■

Inverse rule

THE INVERSE RULE: If f is single valued, and $y = f(x)$, then

$$\frac{df^{-1}(y)}{dy}\frac{df(x)}{dx} \supseteq 1,$$

with equality holding if f is injective.

PROOF. If f is single valued and injective, then $f^{-1}(f(x)) = f^{-1}(y) = x$, and by the chain rule,

$$\frac{df^{-1}(y)}{dy}\frac{df(x)}{dx} = \frac{df^{-1}(f(x))}{df(x)}\frac{df(x)}{dx} = \frac{dx}{df(x)}\frac{df(x)}{dx} = 1.$$

If f is only single valued, then $f^{-1}(y)$ may be multivalued, and $f^{-1}(f(x)) \supseteq x$, and

$$\frac{df^{-1}(y)}{dy}\frac{df(x)}{dx} \supseteq 1. \blacksquare$$

A simple example of function with a multivalued inverse is $f(x) \equiv x^2$.

$$\begin{aligned} f^{-1}(y) &= y^{\frac{1}{2}} = \pm\sqrt{y} \\ (x^2)^{\frac{1}{2}} &= \pm x \\ \frac{d}{dy} y^{\frac{1}{2}} &= \frac{d}{dy}(\pm x) = \frac{\pm 1}{\frac{dx^2}{dx}} \\ &= \frac{\pm 1}{2x} = \frac{\pm 1}{2\sqrt{y}} \end{aligned}$$

Power rule

THE POWER RULE: For any complex n,

$$\boldsymbol{\frac{d}{dx} x^n = n x^{n-1}}$$

PROOF. FOR $n = 1$:

$$\frac{d}{dx} x = \frac{x + 0' - x}{0'} = 1.$$

FOR $n \in \mathbb{Z}^+$: By the **multiproduct rule** (p. 172) with $f_k(x) \equiv x$ and this rule for $n = 1$,

$$\frac{d}{dx} x^n = \frac{d}{dx} \prod_{k=1}^{n} x = \sum_{j=1}^{n} \left(\prod_{\substack{k=1 \\ k \neq j}}^{n} x \right) \cdot 1 = n x^{n-1}.$$

FOR $n = -1$:

$$\frac{d}{dx} \frac{1}{x} = \frac{1}{0'(x+0')} - \frac{1}{0'x} = \frac{x - x - 0'}{0'x(x+0')} = \frac{-0'}{0'x(x+0')} =' \frac{-1}{x^2}.$$

FOR $n \in \mathbb{Z}^-$: By the multiproduct rule with $f_k(x) \equiv x^{-1}$ and $m \equiv -n$, and this rule for $n = -1$,

$$\frac{d}{dx} x^n = \frac{d}{dx} x^{-m} = \frac{d}{dx} \prod_{k=1}^{m} x^{-1} = \sum_{j=1}^{m} \left(\prod_{\substack{k=1 \\ k \neq j}}^{m} x^{-1} \right) (-x^{-2}) = -m x^{-m-1} = n x^{n-1}.$$

For $n \in \frac{1}{\mathbb{Z}^*}$: By the **inverse rule** (p. 174) with $q \equiv \frac{1}{n}$ and $y \equiv x^{\frac{1}{q}}$,

$$\frac{d}{dx}x^n = \frac{d}{dx}x^{\frac{1}{q}} = \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{\frac{dy^q}{dy}} = \frac{1}{qy^{q-1}} = \frac{1}{q}y^{1-q} = \frac{1}{q}x^{\frac{1}{q}-1} = nx^{n-1}.$$

For $n \in \mathbb{Q}^*$: By the multiproduct rule with $f_k(x) \equiv x^{\frac{1}{q}}$, and this rule for $n \in \frac{1}{\mathbb{Z}}^*$, and given any $p \in \mathbb{Z}^+$ and $q \in \mathbb{Z}^*$ such that $n = \dfrac{p}{q}$,

$$\frac{d}{dx}x^n = \frac{d}{dx}x^{\frac{p}{q}} = \frac{d}{dx}\prod k = 1^p x^{\frac{1}{q}} = \sum_{j=1}^{p}\left(\prod_{\substack{k=1\\k\neq j}}^{p} x^{\frac{1}{q}}\right)\left(\frac{1}{q}x^{\frac{1}{q}-1}\right) = \frac{p}{q}x^{\frac{p}{q}-1} = nx^{n-1}.$$

For $n \in \mathbb{R}^*$: n has at least one decimal representation $\sum\limits_{k=-\infty}^{+\infty} a_k 10^k$, where each a_k is a decimal digit $0,1,\ldots,9$. As shown in **Infinite left decimals** (p. 434), this representation is unique only when the decimal representation is not repeating and infinite left decimals are not allowed. We do not require uniqueness here, only that there be at least one such representation.

The following uses an infinite case of the **multiproduct rule** (p. 172). As discussed in **Divergent Series** (p. 301–409), the numeristic theory of infinite series, including equipoint summation, does not have the inconsistencies of the conventional theory of infinite series, so we feel confident using such infinite methods without any special proofs.

Therefore, by the multiproduct rule and this rule for positive and negative integer n,

$$\begin{aligned}
\frac{d}{dx}x^n = \frac{d}{dx}x^{\sum_{k=-\infty}^{+\infty} a_k 10^k} &= \frac{d}{dx}\prod_{k=-\infty}^{+\infty} x^{a_k 10^k} \\
&= \sum_{j=-\infty}^{+\infty}\left(\prod_{\substack{k=-\infty\\k\neq j}}^{+\infty} x a_k 10^k\right)\left(a_j x^{a_j 10^j - 1}\right) \\
&= \sum_{j=-\infty}^{+\infty} a_j 10^j \frac{1}{10}\left(\prod_{k=-\infty}^{+\infty} x^{a_k 10^k}\right) \\
&= \left(\sum_{j=-\infty}^{+\infty} a_j 10^j\right) x^{\left(\sum_{k=-\infty}^{+\infty} a_k 10^k\right)-1} = nx^{n-1}.
\end{aligned}$$

FOR $n \in \mathbb{C}^*$: Let $r \equiv \operatorname{Re} n$ and $s \equiv \operatorname{Im} n$; and let b_k and c_k be the digits of respective digital representations of r and s; and a_k be the complex digits:

$$r = \sum_{k=-\infty}^{+\infty} b_k 10^k$$
$$s = \sum_{k=-\infty}^{+\infty} c_k 10^k$$
$$a_k \equiv b_k + ic_k$$

Then $n = r + is = \sum_{k=-\infty}^{+\infty} a_k 10^k$. By the multiproduct rule and the same calculation as above for real n,

$$\frac{d}{dx} x^n = \frac{d}{dx} x^{r+is} = \frac{d}{dx} x^{\sum_{k=-\infty}^{+\infty} a_k 10^k} = nx^{n-1}. \blacksquare$$

THE QUOTIENT RULE:

$$\boldsymbol{\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{g(x)\frac{d}{dx}f(x) - f(x)\frac{d}{dx}g(x)}{g(x)^2}}$$

PROOF. By the product rule, the power rule for $n = -1$, and the chain rule,

$$\begin{aligned}\frac{d}{dx}\frac{f(x)}{g(x)} &= \frac{1}{g(x)}\frac{d}{dx}f(x) + f(x)\frac{d}{dx}\frac{1}{g(x)} \\ &= \frac{1}{g(x)}\frac{d}{dx}f(x) - f(x)\frac{\frac{d}{dx}g(x)}{g(x)^2} \\ &= \frac{g(x)\frac{d}{dx}f(x) - f(x)\frac{d}{dx}g(x)}{g(x)^2}. \blacksquare\end{aligned}$$

Derivatives of sine and cosine

$$\boldsymbol{\frac{d\cos\theta}{d\theta} = -\sin\theta}$$
$$\boldsymbol{\frac{d\sin\theta}{d\theta} = \cos\theta}$$

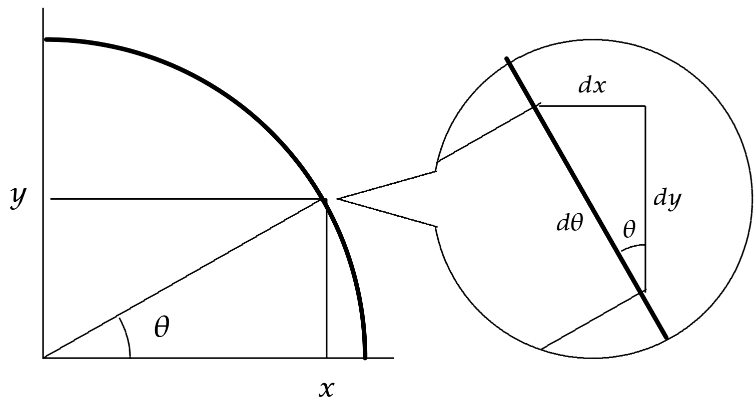

FIG. 54:
Calculation of derivatives
of sine and cosine

PROOF. Figure 54 depicts the calculation of the derivatives of the sine and cosine functions. In this figure, we have

$$x = \cos\theta$$
$$y = \sin\theta$$

and a microscope picture of the point (x, y).

In the microscope, the circle has become a straight line, coincident with the tangent to the circle at (x, y). Outside the microscope, the radius is a single line, but within the microscope, the radius is the class of all lines normal to the tangent. We show two such radius lines that are separated by the distance $d\theta$. The units of this distance must match the units in the tangent and radius, so we must measure $d\theta$, and thus θ itself, in radians.

The line segment along the tangent bounded by the two radii forms a triangle with legs dx and dy and hypotenuse $d\theta$. We then have

$$\frac{y}{x} = -\frac{dx}{dy}$$
$$\frac{dx}{d\theta} = -\sin\theta$$
$$\frac{dy}{d\theta} = \cos\theta. \blacksquare$$

Derivative of exponential function

$$\boldsymbol{\frac{d}{dx}e^x = e^x}$$

PROOF. We start with an equipoint definition of e and compute e^x.

$$\begin{aligned} e &\equiv \left(1 + \frac{1}{\infty'}\right)^{\infty'} \\ e^x &= \left(1 + \frac{1}{\infty'}\right)^{\infty' x} \\ &= \left(1 + \frac{x}{\infty''}\right)^{\infty''}. \end{aligned}$$

The last line comes from the substitution $\infty' x \to \infty''$, or $\infty' \to \dfrac{\infty''}{x}$. Adding to this the substitution $\dfrac{1}{\infty''} \to 0''$, we then have

$$\begin{aligned} e^{0'' x} &= (e^x)^{0''} \\ &= \left(1 + \frac{x}{\infty''}\right)^{\infty'' 0''} \\ &= \left(1 + \frac{x}{\infty''}\right)^{1} \\ &= 1 + \frac{x}{\infty''} \\ &= 1 + 0'' x. \end{aligned}$$

Solving for x we have

$$\begin{aligned} x &= \frac{e^{0'' x} - 1}{0''} \\ 1 &= \frac{e^{0''} - 1}{0''} \\ e^x &= e^x \frac{e^{0''} - 1}{0''} \end{aligned}$$

$$= \frac{e^{x+0''} - e^x}{0''}$$

$$=' \frac{d}{dx} e^x.$$

Natural logarithm as a polynomial

$$\mathbf{\ln t = \frac{t^{0'} - 1}{0'}}$$

PROOF. Start with a result from the previous section and substitute $x = \ln t$.

$$x = \frac{e^{0'x} - 1}{0'}$$

$$\ln t = \frac{t^{0'} - 1}{0'}.$$

$\int_1^t t^{-1}\,dt = \ln t$ is an instance of the general law $\int_0^t t^n\,dt = \frac{t^{n+1}}{n+1}$, not an exception.

PROOF. Integrate t^{-1} with $\int_0^t t^n\,dt = \frac{t^{n+1}}{n+1}$ and obtain

$$\int_1^t t^{-1}\,dt = \frac{t^{0'}}{0'} - \frac{1^{0'}}{0'} = \frac{t^{0'} - 1^{0'}}{0'} = \frac{t^{0'} - 1}{0'} = \ln t.$$

By the **principle of determinacy** (p. 136), unfolded zero must be used, since $\frac{t^0 - 1}{0}$ is indeterminate. By **implied unfolding** (p. 161), the expression $t^{0'} - 1$ means $t^{0'} - 1^{0'}$. ■

This result can be verified with L'Hôpital's rule, which is proved below. For real t, this result is also verified by the following.

$$\int_1^x t^{-1}\,dt = \ln|x|$$

$$\frac{d}{dx}\ln|x| = \frac{\operatorname{sgn} x}{x} = \frac{x}{|x|^2} = \frac{1}{x}.$$

$\dfrac{y^{0'}-1}{0'}$ is a zeroth-order polynomial in unfolded arithmetic, or more accurately a polynomial of order $0'$. Its integrals are unfolded polynomals of higher degrees:

$$\begin{aligned}
\int \ln x\,dx &= x\ln x - x \\
&= \frac{x^{1+0'}-x}{0'} - x \\
\int (x\ln x - x)\,dx &= \frac{x^2}{2}\ln x - \frac{3x^2}{4} \\
&= \frac{x^{2+0'}-x^2}{2\cdot 0'} - \frac{3x^2}{4}
\end{aligned}$$

Constant functions

Given the constant function $q(x) = 5$, the power law does not apply for all x, because $q(x)$ differs from $Q(x) = 5x^0$ for abfinite x, where x^0 is indeterminate. We also cannot differentiate as follows:

$$\frac{dq(x)}{dx} \stackrel{?}{=} \frac{5-5}{0'} \stackrel{?}{=} \frac{0}{0'} = \not\propto.$$

The last equality holds because 0 is a folded number which includes unfolded numbers that are infinitely larger and smaller than $0'$, such as:

$$\begin{array}{lllll}
0 \supset 0'^2 & \dfrac{0'^2}{0'} & = & 0' & =' \ 0 \\
0 \supset 2\cdot 0' & \dfrac{2\cdot 0'}{0'} & & & =' \ 2 \\
0 \supset 0'^{\frac{1}{2}} & \dfrac{0'^{\frac{1}{2}}}{0'} & = & \dfrac{1}{0'^{\frac{1}{2}}} & =' \ \infty
\end{array}$$

The error arises in the first two equalities. By the principle of determinacy, the numerator should be unfolded at the same level as the denominator. The correct derivative is then

$$\frac{dq(x)}{dx} = \frac{5_{[0']} - 5_{[0']}}{0'} = \frac{5 + 0 \cdot 0' - 5 - 0 \cdot 0'}{0'} = \frac{0 \cdot 0'}{0'} = 0.$$

Similarly, for the linear function $s(x) = 5x$, we may not differentiate as follows:

$$\frac{ds(x)}{dx} = \frac{5 + 0' - 5}{0'} = \frac{0'}{0'} \stackrel{?}{=} 0'^0 = \not\infty.$$

Of course we can reduce $\frac{0'}{0'}$ as a fraction to 1, but to reduce it through an exponent, we must unfold the exponent:

$$\frac{0'}{0'} = \frac{0'^{1'}}{0'^{1'}} = 0'^{1'-1'} = 0'^{0 \cdot 0'} = e^{0' \cdot 0' \ln 0'} = e^{0''} =' 1.$$

Antiderivatives

We define the ***semidefinite integral*** of f as a definite integral with a constant lower limit and a variable upper limit:

$$\int_s^x f(u)\, du.$$

By the second fundamental theorem of calculus, the derivative of this function is $f(x)$.

The ***antiderivative*** or ***indefinite integral*** of f is a class of functions, with each member of this class being a semidefinite integral for a fixed s, plus a constant that varies within the class over the real numbers:

$$\int f(x)\, dx \equiv \int_s^x f(u)\, du + \mathbb{R} = \left\{ \int_s^x f(u)\, du + r \;\middle|\; r \in \mathbb{R} \right\}.$$

Without loss of generality, we can often assume $s = 0$ and define as follows:

$$\int f(x)\, dx \equiv \int_0^x f(u)\, du + \mathbb{R} = \left\{ \int_0^x f(u)\, du + r \;\middle|\; r \in \mathbb{R} \right\}.$$

In either case, this is the class of all functions whose derivative is $f(x)$.

This definition assumes that $f(x)$ is integrable over the whole real number line. For exceptions to this condition, see the **Singularities** (p. 203) chapter.

As an application of this definition, we examine a formula for the integral of an inverse function discovered by C.-A. Laisant in 1905.

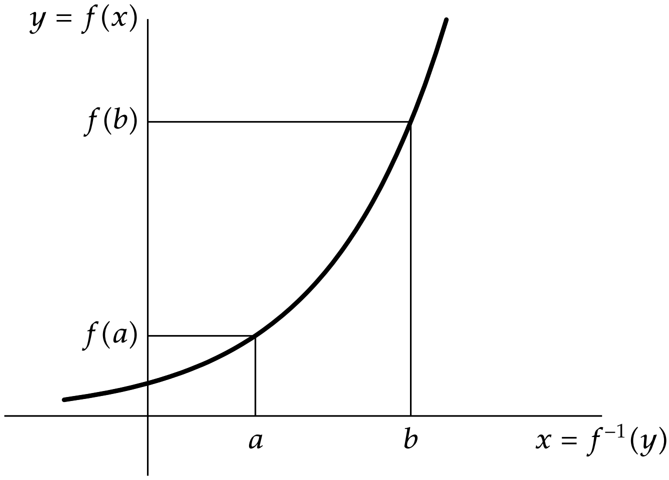

FIG. 55:
Calculation of integral of inverse function

INTEGRAL OF AN INVERSE FUNCTION:

$$\int f^{-1}(y)\,dy = yf^{-1}(y) - \int f(x)\,dx\Big|_{f^{-1}(y)} + \mathbb{R}$$

PROOF. In Fig. 55, the area of the large rectangle consists of three portions: the small rectangle, the area under $f(x)$ from $x = a$ to $x = b$, and the area under $f^{-1}(y)$ from $y = f(a)$ to $y = f(b)$. In symbols:

$$bf(b) = af(a) + \int_a^b f(x)\,dx + \int_{f(a)}^{f(b)} f^{-1}(y)\,dy$$

$$\int_{f(a)}^{f(b)} f^{-1}(y)\,dy = bf(b) - af(a) - \int_a^b f(x)dx$$

Substituting $v = y$, $y = f(b)$, $s = f(a)$, this becomes

$$\int_s^y f^{-1}(v)\,dv = yf^{-1}(y) - sf^{-1}(s) - \int_{f^{-1}(s)}^{f^{-1}(y)} f(x)\,dx$$

$$\begin{aligned}\int_s^y f^{-1}(v)\,dv + r &= yf^{-1}(y) - \int_{f^{-1}(s)}^{f^{-1}(y)} f(x)\,dx - sf^{-1}(s) + r \\ &= yf^{-1}(y) - \int_{f^{-1}(s)}^{f^{-1}(y)} f(x)\,dx + r'\end{aligned}$$

and by the above definition

$$\int f^{-1}(y)\,dy = yf^{-1}(y) - \int f(x)\,dx\bigg|_{f^{-1}(y)} + r'$$

r' is arbitrary, so

$$\int f^{-1}(y)\,dy = yf^{-1}(y) - \int f(x)\,dx\bigg|_{f^{-1}(y)} + \mathbb{R}. \blacksquare$$

Applying this to $\ln y$, we have

$$\begin{aligned}\int \ln y\,dy &= y\ln y - \int e^x\,dx\bigg|_{\ln y} + \mathbb{R} \\ &= y\ln y - e^x\Big|_{\ln y} + \mathbb{R} = y\ln y - y + \mathbb{R}.\end{aligned}$$

LIMITS AND CONTINUITY

Limits

Definition of limit forms

A limit can be defined with an unfolded expression which gives results similar to those given by conventional definitions. In many cases, these expressions can be evaluated where a conventional limit fails to exist. Any syntactically correct statement is meaningful, and so these expressions always have a meaning, which may include multivalued classes or the empty class. We will later see several examples of this.

$$\lim_{x \to a} f(x) \equiv f(a + 0'), \text{where } a \text{ is finite and } 0' \neq 0'^2$$
$$\lim_{x \to a+} f(x) \equiv f(a + 0'), \text{where } a \text{ is finite and } 0' > 0'^2$$
$$\lim_{x \to a-} f(x) \equiv f(a + 0'), \text{where } a \text{ is finite and } 0' < 0'^2$$
$$\lim_{x \to \infty} f(x) \equiv f(\infty'), \text{where } \infty' \neq \infty'^2$$
$$\lim_{x \to +\infty} f(x) \equiv f(\infty'), \text{where } \infty' < \infty'^2$$
$$\lim_{x \to -\infty} f(x) \equiv f(\infty'), \text{where } \infty' > \infty'^2$$

Example

$$\lim_{x \to 0} \frac{e^x - 1}{x} \equiv \frac{e^{0'} - 1}{0'} = 1$$

Offset expressions

A limit in the form $f(a+0')$ or $f(\infty')$ will also be called an ***offset expression***. It is not the result of a process but simply a value of a function at an unfolded point. An expression in the form of $f(a)$ or $f(\infty)$, which is at the origin of the unfolding, will be called an ***original expression***.

Any of the above expressions may be multivalued and/or depend on $0'$ or ∞'. In such cases, we may wish to restrict our attention to those cases in which the expression is single valued and/or independent of $0'$ or ∞'.

Uniform offset expressions

An offset expression $f(a+0')$ is ***uniform*** if it has the same folded value for all $0'$, i.e. if $f(a+0') =' f(a+0'')$ for any $0', 0'' \in' 0$, even when $0'$ and $0''$ have different signs. If f is single valued and $f(a+0')$ is uniform, then the class $f(a+\mathbb{R}0')$ or $f(a+\mathbb{C}0')$ is single valued. A derivative $f'_{0'}(a) = \dfrac{f(a+0')-f(a)}{0'}$ is uniform if it has the same value for all $0'$.

An offset expression $f(a+0')$ is ***semiuniform*** if it is the same for every $0'$ of the same sign. If f is single valued and $f(a+0')$ is semiuniform, then the class $f(a+|\mathbb{R}|0')$ or $f(a+|\mathbb{C}|0')$ is single valued.

An offset expression $f(a+0')$ is ***disuniform*** if it is neither uniform nor semiuniform.

Continuity

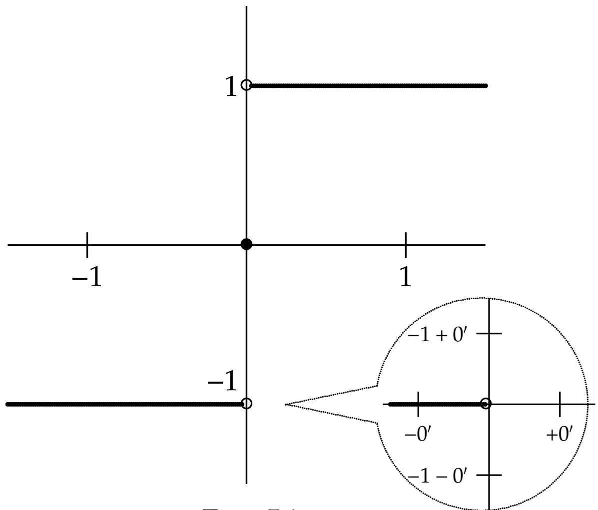

FIG. 56:
Discontinuity of signum function
sgn x at $x = 0$

Definition

A function f is ***continuous*** at x if the offset values are uniform and equal to the original value, i.e. $f(x+a0') =' f(x)$ for every $a \in \mathbb{R}$, or $f(x+\mathbb{R}0') =' f(x)$.

A function f is ***semicontinuous*** at x if the offset values are semiuniform and equal to the original value, i.e. $f(x + a0') =' f(x + b0')$ for every $a, b \in \mathbb{R}$ and $\operatorname{sgn} a = \operatorname{sgn} b$. In this case, if $f(x+a0') =' f(x)$ for positive a, then f is ***right continuous***; for negative a, ***left continuous***.

Figure 56 shows an example of a discontinuity in the signum function

$$\operatorname{sgn} x \equiv \begin{cases} -1 & \text{for } x < 0 \\ 0 & \text{for } x = 0 \\ +1 & \text{for } x > 0. \end{cases}$$

This function is discontinuous at $x = 0$ because $\operatorname{sgn}(x) = 0$ while $\operatorname{sgn}(x - 0') = -1$ and $\operatorname{sgn}(x + 0') = +1$ for $0' > 0^2$. It is semicontinuous but neither left continuous nor right continuous.

If f is continuous at x, then f is locally linear: $f(x+0') - f(x) =' f(x + 0'(k+1)) - f(x+0'k)$ for real k.

If f has a finite derivative at x, then it is continuous at x:

$$\begin{aligned} f'(x) &\equiv \frac{f(x+0') - f(x)}{0'} \\ f(x+0') &= 0'f'(x) + f(x) \\ \text{Since } f'(x) \text{ is finite, } 0'f'(x) &=' 0 \\ f(x+0') &=' f(x). \end{aligned}$$

Continuity for infinite values

Continuity involving infinite values may depend on the choice of infinite element extension. Consider the reciprocal function $f(x) \equiv \dfrac{1}{x}$. At $x = 0$, $f(x + a0') = \dfrac{1}{0 + a0'} =' \dfrac{1}{a0'}$. In the projectively extended real numbers,

$$\frac{1}{a0'} =' \infty$$

for all real a, so $f(x)$ is continuous at $x = 0$. But in the affinely extended real numbers,

$$\frac{1}{a0'} = \begin{cases} +\infty \neq' -\infty & \text{for } a > 0 \\ -\infty \neq' +\infty & \text{for } a < 0, \end{cases}$$

so $f(x)$ is not continuous at $x = 0$, only semicontinuous.

It can be shown that the characteristic function of the rational numbers $[\mathbb{Q}](x)$ is continuous at irrational x and discontinous at rational x. See **Using class counts in derivatives and integrals** (p. 279).

Continuity of multivalued functions

One approach to the continuity of multivalued functions is to examine a single valued branch. A branch can be often be defined by intersecting the multivalued function with a subset of the range. For example, in $\mathbb{R}$, if we take $x^{\frac{1}{2}}$ to mean the multivalued inverse of x^2, we can define $\sqrt{x}$ to be the nonnegative branch of $x^{\frac{1}{2}}$, i.e. $\sqrt{x} \equiv x^{\frac{1}{2}} \cap |\mathbb{R}|$. Then $\sqrt{x}$ is continuous, since $\sqrt{x + \mathbb{R}0'} =' \sqrt{x}$.

Another approach to multivalued continuity is distributed continuity. We define a multivalued function $f(x)$ to be ***conjunctively continuous*** if $(\forall a \in f(x+\mathbb{R}0'))\ a \in f(x)$, and ***disjunctively continuous*** if $(\exists a \in f(x+\mathbb{R}0'))\ a \in f(x)$. Then $x^{\frac{1}{2}}$ is conjunctively continuous everywhere, since $(\forall a \in (x+\mathbb{R}0')^{\frac{1}{2}})\ a \in x^{\frac{1}{2}}$ for all x.

Subcontinuity of multivalued functions

We define a multivalued function to be $f(x)$ ***subcontinuous*** at x if $f(x+\mathbb{R}0') \subseteq f(x)$. We say it is ***semisubcontinuous*** if $f(x+a0') \subseteq f(x)$ for all a of a given sign. Assuming $0'$ is positive, i.e. $0' > 0{\cdot}0'$, then if f is semisubcontinuous, we say that it is ***right subcontinuous*** if a is positive, and ***left subcontinuous*** if a is negative.

In the projectively extended real numbers, we found above that $\dfrac{1}{x}$ is continuous at $x = 0$, but the same does not hold for $e^{\frac{1}{x}}$. At $x = 0$, this function has two values:

$$e^{\frac{1}{0}} = e^{\infty} = \left\{e^{+\infty}, e^{-\infty}\right\} = \{\infty, 0\}$$

and, assuming $0'$ is positive,

$$e^{\frac{1}{a0'}} =' \begin{cases} +\infty & \text{for } a > 0 \\ 0 & \text{for } a < 0, \end{cases}$$

so $e^{\frac{1}{x}}$ is both left and right subcontinuous at $x = 0$.

L'Hôpital's rule

L'Hôpital's rule can be used to evaluate the limit of the quotient of two coabfinite expressions, i.e. the numerator and denominator are either both zero or both infinite.

L'Hôpital's rule for 0/0: If functions f and g are continuous at c and $f(c) = g(c) = 0$, then $\dfrac{f(c+0')}{g(c+0')} = \dfrac{f'(c)}{g'(c)}$.

By the above definition of limit, $\dfrac{f(c+0')}{g(c+0')}$ can also be denoted $\lim\limits_{x\to c} \dfrac{f(x)}{g(x)}$.

PROOF. Since $f(c) = 0$, $f(c+0') = f(c+0') - f(c)$, and similarly for g. Then

$$\frac{f(c+0')}{g(c+0')} = \frac{f(c+0') - f(c)}{g(c+0') - g(c)}$$
$$= \frac{\frac{f(c+0')-f(c)}{0'}}{\frac{g(c+0')-g(c)}{0'}}$$
$$= \frac{f'(c)}{g'(c)}.$$

If both $f'(c)$ and $g'(c)$ are also coabfinite, and f' and g' are continuous at c, then $f'(c) =' f'(c+0')$ and $g'(c) =' g'(c+0')$, and we can iterate the rule, until we find an n for which $f^{(n)}(c)$ and $g^{(n)}(c)$ are noncoabfinite, and $f^{(n)}$ and $g^{(n)}$ are continuous at c. The rule does not apply if there is no such n. ■

L'HÔPITAL'S RULE FOR ∞/∞: If functions f and g are continuous at c, $f(c) = g(c) = \infty$, and f and g are finite in some punctured neighborhood around c, then $\dfrac{f(c+0')}{g(c+0')} = \dfrac{f'(c)}{g'(c)}$.

PROOF. Let $c' \equiv c + 0'$. Since f and g are continuous and infinite at an isolated point c, the unfolded f and g must take on every unfolded infinite value within the unfolded space around c.

Let $c + 0''$ be a point within the unfolded space around c. Within this space, the magnitudes of f and g strictly decrease monotonically as the magnitude of $0''$ increases. It is therefore possible to choose $0''$ so that both

$$\log_{\infty'} |f(c'+0'')| < \log_{\infty'} |g(c')| \text{ and}$$
$$\log_{\infty'} |g(c'+0'')| < \log_{\infty'} |g(c')|.$$

Since the magnitudes of $f(c'+0'')$ and $g(c'+0'')$ are less than that of $g(c')$, we have

$$\frac{f(c'+0'')}{g(c')} =' 0$$
$$\frac{g(c'+0'')}{g(c')} =' 0$$

We then compute

$$\begin{aligned}
\frac{f(c')}{g(c')} &= \frac{\frac{f(c')}{g(c')}}{1} \\
&=' \frac{\frac{f(c')}{g(c')} - \frac{f(c'+0'')}{g(c')}}{1 - \frac{g(c'+0'')}{g(c')}} \\
&= \frac{f(c'+0'') - f(c')}{g(c'+0'') - g(c')} \\
&= \frac{f'(c')}{g'(c')}.
\end{aligned}$$

$\dfrac{f'(c')}{g'(c')} =' \dfrac{f'(c)}{g'(c)}$ as long as $f'(c)$ and $g'(c)$ are noncoabfinite. Otherwise the rule can be interated until $f^{(n)}(c)$ and $g^{(n)}(c)$ are noncoabfinite. ■

This proof only requires that $g(c)$ be infinite. If $f(c)$ is finite, then the rule still applies but is not needed, since $\dfrac{f(c)}{g(c)} = 0$ by folded extended arithmetic.

Since numeristic division and logarithms are total (unrestricted) functions, it is easy to extend the rule to other indeterminate forms.

- If $f(c)g(c)$ is of the form $0 \cdot \infty$, then use the rule on $\dfrac{f(x)}{\frac{1}{g(x)}}$ or $\dfrac{\frac{1}{f(x)}}{g(x)}$.
- If $f(c) - g(c)$ is of the form $\infty - \infty$, then use the rule on $e^{f(c)-g(c)} = \dfrac{e^{f(c)}}{e^{g(c)}}$.
- If $f(c)^{g(c)}$ is of the form 0^0, 1^∞, or ∞^0, then use the rule on $\ln f(c)^{g(c)} = g(c) \ln f(c)$.

DIFFERENTIAL AND INTEGRAL OPERATORS

Differentials and integrants

Differentials

As suggested by the Leibniz notation $\frac{dy}{dx}$, a derivative is an arithmetic quotient of differentials. The differential of an independent variable is an infinitesimal, as is the differential of a dependent variable when the derivative is finite. Infinitesimals are unfolded members of folded zero, which are exactly equal to zero in folded aritmetic but distinct in unfolded arithmetic.

A ***differential*** is an operator on a function with respect to a member of zero. We define

$$^{0'}d_a f(x) \equiv f(a + 0') - f(a),$$

from which follows

$$^{0'}d_a x \equiv a + 0' - a = 0'.$$

A derivative with respect to an infinitesimal $0'$ can therefore be defined as:

$$f'_{0'}(a) \equiv \frac{^{0'}d_a f(x)}{^{0'}d_a x} = \frac{f(a + 0') - f(a)}{0'}.$$

If the derivative is independent of the infinitesimal, we write:

$$f'(a) \equiv \frac{d_a f(x)}{d_a x} = \frac{f(a + 0') - f(a)}{0'}.$$

This occurs when $f(x)$ is analytic, since, for $f(x) = x^n$,

$$^{0'}d_a f(x) = nx^{n-1}0' + \sum_{k=2}^{n} \binom{n}{k} a^{n-k} 0'^k =' nx^{n-1}0'.$$

The notations $f'(x)$ and $\dfrac{df(x)}{dx}$, of course, mean the function $\dfrac{d_x f(x)}{d_x x}$, and $\dfrac{d_a f(x)}{d_a x}$ can be written $f'(x)\big|_a$ or $\left.\dfrac{df(x)}{dx}\right|_a$.

Integrants

We define an ***integrant*** as an operator on a function:

$$\int^{0'}\!\!{}^{a} f(x) \equiv \sum_{k=1}^{\frac{a}{0'}} f(0'k).$$

An integrant is infinite whenever the corresponding integral is finitesimal.

The definite integral can be defined in terms of an integrant and a differential:

$$\int_a^{0'\,b} f(x)\,dx \equiv \int^{0'\,b} f(x)0' - \int^{0'\,a} f(x)0' = \sum_{k=1}^{\frac{b}{0'}} f(0'k)0' - \sum_{k=1}^{\frac{a}{0'}} f(0'k)0'$$

$$\left(\text{substituting } \infty' \equiv \tfrac{b-a}{0'}\right)$$

$$= \sum_{k=1}^{\frac{\infty' b}{b-a}} f\left(k\frac{b-a}{\infty'}\right)\frac{b-a}{\infty'} - \sum_{k=1}^{\frac{\infty' a}{b-a}} f\left(k\frac{b-a}{\infty'}\right)\frac{b-a}{\infty'}$$

$$= \sum_{k=\frac{\infty' a}{b-a}+1}^{\frac{\infty' b}{b-a}} f\left(k\frac{b-a}{\infty'}\right)\frac{b-a}{\infty'} \qquad \left(\text{substituting } j \equiv k - \tfrac{\infty' a}{b-a}\right)$$

$$= \sum_{j=1}^{\infty'} f\left(a + j\frac{b-a}{\infty'}\right)\frac{b-a}{\infty'}$$

If the integral is independent of the infinitesimal, we write:

$$\int_a^b f(x)\,dx \equiv \sum_{k=1}^{\frac{b}{0'}} f(0'k)0' - \sum_{k=1}^{\frac{a}{0'}} f(0'k)0'.$$

We can define the indefinite integral operator in terms of the definite integral in two ways. The first way is as a definite integral plus an arbitrary constant:

$$\overset{0'}{\int} f(x)\,dx \equiv \overset{0'}{\int}{}^{x} f(t)\,dt + \mathbb{R} = \left\{ \overset{0'}{\int}{}^{x} f(t)\,dt + a \mid a \in \mathbb{R} \right\}$$

or

$$\int f(x)\,dx \equiv \int^{x} f(t)\,dt + \mathbb{R} = \left\{ \int^{x} f(t)\,dt + a \mid a \in \mathbb{R} \right\}.$$

The second way to define the indefinite integral is as a class of definite integrals with an arbitrary lower limit:

$$\overset{0'}{\int} f(x)\,dx \equiv \overset{0'}{\int}{}_{\mathbb{R}}^{x} f(t)\,dt = \left\{ \overset{0'}{\int}{}_{a}^{x} f(t)\,dt \mid a \in \mathbb{R} \right\}$$

or

$$\int f(x)\,dx \equiv \int_{\mathbb{R}}^{x} f(t)\,dt = \left\{ \int_{a}^{x} f(t)\,dt \mid a \in \mathbb{R} \right\}.$$

Either of these is a class of functions. If we denote the first as A and the second as B, then given any two $F_1, F_2 \in A$, we have $F_2(x) = F_1(x) + c$, where c is a constant, and conversely. Similarly, given any two $F_1, F_2 \in B$ and their corresponding a_1, a_2, we have $F_2(x) = F_1(x) - F(a_1) + F(a_2)$. Thus $A \supseteq B$, with equality holding if all the members of A are surjective.

The integrant is the left inverse of the differential, which is essentially the first fundamental theorem of calculus:

$$\begin{aligned}
\int_0^x df(x) &= \sum_{k=1}^{\frac{x}{0'}} f(0'k + 0') - f(0'k) \\
&= \sum_{k=1}^{\frac{x}{0'}} f\left(0'(k+1)\right) - f(0'k) \\
&= f\left(0'\left(\frac{x}{0'} + 1\right)\right) - f(0') \\
&= f(x + 0') - f(0') \\
&=' f(x) - f(0').
\end{aligned}$$

The integrant is also the right inverse of the differential, which is essentially the second fundamental theorem of calculus:

$$\begin{aligned}
d\int_0^x f(x) &= \int^{x+0'} f(x) + \int^{x} f(x) \\
&= \int^{x} f(x) + \int_{x}^{x+0'} f(x) + \int^{x} f(x)
\end{aligned}$$

$$= \sum_{k=1}^{\frac{x+0'-x}{0'}} f(x)$$
$$=' f(x).$$

Partial differentials

The ***partial differential*** is defined analogously to the differential. Here we define a partial differential on a function of two independent variables:

$$^{0'}\partial_x f(x,y) \equiv f(x+0',y) - f(x,y),$$

or, if the result is independent of $0'$:

$$\partial_x f(x,y) \equiv f(x+0',y) - f(x,y).$$

The total differential is then easily seen to be the sum of partial differentials:

$$\begin{aligned} df(x,y) = d_{x,y} f(x,y) & = f(x+0',y+0') - f(x,y) \\ & = \left[f(x+0',y+0') - f(x,y+0')\right] + \left[f(x,y+0') - f(x,y)\right] \\ & = \left[f(x+0',y) - f(x,y)\right] + \left[f(x,y+0') - f(x,y)\right] \\ & = \partial_x f(x,y) + \partial_y f(x,y) \\ & = (\partial_x + \partial_y) f(x,y) \end{aligned}$$

Quotiential and prodegrant operators

Closely related to the differential is its multiplicative equivalent, the ***quotiential***:

$$^{0'}q_a f(x) \equiv \frac{f(x+0')}{f(x)} = e^{d \ln f(x)}.$$

The inverse of the quotiential is the ***prodegrant***:

$$^{0'}\!\!\int^{a} f(x) \equiv \prod_{k=1}^{\frac{a}{0'}} f(0'k) = e^{\int \ln x}.$$

From the quotiential and differential we derive two ***quotient derivatives***, the ***geometric derivative*** and the ***bilogarithmic derivative***:

$$\sqrt[dx]{qf(x)} = qf(x)^{\frac{1}{dx}} = e^{\frac{d \ln f(x)}{dx}} = e^{\frac{df(x)}{f(x)dx}},$$

$$\log_{qx} qf(x) = e^{\frac{df(x)}{d\ln x}} = e^{\frac{x df(x)}{dx}}.$$

We derive two ***product integrals***: the ***geometric integral*** or ***type 1 product integral***, and the ***bilogarithmic integral***:

$$\begin{aligned} \int_a^b f(x)^{dx} &\equiv \frac{\int^b f(x)^{dx}}{\int^a f(x)^{dx}} \\ &= e^{\int_a^b \ln f(x)\, dx} \\ \int_a^b qx^{f(x)} &\equiv \frac{\int^b qx^{f(x)}}{\int^a qx^{f(x)}} \\ &= e^{\int_a^b f(x)\, d\ln x} \end{aligned}$$

Somewhat ambiguously, the symbol $\prod$ is sometimes used elsewhere instead of $\int$.

Volterra, who first investigated product integrals [V87], originally defined what is now called the ***type 2 product integral***:

$$\prod_a^b [1 + f(x)\, dx] \equiv e^{\int_a^b f(x)\, dx} = \int_a^b e^{f(x)\, dx}$$

The inverse of the type 2 product integral is the ***logarithmic derivative***:

$$\frac{f'(x)}{f(x)} = \frac{d\ln f(x)}{dx} = \frac{df(x)}{x\, dx}.$$

The ***partial quotiential*** is given by:

$${}^{0'}\varphi_x f(x,y) \equiv \frac{f(x+0',y)}{f(x,y)} = e^{{}^{0'}\partial_x \ln f(x,y)}.$$

Higher order derivatives and integrals

The simple form of the equipoint derivative lends itself to direct calculaton of higher order derivatives. These derivatives are are also simple quotients, with $dx^n \equiv (dx)^n$ in the denominator.

HIGHER ORDER DERIVATIVE FORMULA:

$$\boldsymbol{f^{(n)}(x) = \frac{\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(x + 0'k)}{0'^n}}$$

PROOF. Computing higher order derivatives is mainly a matter of computing the numerator $d^n f(x)$, which is an iterated application of the differential operator:

$$\begin{aligned} f''(x) &= \frac{d^2}{dx^2} f(x) \\ &= \frac{d[d[f(x)]]}{0'^2} \\ &= \frac{d[f(x+0') - f(x)]}{0'^2} \\ &= \frac{[f(x+2\cdot 0') - f(x+0')] - [f(x+0') - f(x)]}{0'^2} \\ &= \frac{f(x+2\cdot 0') - 2f(x+0') + f(x)}{0'^2}. \end{aligned}$$

The expansion of these operators is similar to expansion of the binomial power

$$\begin{aligned} (a-b)^n &= \sum_{k=0}^{n} (-1)^k \binom{n}{k} a^{n-k} b^k \\ &= \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} a^k b^{n-k}. \end{aligned}$$

In derivatives, $f(x + 0'k)$ corresponds to $a^k b^{n-k}$: the n-th derivative is

$$f^{(n)}(x) = \frac{d^n}{dx^n} f(x)$$

$$= \frac{\sum_{k=0}^{n}(-1)^k \binom{n}{k} f\left(x + [n-k]0'\right)}{0'^n}$$

$$= \frac{\sum_{k=0}^{n}(-1)^{n-k} \binom{n}{k} f(x + 0'k)}{0'^n},$$

which can be proved by induction as follows:

$$f^{(0)}(x) = f(x) = \binom{0}{0}(-1)^0 f(x - 0'),$$

and

$$\frac{d^{n+1}}{dx^{n+1}} f(x) = \frac{d}{dx}\frac{d^n}{dx^n} f(x) = \frac{1}{0'}\left[\frac{d^n}{dx^n} f(x - 0') - \frac{d^n}{dx^n} f(x)\right]$$

$$= \frac{1}{0'^{n+1}}\left[\sum_{k=0}^{n}(-1)^k \binom{n}{k} f\left(x + [n+1-k]0'\right)\right.$$

$$\left. - \sum_{k=0}^{n}(-1)^k \binom{n}{k} f\left(x + [n-k]0'\right)\right]$$

$$= \frac{1}{0'^{n+1}}\left[\binom{n}{0} f\left(x + [n+1]0'\right)\right.$$

$$+ \sum_{k=1}^{n}(-1)^k \left[\binom{n}{k} + \binom{n}{k-1}\right] f\left(x + [n+1-k]0'\right)$$

$$\left. + \binom{n}{n} f(x)\right]$$

$$= \frac{1}{0'^{n+1}}\left[\binom{n+1}{0} f\left(x + [n+1]0'\right)\right.$$

$$+ \sum_{k=1}^{n}(-1)^k \binom{n+1}{k} f\left(x + [n+1-k]0'\right)$$

$$\left. + \binom{n+1}{n+1} f(x)\right]$$

$$= \frac{1}{0'^{n+1}}\sum_{k=0}^{n+1}(-1)^k \binom{n+1}{k} f\left(x + [n+1-k]0'\right). \blacksquare$$

A simple example of this theorem:

$$\begin{aligned}
\frac{d^2(x^3)}{dx^2} &= \frac{(x+2\cdot 0')^3 - 2(x+0')^3 + x^3}{0'^2} \\
&= \frac{x^3 + 6\cdot 0'x^2 + 12\cdot 0'^2 x + 0'^3 - 2x^3 - 6\cdot 0'x^2 - 6\cdot 0'^2 x - 0'^3 + x^3}{0'^2} \\
&=' 6x
\end{aligned}$$

HIGHER ORDER INTEGRAL FORMULA:

$$\boldsymbol{\underbrace{\int\cdots\int}_{n} f(x)\,dx^n = f^{(-n)}(x) = \sum_{k=n}^{\infty'} \binom{k-1}{k-n} f(x-0'k)\,0'^n.}$$

PROOF. Since the binomial theorem extends to negative exponents, we can extend the previous theorem to integrals. In this case, the upper limit on the summation is infinite:

$$\begin{aligned}
(a-b)^{-n} &= \sum_{k=0}^{\infty'} (-1)^k \binom{-n}{k} a^{-n-k} b^k \\
&= \sum_{k=0}^{\infty'} (-1)^{2k} \binom{n+k-1}{k} a^{-n-k} b^k \\
&= \sum_{k=0}^{\infty'} \binom{n+k-1}{k} a^{-n-k} b^k.
\end{aligned}$$

For $n = 1$, this becomes

$$(a-b)^{-1} = \sum_{k=0}^{\infty'} a^{-1-k} b^k$$

and

$$\begin{aligned}
f^{(-1)}(x) &= \frac{d^{-1}}{dx^{-1}} f(x) \\
&= \sum_{k=0}^{\infty'} f(x-(k+1)0')\,0' \\
&= \sum_{k=1}^{\infty'} f(x-0'k)\,0'.
\end{aligned}$$

Taking $0' = \frac{a-x}{\infty'}$, the above summation matches the definition of the definite integral:

$$f^{(-1)}(x) = \sum_{k=1}^{\infty'} f(x - 0'k)\; 0' = \int_a^x f(t)\,dt.$$

Since ∞' is independent of $0'$, a is arbitrary, and this expression is actually a class of functions of x, each expressed as a definite integral with a fixed lower limit and a variable upper limit. This matches the second definition of the indefinite integral $\int f(x)\,dx$ given in **Differentials and integrants** (p. 192) above.

Higher order integrals are obtained through other negative powers of binomials:

$$\underbrace{\int \cdots \int}_{n} f(x)\,dx^n = f^{(-n)}(x) = \frac{d^{-n}}{dx^{-n}} f(x)$$

$$= \sum_{k=0}^{\infty'} (-1)^k \binom{-n}{k} f(x - (k+n)0')\; 0'^n$$

$$= \sum_{k=0}^{\infty'} \binom{k+n-1}{k} f(x - (k+n)0')\; 0'^n$$

$$= \sum_{k=n}^{\infty'} \binom{k-1}{k-n} f(x - 0'k)\; 0'^n. \;\blacksquare$$

Power series

In the following, we define σ as an integration operator with a fixed lower bound and a variable upper bound:

$$\sigma_a f(t) \equiv \int_a^t f(t)\,dt$$

and

$$\sigma f(t) \equiv \sigma_0 f(t) = \int_0^t f(t)\,dt$$

Powers of σ denote repeated integration or differentiation:

$$\sigma^n f(t) = \underbrace{\int_0^t \int_0^u \cdots \int_0^u}_{n} f(u)\,du^n$$

$$\sigma^{-n} f(t) = \frac{d^n}{du^n} f(u)\Big|_{u=t}$$
$$\sigma^0 f(t) = f(t).$$

We are now ready to derive a compact formula for power series of an analytic function.

POWER SERIES: For an analytic function f,

$$\boldsymbol{f(t) = e^{(t-a)\sigma^{-1}} f(a).}$$

PROOF. We start by integrating and differentiating f repeatedly.

$$\begin{aligned}
\sigma_a\sigma^{-1} f(t) &= f(t) - f(a)\\
\sigma_a\sigma^{-2} f(t) &= \sigma^{-1} f(t) - \sigma^{-1} f(a)\\
\sigma_a^2\sigma^{-2} f(t) &= f(t) - f(a) - (t-a)\sigma^{-1} f(a)\\
\sigma_a^2\sigma^{-3} f(t) &= \sigma^{-1} f(t) - \sigma^{-1} f(a) - (t-a)\sigma^{-2} f(a)\\
\sigma_a^3\sigma^{-3} f(t) &= f(t) - f(a) - (t-a)\sigma^{-1} f(a) - \frac{1}{2}(t-a)^2\sigma^{-2} f(a)\\
&\dots\\
\sigma_a^n\sigma^{-n} f(t) &= f(t) - f(a) - (t-a)\sigma^{-1} f(a) - \frac{1}{2}(t-a)^2\sigma^{-2} f(a) - \dots\\
&\quad - \frac{1}{n!}(t-a)^n\sigma^{-n} f(a).
\end{aligned}$$

We then take the infinite case of this series and regard it as an operator ψ on f. We do similar operations on this series and find that it leaves the series unchanged.

$$\begin{aligned}
\psi f(t) &\equiv \sigma_a^\infty\sigma^{-\infty} f(t)\\
&= f(t) - f(a) - (t-a)\sigma^{-1} f(a) - \frac{1}{2}(t-a)^2\sigma^{-2} f(a) - \dots\\
\psi\sigma f(t) &= \sigma f(t) - \sigma f(a) - (t-a)\sigma^{-2} f(a) - \frac{1}{2}(t-a)^2\sigma^{-3} f(a) - \dots\\
\sigma_a\psi\sigma f(t) &= f(t) - f(a) - (t-a)\sigma^{-1} f(a) - \frac{1}{2}(t-a)^2\sigma^{-2} f(a) - \dots\\
&= \psi f(t).
\end{aligned}$$

For all infinitely differentiable f and all a, we now have $\psi f(t) = \sigma_a \psi \sigma f(t)$, or $\psi \sigma f(t) = \sigma_a \psi f(t)$. Since $\sigma_a \sigma f(t) = \sigma \sigma_a f(t)$, by the definition of ψ we have $\psi \sigma f(t) = \sigma \psi f(t) = \sigma_a \psi f(t) = \sigma \psi f(t) - \sigma \psi f(a)$. Subtracting, $\sigma \psi f(a) = 0$ for all a, i.e. $\sigma \psi f$ is the zero function. Hence $\psi f(t)$ must also be the zero function for all f, i.e. ψ is the zero operator. So

$$\begin{aligned} f(t) &= f(a) - (t-a)\sigma^{-1} f(a) - \frac{1}{2}(t-a)^2 \sigma^{-2} f(a) - \dots \\ &= \sum_{n=0}^{\infty} \frac{(t-a)^n}{n!} \sigma^{-n} f(a) \\ &= e^{(t-a)\sigma^{-1}} f(a). \ \blacksquare \end{aligned}$$

SINGULARITIES

Offset derivatives

In **Differentials and integrants** (p. 192), we defined the differential of a function f at a finite point x with respect to a zero $0'$, denoted ${}^{0'}df(x)$, is the difference $f(x+0') - f(x)$. The equipoint handling of singularities sometimes requires a variant of this differential.

If f is finite and continuous at x, the differential is zero: By continuity, $f(x+0') =' f(x)$, so ${}^{0'}df(x) \equiv f(x+0') - f(x) =' 0$. Since dx is the differential of the identity function $f(x) = x$, it too is always zero for finite x.

If $f(x)$ is infinite or discontinuous, $df(x)$ may be finitesimal. Previous chapters have assumed that differentials of dependent variables are zero, but most results continue to hold if they are finitesimal. Exceptions include the two **fundamental theorems of calculus** (p. 168), which do not hold at **poles** (p. 223), as described below.

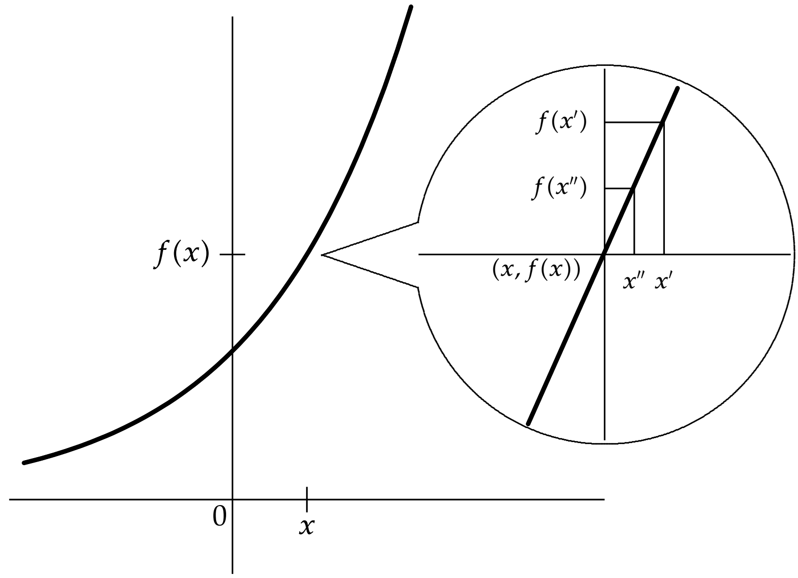

FIG. 57:
Calculation of derivative
with offset differentials

Occasionally the differential ${}^{0'}df(x) \equiv f(x+0') - f(x)$, or derivatives that use it, do not yield a determinate result. In such cases, we may use the fact that the slope of an analytic curve at a finite point x can be computed with any two points within the microscope. This is shown in Figure 57, where we use the two points

$$\left(x', f(x')\right) \equiv \left(x+0', f(x+0')\right)$$
$$\left(x'', f(x'')\right) \equiv \left(x+0'', f(x+0'')\right)$$

The differentials along the two axes in the microscope are called ***offset differentials***, and the derivative using them is called an ***offset derivative***, with the following notations:

$$ {}^{0'}_{0''}df(x) \equiv f(x+0') - f(x+0'')$$
$$ {}^{0'}_{0''}f'(x) \equiv \frac{{}^{0'}_{0''}df(x)}{{}^{0'}_{0''}dx} = \frac{f(x+0') - f(x+0'')}{0' - 0''}$$

The quantity $0'$ is called the ***upper displacement*** and $0''$ the ***lower displacement***. The first type of differential, with only an upper displacement, is called a ***original differential***, since the lower displacement is the origin of the microscope. As shown in Figure 57, for a finite analytic function, the curve becomes a straight line in the microscope, so a original derivative and an offset derivative yield the same result.

Letting $0''' \equiv 0' - 0''$, we have

$$\frac{{}^{0'}_{0''}df(x)}{{}^{0'}_{0''}dx} = \frac{f(x+0') - f(x+0'')}{0' - 0''} = \frac{f(x+0''+0''') - f(x+0'')}{0'''} = {}^{0'''}f'(x+0'').$$

This form of an offset derivative shows that it can be considered as the derivative of an **offset** (p. 185). As with original derivatives, if an offset derivative in this form is independent of its upper displacement, we omit it and write $f'(x+0'')$.

The above definitions apply only to finite x. For infinte x, we use the fact that for $x = 0$, ${}^{0'}_{0''}df(x) \equiv f(0') - f(0'')$. For infinite x, then, we define

$$\begin{aligned}
\infty''' &\equiv \frac{1}{0'''} \equiv \frac{1}{0'} - \frac{1}{0''} \equiv \infty' - \infty'' \\
{}^{0'}_{0''}df(x) &\equiv f\left(\frac{1}{0'}\right) - f\left(\frac{1}{0''}\right) = f(\infty') - f(\infty'') \\
&= f\left(\frac{1}{0''} + \frac{1}{0'''}\right) - f\left(\frac{1}{0''}\right) = f(\infty'' + \infty''') - f(\infty'') \\
{}^{0'}_{0''}f'(x) &\equiv \frac{{}^{0'}_{0''}df(x)}{{}^{0'}_{0''}dx} = \frac{f\left(\frac{1}{0'}\right) - f\left(\frac{1}{0''}\right)}{\frac{1}{0'} - \frac{1}{0''}} = \frac{f(\infty') - f(\infty'')}{\infty' - \infty''} \\
&= \frac{f\left(\frac{1}{0''} + \frac{1}{0'''}\right) - f\left(\frac{1}{0''}\right)}{\frac{1}{0'''}} = \frac{f(\infty'' + \infty''') - f(\infty'')}{\infty'''} \\
&= {}^{0'''}f'\left(\frac{1}{0''}\right) = {}^{0'''}f'(\infty'') \\
&= f'\left(\frac{1}{0''}\right) = f'(\infty'') \text{ if independent of } 0'''.
\end{aligned}$$

Offset derivatives are not always inverse with integrals and should only be used when original derivatives do not yield a determinate result. This is clarified further in following sections, especially **Poles** (p. 223).

Definition of singularity

A class x is ***integrous*** if there is a bijection between the elements of x and some subset of the integers. Examples are 5, ± 1 and $2\pi\mathbb{N}$. This concept is further discussed in **Class count comparisons** (p. 274).

A class x is ***determinate*** if it is nonempty and integrous.

A class is ***semideterminate*** if it is not empty, not determinate, and not full. An example is the interval $[-1, +1]$.

A class is ***indeterminate*** if it is full.

A function f is ***regular*** or ***analytic*** on a region A if:

- $f(x)$ and its original derivatives $f^{(n)}(x)$ are determinate and continuous for every $x \in A$; and
- $f(x)$ is equal to some value of the power series $e^{(x-a)\sigma^{-1}} f(a)$ for every $x, a \in A$.

The numeristic theory of infinite series shows how most infinite series, even convergent ones, are multivalued. See **Divergent Series** (p. 301–409).

An ***ordinary point*** of a function f is any point x in a region where where f is regular. A ***singularity*** of f is any other point, i.e. where any of the above conditions fails.

A function f is ***semiregular*** on a region A if:

- $f(x)$ and $f^{(n)}(x)$ are determinate and continuous for every nonsingular x in A; and
- The offset $f(x + 0')$ and offset derivatives $f^{(n)}(x + 0')$ are semideterminate and semiuniform for every singular x in A; and
- $f(x)$ is equal to some value of the power series $e^{(x-a)\sigma^{-1}} f(a)$ for every $x, a \in A$, where the power series is calculated with original values and derivatives for nonsingular a and offset values and derivatives for singular a.

A ***semiordinary point*** of a function f is any point x in a region where f is semiregular. An ***irregularity*** of f is any other point.

Types of singularity

A singularity is ***isolated*** if there is a punctured perfinite-size neighborhood that contains no singularies. This means that the unfolding of the singularity contains only one singularity. In this chapter we discuss the following four types of isolated singularity:

- ***Removable discontinuity***: f has a removable discontinuity at p if the offset values $f(p + 0')$ is uniform, but the function is discontinuous, i.e. $f(p + 0') \neq' f(p)$. Examples discussed below are the **punctured constant function** (p. 210), the **Kronecker delta function** (p. 216), and the **Dirac delta function** (p. 217).

- ***Jump discontinuity***: f has a jump discontinuity at p if the offset value $f(p+0')$ is semiuniform but not uniform, and the function is discontinuous. Examples discussed below are the **absolute value function** (p. 208) and its derivative, the step function.

- ***Pole***: f has a pole at p if $f(x) = \dfrac{g(x)}{h(x)}$, g and h are regular, h has a root (zero) at p, and the multiplicity of the root p of h is finite. An example is the reciprocal function, discussed below in **Poles** (p. 223).

- ***Essential singularity***: f has an essential singularity at p if it has a singularity that is not any of the above three types. An example is the function $\sin \frac{1}{x}$, discussed below in **Function $\sin \frac{1}{x}$** (p. 227).

There are many types of nonisolated singularities. Some examples are given in **Other singularities** (p. 232), but they are not analyzed in detail.

This chapter also gives an example of a function which is singular everywhere in conventional analysis but is regular in equipoint analysis. See **Weierstrass function** (p. 229).

Absolute value function

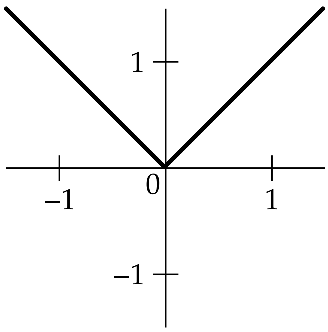

FIG. 58: Absolute value function $a(x) \equiv |x|$

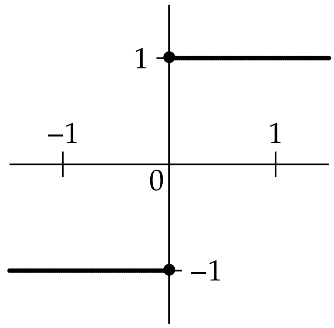

FIG. 59: Derivative of absolute value function $a(x) \equiv |x|$, $a'(x) = \operatorname{sgn}_3 x$

The absolute value function $a(x) \equiv |x|$ is shown in Figure 58. Its derivative is the step function shown in Figure 59. The derivative has a **jump discontinuity** (p. 207) at 0.

In the region $x > 0$, we have $a(x) = x$, $a'(x) = 1$, and the power series about any p in this region is $e^{(x-p)\sigma^{-1}}a(p) = p + (x - p) = x$. The function is therefore regular in this region. Similarly, it is regular in the region $x < 0$.

For any region that includes $x = 0$, the derivative is not uniform, since it has two values at 0:

$$\begin{aligned} a'_{0'}(0) &= \frac{a(0') - a(0)}{0'} = \frac{0'}{0'} = 1 \\ a'_{-0'}(0) &= \frac{a(-0') - a(0)}{-0'} = \frac{0'}{-0'} = -1 \\ a'(0) &= \pm 1. \end{aligned}$$

$a(x)$ is therefore not regular for any region which includes $x = 0$. However, $a'(x)$ is semiuniform, and $a(x)$ is therefore semiregular everywhere.

The derivative of a similar step function is discussed below in **Dirac delta function** (p. 217). As discussed in that section, a step function can be made analytic at the unfolded level. In the same way, the absolute value function, as the integral of a step function, can also be made unfolded analytic.

As a complex function, the derivative of a is the unit circle: for any $0' \in' 0$,

$$a'_{0'}(0) = \frac{a(0') - a(0)}{0'} = \frac{|0'|}{0'} = \operatorname{sgn} 0'$$

$$a_{\mathbb{C}0'} = e^{i\mathbb{R}}.$$

Multivalued complex derivatives are discussed further in **Complex derivative** (p. 234).

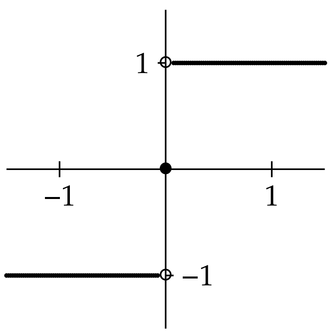

FIG. 60:
Conventional signum function $f(x) = \operatorname{sgn}_1 x$

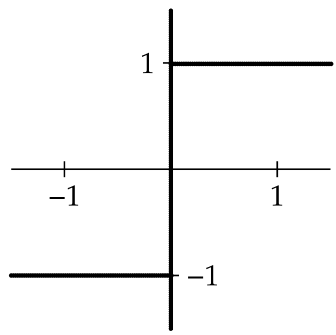

FIG. 61:
Alternate signum function $f(x) = \operatorname{sgn}_2 x$

The derivative a' is an alternate form of the signum function. The standard form, shown in Figure 60, is

$$\operatorname{sgn}_1 x \equiv \begin{cases} -1 & \text{for } x < 0 \\ 0 & \text{for } x = 0 \\ +1 & \text{for } x > 0. \end{cases}$$

In **Signum function** (p. 86) we developed an alternate form, shown in Figure 61:

$$\operatorname{sgn}_2 x \equiv \frac{|x|}{x}$$

From the above derivative we can define

$$\operatorname{sgn}_3 x \equiv f'(x) = \frac{|x \pm 0'|}{x \pm 0'}$$

$$\operatorname{sgn}_3 0 = \pm 1.$$

This third form allows us to calculate signum for infinite numbers:

$$\begin{aligned}
\text{Projectively extended real numbers } (\widehat{\mathbb{R}}):&\quad \operatorname{sgn}_3 \infty = \pm 1\\
\text{Affinely extended real numbers } (\overline{\mathbb{R}}):&\quad \operatorname{sgn}_3(+\infty) = +1\\
&\quad \operatorname{sgn}_3(-\infty) = -1\\
\text{Single projectively extended complex numbers } (\widehat{\mathbb{C}}):&\quad \operatorname{sgn}_3 \infty = e^{i\mathbb{R}}\\
\text{Double projectively extended complex numbers } (\widetilde{\mathbb{C}}):&\quad \operatorname{sgn}_3\left(\infty e^{ir}\right) = \pm e^{ir}\\
\text{Affinely extended complex numbers } (\overline{\mathbb{C}}):&\quad \operatorname{sgn}_3\left(\infty e^{ir}\right) = e^{ir}
\end{aligned}$$

Punctured constant function

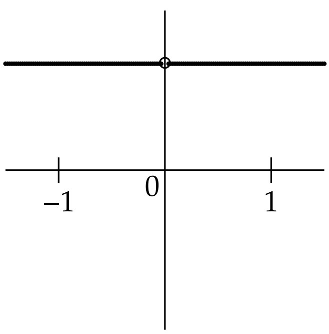

FIG. 62: Punctured constant function $p(x)$

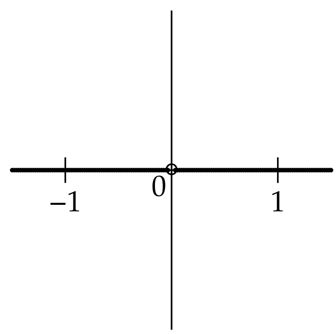

FIG. 63: Derivative of punctured constant function $p'(x)$

A function with a missing point, a point where the function has no value, is shown in Figure 62. This is a punctured constant function:

$$p(x) \equiv \begin{cases} 1 & \text{for } x \neq 0 \\ \emptyset & \text{for } x = 0. \end{cases}$$

The function p has a **removable discontinuity** (p. 207) at 0, since $p(0') = p(0'') = 1$ for all unfolded elements $0'$ and $0''$, but $1 = p(0') \neq p(0) = \emptyset$.

The derivative p', shown in Figure 63, also has a missing point:

$$p'(0) = \frac{p(0') - p(0)}{0'} = \frac{1 - \emptyset}{0'} = \emptyset.$$

The offset derivative, as defined in **Offset derivatives** (p. 203), yields a value everywhere:

$$ {}_{0''}p'(0) = \frac{p(0') - p(0'')}{0'} = \frac{1-1}{0' - 0''} = 0. $$

Since $p(0)$ is empty, it is not determinate, and $p(x)$ cannot be regular or semiregular in any any region that includes the singularity at $x = 0$.

Vertical line

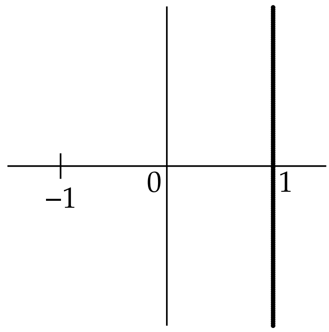

FIG. 64: Vertical line functions $v(x)$ and $V(x)$ for $a = 1$ in finite space

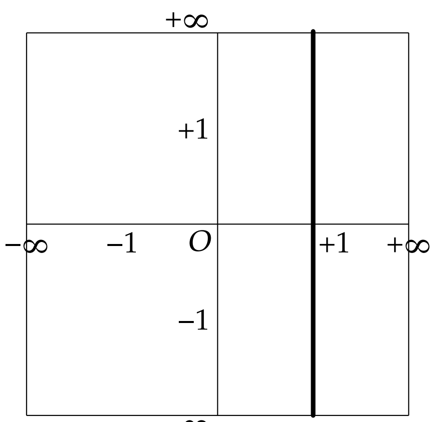

FIG. 65: Tangent scale plot of $v(x)$ for $a = 1$

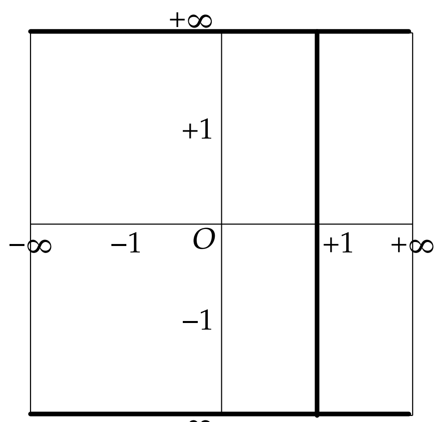

FIG. 66: Tangent scale plot of $V(x)$ for $a = 1$

We examine vertical lines here through two functions:

- The partial function $v(x)$, which has a value only at the x-intercept a.

- The total function $V(x)$, which has a value everywhere.

  The partial function is defined thus:

$$v(x) \equiv \begin{cases} \not\infty & \text{for } x = a\text{, where } a \text{ is the } x\text{-intercept} \\ \emptyset & \text{for } x \neq a \end{cases}$$

$v(x)$ is diagrammed in a finite region for $a = 1$ in Figure 64, and as a **tangent scale plot** (p. 82) for finite and infinite regions in Figure 65. If $y = v(x)$, then it is a constant as a function of y, but as a function of x, it is indeterminate at $x = a$ and empty elsewhere, and not regular or semiregular anywhere.

The total function can be defined in several ways which are similar to the usual equations for horizontal and oblique lines. The point-slope form is

$$V(x) \equiv \infty(x - a),$$

which uses only folded arithmetic, and the slope-intercept form is

$$V(x) \equiv \infty' x - \infty' a,$$

which requires unfolded arithmetic. If $y = V(x)$, we can also express it in intercept form as a relation:

$$\frac{x}{a} - \frac{y}{\infty' a} = 1.$$

$V(x)$ is diagrammed in a finite region for $a = 1$ in Figure 64, and as a tangent scale plot in Figure 66. It is indeterminate at $x = a$, integrous and semideterminate elsewhere, not regular or semiregular in any region that includes $x = a$, and regular elsewhere.

Singularities at infinity

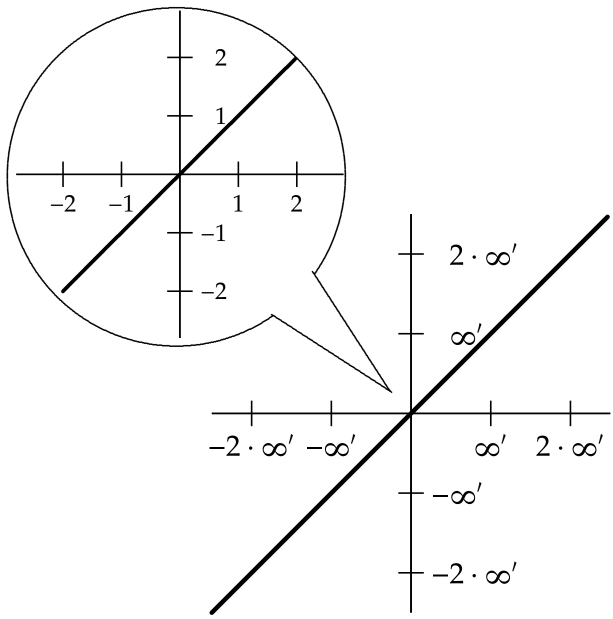

FIG. 67: Identity function $I(x) \equiv x$
with microscope view of finite plane
within origin of unfolded infinite plane

Even a very simple function such as $I(x) \equiv x$ has a singularity at infinite values. This function is shown in Figure 67, which shows the finite plane in a microscope and the unfolded infinite plane in the macroscope. For clarity, we use the affinely extended real numbers, and set $\infty' \equiv \dfrac{1}{0'}$. The origin of the unfolded infinite line is $\dfrac{1}{0 \cdot 0'} = \infty \cdot (\pm\infty')$. This is a pair of points infinitely removed from the origin of the macroscope.

The original derivative at $x = +\infty$ therefore uses $\infty \cdot \infty'$ as a lower displacement, but this yields an indeterminate result:

$$^{\infty'}I'(x) = \frac{(\infty + \infty') - \infty}{\infty'} = \frac{(\infty - \infty) + \infty'}{\infty'} = \not\infty.$$

An offset derivative yields

$$ {}^{\infty'}_{\infty''}I'(x) = \frac{\infty'' - \infty'}{\infty'' - \infty'} = 1. $$

Since the original derivative is indeterminate but the offset derivative is determinate, $I(x)$ is only semiregular in any region that includes an infinite value.

For another example, we take the exponential function $\exp(x) \equiv e^x$. At $x = -\infty$, an original derivative is sufficient:

$$ {}^{\infty'}\exp'(x) = \frac{e^{-\infty+\infty'} - e^{-\infty}}{\infty'} =' \frac{e^{-\infty} - e^{-\infty}}{\infty'} = \frac{0-0}{\infty'} =' 0. $$

But at $x = +\infty$, an offset derivative is required:

$$ {}^{\infty'}\exp'(x) = \frac{e^{\infty+\infty'} - e^{\infty}}{\infty'} =' \frac{e^{\infty} - e^{\infty}}{\infty'} = \frac{\infty - \infty}{\infty'} = \not\infty, $$
$$ {}^{\infty''}_{\infty'}\exp'(x) = \frac{e^{\infty''} - e^{\infty'}}{\infty'' - \infty'} = \frac{\infty'''}{\infty'' - \infty'} =' \infty. $$

Therefore, in the affinely extended real numbers, $\exp(x)$ has a singularity at $+\infty$ but not at $-\infty$, and is regular in any region that includes $-\infty$ but only semiregular in a region that includes $+\infty$.

Every singularity at an infinite value is nonisolated, since $\infty + r = \infty$ for all perfinite r, and any punctured perfinite size neighborhood of the infinite value is still within the same infinite value.

Kronecker delta function

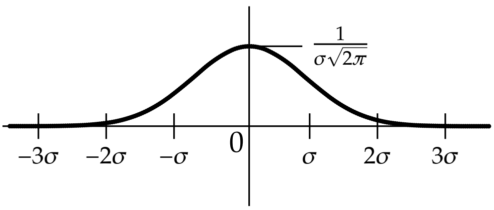

FIG. 68: Normal distribution function $\phi(x)$ with standard deviation σ

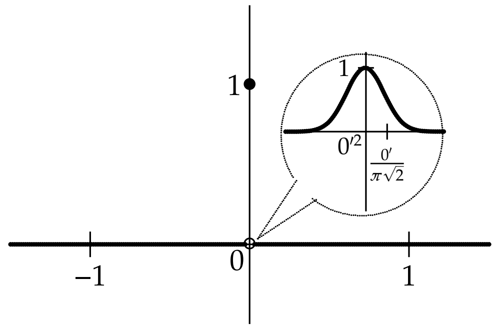

FIG. 69: Kronecker delta function $\delta_{0,x}$ as proper unfolded normal distribution

The ***Kronecker delta function*** has a very simple definition:

$$\delta_{a,b} \equiv \begin{cases} 1 & \text{for } a = b \\ 0 & \text{for } a \neq b. \end{cases}$$

The function $\delta_{x,0}$ has a **removable discontinuity** (p. 207) at 0, since $f(0') = 0$ for all unfolded elements $0'$, but $f(0) = 1$. Put another way, $\lim_{x \to 0} \delta_{x,0}$ exists and is 0. In equipoint terms, $\lim_{x \to 0}$ means $f(x + 0')$, and to say it exists means that $f(x + 0')$ is single valued and independent of $0'$. See **Limits** (p. 185) above.

The Kronecker delta function is not regular function in any region that includes the singularity at $x = 0$, since the original and offset derivatives there do not agree: the original derivative is infinite while the offset derivatives are zero. The function is not semiregular in these regions, since the power series using offset derivatives at the singularity do not equal the function. Hence the function is irregular in these regions.

The Kronecker delta function, and any function with this type of discontinuity, can be made regular at the unfolded level, by constructing a proper unfolded regular function which folds into this function. Figure 69 shows one way of doing this, by constructing a normal distribution with an infinitesimal standard deviation.

Figure 68 shows the standard normal distribution $\phi(x)$ with standard deviation σ:

$$\begin{aligned}\phi_\sigma(x) &\equiv \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}} \\ &= \frac{1}{\sigma\sqrt{2\pi}} \sum_{n=0}^{\infty} \frac{x^{2n}}{n!2^n\sigma^{2n}} \\ &= \frac{1}{\sigma\sqrt{2\pi}} \left(1 - \frac{x^2}{2\sigma^2} + \frac{x^4}{8\sigma^4} - \frac{x^6}{48\sigma^6} + \ldots\right).\end{aligned}$$

We can then define the Kronecker delta in terms of $\phi(x)$, as graphed in Figure 69:

$$\begin{aligned}\delta_{x,0} &\equiv 0'\sqrt{\pi}\ \phi_{\frac{0'}{\pi\sqrt{2}}}(x) \\ &= e^{-\frac{x^2}{0'^2}} \\ &= \sum_{n=0}^{\infty} \frac{x^{2n}}{n!0'^{2n}} \\ &= 1 - \frac{x^2}{0'^2} + \frac{x^4}{2 \cdot 0'^4} - \frac{x^6}{6 \cdot 0'^6} + \ldots.\end{aligned}$$

Dirac delta function

Definitions of the Dirac delta function

The ***Dirac delta function*** or ***unit impulse function*** $\delta(\cdot)$ has many definitions. Two qualities of δ which should follow from any definition are:

$$\delta(\mathbb{R}^*) = 0 \qquad (\mathbb{R}^* \equiv \mathbb{R} \setminus 0)$$

$$\int_{-\infty}^{+\infty} \delta(x)\, dx = 1.$$

In equipoint terms, we should refine these conditions as follows:

$$\delta(\mathbb{R}^*) =' 0'^2$$

$$\int_{-\sqrt{\frac{1}{\infty'}}}^{+\sqrt{\frac{1}{\infty'}}} \delta(x)\, dx = 1.$$

These two conditions imply an infinite value for $\delta(0)$. In conventional analysis, this does not allow δ to be a function. δ is instead defined as a distribution or generalized function. Here we consider the Dirac delta to be a function with an infinite value at 0, a **removable discontinuity** (p. 207).

Here we give three definitions of the Dirac delta.

1. $\delta_{\infty'}(x)$ is the class of **proper unfolded functions** (p. 158) such that
$$\max \delta_{\infty'}(x) =' \infty'$$
$$\delta_{\infty'}(\mathbb{R}^*) =' \frac{1}{\infty'^2}$$
$$\int_{-\infty'}^{+\infty'} \delta_{\infty'}(x)\,dx = \int_{-\sqrt{\frac{1}{\infty'}}}^{+\sqrt{\frac{1}{\infty'}}} \delta_{\infty'}(x)\,dx = 1$$

2. $\delta_{\infty'}(x) \equiv \int_{-\infty'}^{+\infty'} e^{2\pi ixy}\,d_{\frac{1}{2\infty'}}y$. This integral yields the class of unfolded functions in definition 1.

3. $\delta_{\frac{1}{0'}}(x)$ is the derivative of the ***Heaviside step function***, also called the ***unit step function***. This also has several definitions, but for the moment, we will use the left-continuous form:
$$H(x) \equiv \begin{cases} 0 & \text{for } x \le 0 \\ 1 & \text{for } x > 0. \end{cases}$$

The derivative is
$$\delta_{\frac{1}{0'}}(x) = H'_{0'}(x) = \frac{{}^{0'}dH(x)}{{}^{0'}dx}.$$

From any of the definitions, it easily follows that, for any finite function $f(x)$,
$$\int_{-\infty}^{+\infty} f(x)\,\delta(x)\,dx = f(0)$$

In particular,
$$\int_{-\infty}^{+\infty} 0\,\delta(x)\,dx = 0.$$

Definition 3

We now examine definition 3 in more detail. In the derivative expression, ${}^{0'}dH(x) = 1$ for any $0' >' 0'^2$, i.e. for any $0'$ on the right side of unfolded 0. When it is divided by ${}^{0'}dx = 0'$, the result is infinite.

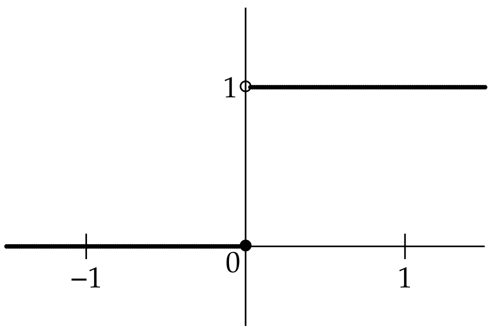

FIG. 70:
Heaviside step function $H(x)$

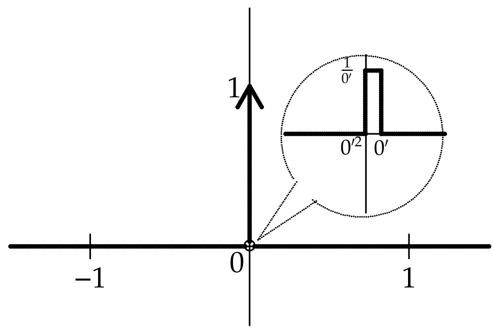

FIG. 71:
Dirac delta function $\delta(x)$

Figure 71 shows the infinite value at $\delta(0)$. The microscope in this figure expands infinitely in the x direction and *contracts* infinitely in the y direction. The rectangle in the microscope is infinitely tall and infinitely narrow, and its total area is 1.

At the unfolded level, $\delta(x)$ is a class of single valued functions, but at the folded level, it becomes multivalued and loses other properties:

$$\delta(\mathbb{R}^*) = 0$$

$$\delta(0) = \widehat{\mathbb{R}}^+ \text{ or } \overline{\mathbb{R}}^+$$

In the unfolded form, the properties of $\delta_0(x)$ are independent of $0'$. We can regard δ as a class of proper unfolded functions, and we can drop the subscripts and write

$$\delta(x) \equiv H'(x) = \frac{dH(x)}{dx}.$$

Figures 70 and 71 show the left-continuous form of $H(x)$ and the corresponding $\delta(x)$. There are several alternatives, a few of which are these:

1. In the right-continuous form of $H(x)$, $H(0) = 1$, and the rectangle in the microscope of $\delta(0)$ is to the left of $0'^2$ instead of the right. The difference is only in the unfolded arithmetic; the folded properties of $\delta(x)$ remain the same.

2. If we define $H(0) = \frac{1}{2}$, then $H(x) = \dfrac{1 + \operatorname{sgn} x}{2}$, and the microscope rectangle of $\delta(0)$ is half on the left and half on the right of $0'^2$. Again, this makes no difference to the folded properties of $\delta(x)$.

3. If we allow $H(x)$ to be multivalued and set $H(0) = [0,1]$, the unit interval, then the graph of $H(x)$ is a continuous path and can be parameterized with a single valued function. Since $H(0)$ is a multivalued class, then $\delta(0)$ is multivalued also, the class $\{[0,1]\delta_1(0)\}$, where $\delta_1(x)$ is the single valued $\delta(x)$ defined above. One of the members of this class, $0\delta_1(0)$, is itself multivalued, since $0\delta_1(0) = 0'\mathbb{R}\frac{1}{0'} = \mathbb{R}$. The other values, $\{(0,1]\delta_1(0)\}$, yield all the infinite multiples of $\delta_1(0)$ up to $\delta_1(0)$ itself. Therefore the graph of $\delta(x)$ is also a continuous path and can be parameterized with a single valued function.

4. Define $\delta(x)$ as a proper unfolded normal distribution, and $H(x)$ as its integral, as discussed below.

Under the first definition, the derivative of $\delta(x)$ can be computed in superunfolded arithmetic. We must compute the derivative at the two sides of the rectangle, first at the infinitely increasing step function at $0'^2$, and secondly at the infinitely decreasing step function at $\dfrac{1}{0'}$. The result, $\dfrac{d\delta(x)}{dx}$, is a second order proper unfolded function, and $\dfrac{d^{(n)}\delta(x)}{dx^n} = \dfrac{d^{(n+1)}H(x)}{dx^{n+1}}$, is $(n+1)$-th order noncanonical unfolded.

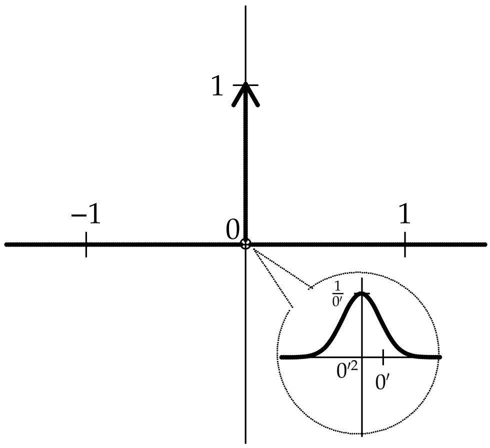

FIG. 72: Dirac delta function $\delta(x)$ as proper unfolded normal distribution

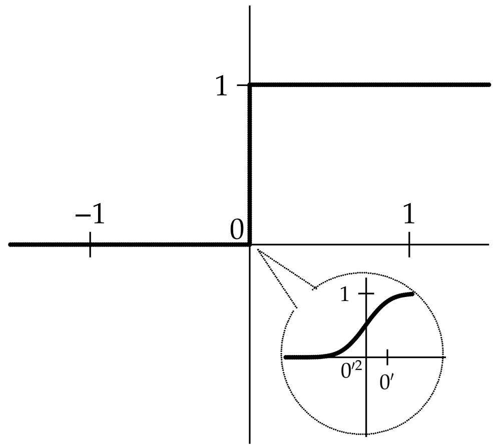

FIG. 73: Heaviside step function $H(x)$ as noncanonical unfolded cumulative normal distribution

Unfolded regularity of the Dirac delta function

Like the **Kronecker delta function** (p. 216), the Dirac delta function is not regular in any region that includes the singularity at $x = 0$, since the original and offset derivatives there do not agree, and it is not semiregular, since power series using offset derivatives at the singularity do not equal the function. Hence the function is irregular in these regions.

We made the Kronecker delta function regular at the unfolded level by constructing it as a normal distribution with an infinitely small standard deviation. A similar technique can be used with the Dirac delta function, as shown in Figure 72. In this case, we want the integral under the function to remain unity, so again we use $\sigma = \dfrac{0'}{\pi\sqrt{2}}$, but without any additional scaling:

$$\begin{aligned}\delta(x) &\equiv \phi_{\frac{0'}{\pi\sqrt{2}}}(x)\\ &= \frac{1}{0'\sqrt{\pi}}e^{-\frac{x^2}{0'^2}}\\ &= \frac{1}{0'\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{x^{2n}}{n!0'^{2n}}\end{aligned}$$

$$= \frac{1}{0'\sqrt{\pi}} \left(1 - \frac{x^2}{0'^2} + \frac{x^4}{2 \cdot 0'^4} - \frac{x^6}{6 \cdot 0'^6} + \dots \right).$$

To do the same for the Heaviside step function, we naturally choose the integral of the normal distribution, the cumulative normal distribution:

$$\begin{aligned}
\Phi_\sigma(x) &\equiv \int_{-\infty}^{x} \phi_\sigma(u)\, du \\
&= \int_{-\infty}^{x} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{u^2}{2\sigma^2}}\, du \\
&= \frac{1}{2} + \frac{1}{\sigma\sqrt{2\pi}} \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)n!2^n\sigma^{2n}} \\
&= \frac{1}{2} + \frac{1}{\sigma\sqrt{2\pi}} \left(x - \frac{x^3}{6\sigma^2} + \frac{x^5}{40\sigma^4} - \frac{x^7}{336\sigma^6} + \dots \right).
\end{aligned}$$

We can then redefine the Heaviside step function in terms of $\Phi(x)$, as graphed in Figure 73:

$$\begin{aligned}
H(x) &\equiv 0'\sqrt{\pi}\, \Phi_{\frac{0'}{\pi\sqrt{2}}}(x) \\
&= \frac{1}{0'\sqrt{\pi}} \int_{-\infty}^{x} e^{-\frac{u^2}{0'^2}}\, du \\
&= \frac{1}{2} + \frac{1}{0'\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)n!0'^{2n}} \\
&= \frac{1}{2} + \frac{1}{0'\sqrt{\pi}} \left(x - \frac{x^3}{3 \cdot 0'^2} + \frac{x^5}{10 \cdot 0'^4} - \frac{x^7}{42 \cdot 0'^6} + \dots \right).
\end{aligned}$$

Poles

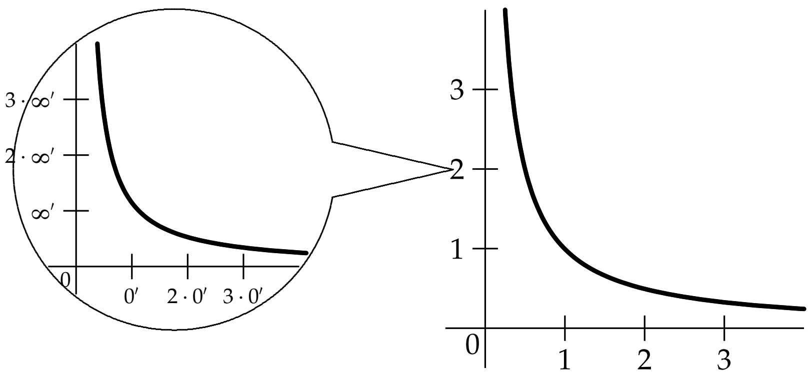

FIG. 74: Reciprocal function $r(x) \equiv \frac{1}{x}$
with microscope view of y axis infinitely expanded
in x direction and infinitely compressed in y direction

In **Types of singularity** (p. 207), we defined a pole of a function f as a point p such that $f(x) = \dfrac{g(x)}{h(x)}$, g and h are analytic, $h(x)$ has a root (zero) at p, and the multiplicity of the root is finite.

Here we discuss the simplest pole, the function $r(x) \equiv \frac{1}{x}$ at the point $x = 0$. Figure 74 shows a graph of $r(x)$ and a microscope of the y-axis, which is infinitely expanded in the x direction and infinitely compressed in the y direction. In curves like the one in Figure 51, the curve becomes straight in the microscope, but in Figure 74, the curve keeps its asymptote along the vertical axis. This remains the case no matter how many times the curve is superunfolded.

This leads to an indeterminacy in the derivative:

$$r'(0) \equiv \frac{r(0+0') - r(0)}{0'} = \frac{\frac{1}{0+0'} - \frac{1}{0}}{0'} =' \frac{\infty - \infty}{0} = \not 0.$$

The **offset derivative** (p. 203) however is determinate:

$$r'(x+0'') \equiv \frac{r(0''+0') - r(0'')}{0'} = \frac{\frac{1}{0''+0'} - \frac{1}{0''}}{0'}$$

$$= \frac{0'' - 0'' - 0'}{0'0''(0'' + 0'')} = \frac{-1}{0''(0'' + 0')} = \begin{cases} \infty & \text{in } \widehat{\mathbb{R}} \\ -\infty & \text{in } \overline{\mathbb{R}} \end{cases} .$$

$r(x)$ is therefore not regular in any region that includes the pole, but it is semiregular, since the offset derivatives are determinate and semiuniform, and the power series using them yields the function. For perfinite a and a zero $0'$ we have:

$$f(x) = e^{(x-a)\sigma^{-1}} f(a) = \frac{1}{a} - \frac{x-a}{a^2} + \frac{(x-a)^2}{a^3} - \ldots = \sum_{k=0}^{\infty} \frac{(-1)^k (x-a)^k}{a^{k+1}}$$

$$f(x) = e^{(x-0')\sigma^{-1}} f(0') = \frac{1}{a} - \frac{x-0'}{0'^2} + \frac{(x-0')^2}{0'^3} - \ldots = \sum_{k=0}^{\infty} \frac{(-1)^k (x-0')^k}{0'^{k+1}}$$

An indeterminacy problem also occurs in the integral. A curve like the one in Figure 52 becomes flat in the microscope, but $r(x)$ keeps its asymptote in all unfoldings. If we integrate $r(x)$ from $0'$ to a positive point p, we obtain the infinite result $\ln p - \ln 0' = +\infty$ without a problem, but if we try to integrate through the pole, we run into an indeterminacy. An attempt to integrate from $-p$ to $+p$, for example, would lead to the following:

$$\int_{-p}^{+p} r(x)dx = \int_{-p}^{-0'} r(x)dx + \int_{-0'}^{0\cdot 0'} r(x)dx + \int_{0\cdot 0'}^{+0'} r(x)dx + \int_{+0}^{+p} r(x)dx.$$

The indeterminacy occurs with either of the middle two pieces, $\int_{-0'}^{0\cdot 0'} r(x)dx$ and $\int_{0\cdot 0'}^{+0'} r(x)dx$. The second of these two we can see in the microscope of Figure 74 as the area under the curve from the origin $0 \cdot 0'$ to $0'$. If the curve were flat, we could use a rectangle with the right side as the height, $0' \cdot \infty' = 1$, but this value is clearly too small in this case. Using the left side as the height gives $0' \cdot \infty \cdot \infty' = 0' \cdot \infty = \not{0}$ in the projectively extended real numbers and $0' \cdot \infty \cdot \infty' = 0' \cdot \infty = |\not{0}|$ in the affinely extended real numbers. If we use the trapezoidal estimate, we still obtain an indeterminacy: $0' \left(\frac{\infty \cdot \infty' + \infty'}{2} \right) = 0' \left(\frac{(\infty + 1)\infty'}{2} \right) = 0' \cdot \infty = \not{0}$ or $|\not{0}|$. Further unfoldings yield the same indeterminate result, since $0'^n \frac{1}{0 \cdot 0'^n} = \frac{1}{0} = \infty$ for any n.

Any approximation to this area that involves the left endpoint gives in an indeterminacy, and any approximation that does not is inaccurate. Therefore we cannot integrate directly through this pole. The same problem occurs with any other pole.

This leaves us with two alternatives:

1. In real space, integrate piecewise, once to the left of the pole, and once to the right.

2. In complex space, integrate around the pole.

The antiderivative of $\frac{1}{x}$ is $\ln x$, but since this is imaginary for negative x, it cannot be used as an integral in real analysis. Instead we use the fact that $\ln x = \ln(|x| \operatorname{sgn} x) = \ln|x| + \ln \operatorname{sgn} x$ and integrate either completely on the positive side of the real axis or completely on the negative side. In this case, the $\ln \operatorname{sgn} x$ terms cancel, and the effective antiderivative is $\ln|x|$.

In complex analysis, the antiderivative is $\ln x$, and the path of integration is connected. This is discussed in detail in **Complex poles** (p. 243).

Axial function

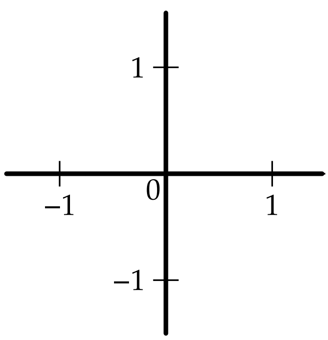

FIG. 75:
Axial function $A(x)$

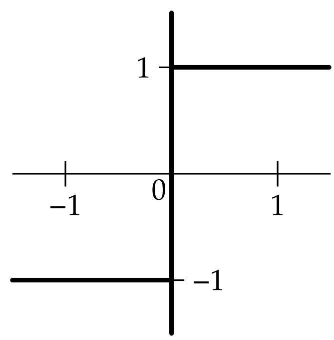

FIG. 76:
Sample of integral of axial function $A^{(-1)}(x)$

Figure 75 shows the ***axial function*** $A(x) \equiv \frac{0}{x}$, a multivalued function whose graph coincides with both the horizontal and vertical axes:

$$A(x) = \begin{cases} 0 & \text{for } x \neq 0 \\ \not\infty & \text{for } x = 0 \end{cases}$$

As a multivalued function, the continuity of $A(x)$ is of three types, as defined in **Continuity** (p. 187):

- ***Classwise continuity***: The axial function is classwise discontinuous at 0 because $A(0') = \{0\}$ and $A(0) = \varnothing$.

- ***Conjunctive continuity***: The axial function is conjunctively discontinuous at 0 because $\{0\}$ cannot be mapped bijectively to $\varnothing$.

- ***Disjunctive continuity***: The axial function is disjunctively continuous at 0 because $A(0') = 0 \in \varnothing = A(0)$.

The singularity of $A(x)$ at 0 is a **removable singularity** (p. 207). $A(x)$ is not regular in any region that includes the singularity at $x = 0$, since $A(0)$ is indeterminate. The function is not semiregular in these regions, since the power series using offset derivatives at the singularity equal zero. Hence the function is irregular in these regions.

The derivative of $A(x)$ is $A(x)$, since $A'(x) = \dfrac{dA(x)}{dx} = \dfrac{d}{dx}\dfrac{0}{x} = \dfrac{0}{-x^2} = \dfrac{0}{x^2} = \dfrac{0}{x}$. Since the original derivative is indeterminate at 0, as it is at a pole, the Fundamental Theorems of Calculus do not hold here. See **Poles** (p. 223) for a detailed discussion of this point.

To compute the integral of $A(x)$:

- In real space, integrate piecewise on the left and right sides. This allows us to choose independent constants of integration for the right and left integrals. One possible integral of $A(x)$ is shown in Figure 76.

- In complex space, integrate around the singularity. See **Complex axial function** (p. 245).

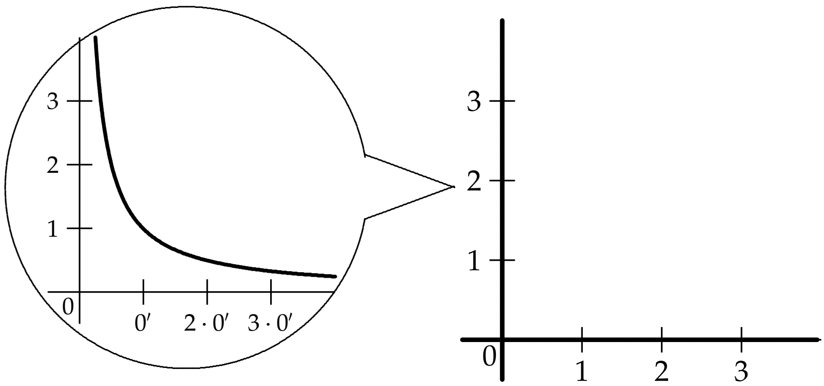

FIG. 77: Axial function $A(x)$
as proper unfolded reciprocal function

The axial function can be made semiregular at the unfolded level by choosing a proper unfolding. In this case, we use the proper unfolding $A(x) \equiv \frac{0'}{x}$, shown in Figure 77.

Function $\sin \frac{1}{x}$

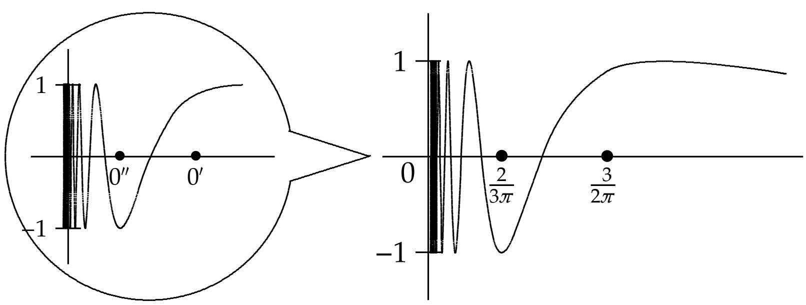

FIG. 78: Essential singularity of $S(x) \equiv \sin \frac{1}{x}$
with microscope view of y axis infinitely expanded in x
direction and unchanged scale in y direction

The function $S(x) \equiv \sin \dfrac{1}{x}$ is graphed in Figure 78. Like the pole in Figure 74, $S(x)$ at 0 maintains its shape from macroscope to microscope. Within the microscope, $S(0)$ takes on every value within the interval $[-1, +1]$. Algebraically we can see this by observing that $\infty + r = \infty$ for every real perfinite r, so $\sin\infty = S(0) = [-1, +1]$.

Since $S(0)$ is not determinate, it is a singularity. An offset value $S(0')$ can be any point within $[-1, +1]$, so the offset values are not uniform or semiuniform, and the singularity is not a removable discontinuity or jump discontinuity.

The following calculation shows that the singularity is also not a pole. As defined in **Types of singularity** (p. 207), a pole of a function f is a point p such that $f(x) = \dfrac{g(x)}{h(x)}$, g and h are analytic, $h(x)$ has a root (zero) at p, and the multiplicity of the root is finite. The following converts the power series for $S(x)$ to a fraction, using the ***Pochhammer symbol*** $(n)_r$ to denote the ***falling factorial*** function $\dfrac{n!}{(n-r)!}$, where $(n)_n = (n)_{n-1} = n!$, $(n)_1 = n$, $(n)_0 = 1$.

$$\begin{aligned}
\sin\frac{1}{x} &= \frac{1}{x} - \frac{1}{3!x^3} + \frac{1}{5!x^5} - \ldots \\
&= \frac{3!x^2 - 1}{3!x^3} + \frac{1}{5!x^5} - \ldots \\
&= \frac{5!x^4 - (5)_2x^2 + 1}{5!x^5} - \frac{1}{7!x^7} + \ldots \\
&= \frac{7!x^6 - (7)_4x^4 + (7)_2x^2 - 1}{7!x^7} + \frac{1}{9!x^9} - \ldots \\
&= \frac{\sum_{k=0}^{\infty'}(-1)^k(2\infty' + 1)_{2(\infty'-k)}x^{2(\infty'-k)}}{(2\infty' + 1)!x^{2\infty'+1}}
\end{aligned}$$

The denominator of the final fraction has a root at $x = 0$ of infinite multiplicity. Since the singularity is not a removable discontinuity, jump discontinuity, or pole, it is an essential singularity.

The derivative $S'(x) = -\dfrac{\cos\frac{1}{x}}{x^2}$ is indeterminate at the singularity, but the antiderivative $\displaystyle\int S(x)dx = x\sin x + \int_{\frac{1}{x}}^{\infty}\frac{\cos t}{t}dt + k$ is determinate.

Weierstrass function

Weierstrass gave an example of a class of functions that, in conventional analysis, are continuous everywhere but differentiable nowhere. We now examine a function which is simpler but still shows the essential features of the original Weierstrass functions:

$$W(x) \equiv \sum_{n=0}^{\infty} \frac{\sin(2^n x)}{2^n} = \sin x + \frac{\sin 2x}{2} + \frac{\sin 4x}{4} + \ldots$$

Conventional analysis cannot differentiate this function because

$$\liminf_{\delta\to 0} \frac{W(x+\delta) - W(x)}{\delta} > \limsup_{\delta\to 0} \frac{W(x+\delta) - W(x)}{\delta},$$

at every point, and thus

$$W'(x) = \lim_{\delta\to 0} \frac{W(x+\delta) - W(x)}{\delta}$$

does not exist anywhere.

Equipoint analysis does not have any such requirement. It requires only that a function be defined on an interval. Then, using an unfolded $0'$-level arithmetic, it computes

$$W'(x) = \frac{W(x+0') - W(x)}{0'}.$$

We can easily approach this function through its series definition. The conventional theory of infinite series allow us to use the commutative, associative, and distributive properies of addition and multiplication on infinite series only in restricted cases. ***Equipoint summation***, on the other hand, developed in the third part of this book, **Divergent Series** (p. 301–409), allows unrestricted use of the these properties with no known inconsistencies. Therefore, when coupled with the algebraic definition of derivatives developed here, we feel confident that we can differentiate W term by term as we would a finite sum, and so:

$$W'(x) = \sum_{n=0}^{\infty} \cos(2^n x).$$

Another way to differentiate this function is to define it with an unfolded upper limit:

$$W(x) \equiv \sum_{n=0}^{\infty'} \frac{\sin(2^n x)}{2^n} = \sin x + \frac{\sin 2x}{2} + \frac{\sin 4x}{4} + \ldots$$

and differentiate at an unfolded level beyond the unfolding of the upper limit, where $W(x)$ is smooth, just as $\sin x$ is smooth at an unfolded level:

$$0' \equiv \frac{1}{\infty'^2}$$

$$W'_{0'}(x) = \frac{{}^{0'}dW(x)}{{}^{0'}dx}$$

$$= \sum_{n=0}^{\infty'} \cos(2^n x).$$

Figures 79 and 80 show $W(x)$ and $W'(x)$.

In the first calculation, the derivative $W'(x)$ is can be calculated at every point within an unfolding of x and is single valued everywhere, but the offset values $W(x+0')$ vary with each $0'$, so $W(x)$ is not continuous anywhere. Thus every point is a singularity, each of which is nonisolated.

In the second calculation, $W(x+0')$ matches $W(x)$, but only at levels of unfolding beyond $0'$. It can be said to be continuous at those unfoldings.

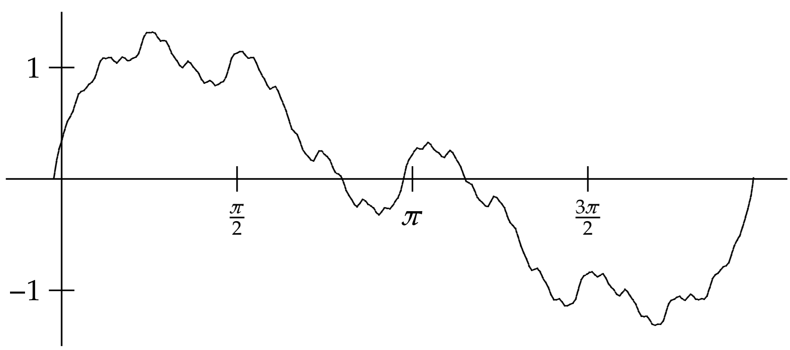

FIG. 79:
Weierstrass-like function $W(x)$

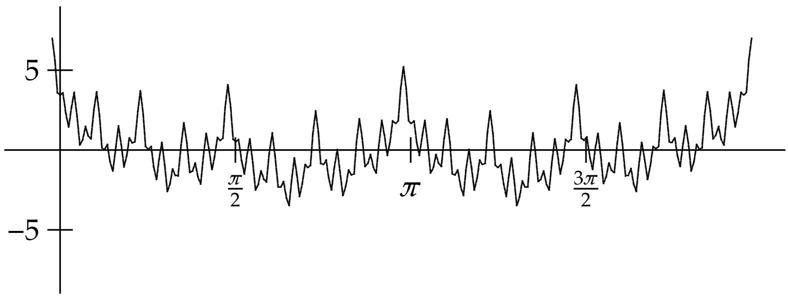

FIG. 80:
Derivative of Weierstrass-like function

Fourier transform

The Fourier transform, in the unitary asymmetric form, maps a function $f(x)$ to the transform $\widehat{f}(k)$ by

$$\widehat{f}(k) \equiv \int_{-\infty}^{+\infty} f(x) e^{-2\pi i k x}\, dx.$$

We will not redevelop Fourier theory here but only note the Fourier transform of some proper unfolded functions. These are variations of the two elementary transforms

$\mathbf{f(x)}$	$\widehat{\mathbf{f}}\mathbf{(k)}$
1	$\delta(k)$
$e^{2\pi i a x}$	$\delta(k-a)$

The noncanonical unfolded variations are

$\mathbf{f(x)}$	$\widehat{\mathbf{f}}\mathbf{(k)}$
0	$0'\delta(k) = \delta_{k,0}$
$0e^{2\pi i a x}$	$0'\delta(k-a) = \delta_{k,a}$

These two results assume that the width and height of $\delta(0)$ are $0'$ and $\frac{1}{0'}$. These connect the **Dirac delta** (p. 217) $\delta(x-a)$ with the **Kronecker delta** (p. 216) $\delta_{x,a}$.

Other singularities

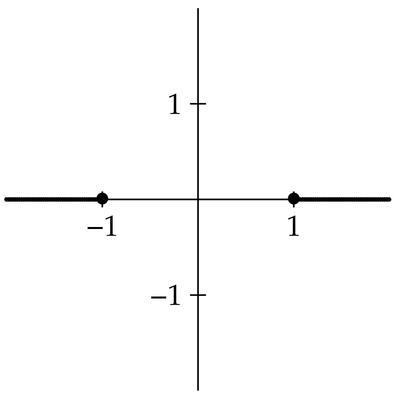

FIG. 81:
Gapped interval in $G(x)$

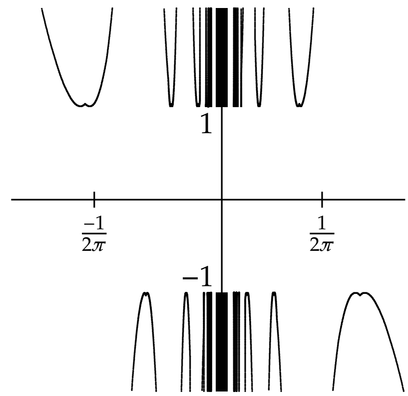

FIG. 82:
Accumulation point
of poles of $C(x) = \csc \frac{1}{x}$

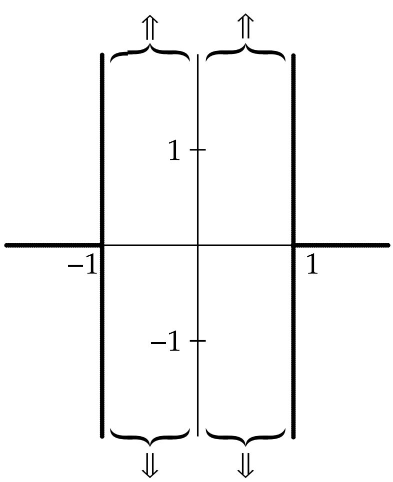

FIG. 83:
Interval of poles
in $J(x) = \frac{1}{x^\infty}$

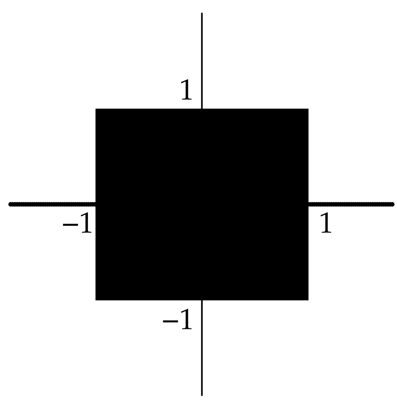

FIG. 84:
Interval of intervals
in $T(x) = \sin \frac{1}{x^\infty}$

Nonisolated singularities. Examples of nonisolated singularities are shown in Figures 81 through 84.

- Figure 81: A gapped interval in the function
$$G(x) \equiv \begin{cases} 0 & \text{for } |x| \geq 1 \\ \emptyset & \text{for } |x| < 1 \end{cases}$$

- Figure 82: An accumulation point of poles at $x = 0$ in the function $C(x) \equiv \csc \dfrac{1}{x}$. Every neighborhood around $x = 0$, and the unfolded point itself, has an infinite number of poles.

- Figures 83: An interval of poles in the function $J(x) \equiv \dfrac{1}{x^\infty}$. For every $x \in [-1, +1]$, $J(x)$ is a pole. For $x = \pm 1$, $J(x) = \not\infty$, and elsewhere $J(x) = 0$.

- Figure 84: An interval of intervals in the function $T(X) = \sin \dfrac{1}{x^\infty}$. For every point $x \in [-1, +1]$, $T(x) = [-1, +1]$. Elsewhere, $T(x) = 0$.

- Characteristic function of the rational numbers. See **Using class counts in derivatives and integrals** (p. 279).

Complex singularities. The following are analyzed in the **Complex functions** (p. 234) chapter.

- **Complex poles** (p. 243), the complex ananlogs of **real poles** (p. 223) described above.

- The **complex axial function** (p. 245), the complex analog of the **real axial function** (p. 225) described above.

- The **complex function $e^{\frac{1}{x}}$** (p. 247), which includes the **real function $\sin \frac{1}{x}$** (p. 227) described above.

COMPLEX FUNCTIONS

Complex derivative

The complex derivative is similar to the real derivative but allows folded and unfolded complex numbers and extended complex numbers and functions.

It is single valued and finite if the real and imaginary partial derivatives are single valued, finite, and analytic. It may be multivalued otherwise. For example, at $x = 0$, for real x,

$$\frac{d|x|}{dx} = \pm 1,$$

while for complex z,

$$\frac{d|z|}{dz} = e^{i\mathbb{R}}.$$

The general complex derivative (possibly multivalued, infinite, and/or non-analytic) is as follows. For a complex function $f(z)$ we first define

$$\begin{aligned} f(z) &= \operatorname{Re} f(\operatorname{Re} z + i \operatorname{Im} z) + i \operatorname{Im} f(\operatorname{Re} z + i \operatorname{Im} z) \\ &= g(x, y) + i h(x, y). \end{aligned}$$

We then have

$$\begin{aligned} \frac{{}^{0'}df(z)}{{}^{0'}dz} &= \frac{g(\operatorname{Re}(z + 0'), \operatorname{Im}(z + 0')) - g(\operatorname{Re} z, \operatorname{Im} z)}{0'} \\ &+ i \frac{h(\operatorname{Re}(z + 0'), \operatorname{Im}(z + 0')) - h(\operatorname{Re} z, \operatorname{Im} z)}{0'} \\ &= \frac{g(x + \operatorname{Re} 0', y + \operatorname{Im} 0') - g(x, y)}{0'} \\ &+ i \frac{h(x + \operatorname{Re} 0', y + \operatorname{Im} 0') - h(x, y)}{0'} \\ &= \frac{g(x + \operatorname{Re} 0', y + \operatorname{Im} 0') - g(x, y + \operatorname{Im} 0')}{\operatorname{Re} 0'} \cdot \frac{\operatorname{Re} 0'}{0'} \end{aligned}$$

$$
\begin{aligned}
&+\frac{g(x,y+\operatorname{Im}0')-g(x,y)}{\operatorname{Re}0'}\cdot\frac{\operatorname{Re}0'}{0'}\\
&+i\frac{h(x+\operatorname{Re}0',y+\operatorname{Im}0')-h(x,y+\operatorname{Im}0')}{\operatorname{Im}0'}\cdot\frac{\operatorname{Im}0'}{0'}\\
&+i\frac{h(x,y+\operatorname{Im}0')-h(x,y)}{\operatorname{Im}0'}\cdot\frac{\operatorname{Im}0'}{0'}\\
&=\frac{\partial g(x,y)}{\partial x}\cdot\frac{\operatorname{Re}0'}{0'}+\frac{\partial g(x,y)}{\partial y}\cdot\frac{\operatorname{Im}0'}{0'}\\
&+i\frac{\partial h(x,y)}{\partial x}\cdot\frac{\operatorname{Re}0'}{0'}+i\frac{\partial h(x,y)}{\partial y}\cdot\frac{\operatorname{Im}0'}{0'}\\
&=\frac{\partial g(x,y)+i\partial h(x,y)}{\partial x}\cdot\frac{\operatorname{Re}0'}{0'}+\frac{\partial g(x,y)+i\partial h(x,y)}{\partial y}\cdot\frac{\operatorname{Im}0'}{0'}\\
&=\frac{\partial f(z)}{\partial\operatorname{Re}z}\cdot\frac{\cos\arg 0'}{\operatorname{sgn}0'}+\frac{\partial f(z)}{\partial\operatorname{Im}z}\cdot\frac{\sin\arg 0'}{\operatorname{sgn}0'}.
\end{aligned}
$$

If f is analytic, then this becomes

$$
\begin{aligned}
\frac{df(z)}{dz}&=\frac{\partial f(z)}{\partial\operatorname{Re}z}\cdot\frac{\cos\arg 0'}{\operatorname{sgn}0'}+i\frac{\partial f(z)}{\partial\operatorname{Re}z}\cdot\frac{\sin\arg 0'}{\operatorname{sgn}0'}\\
&=\frac{\partial f(z)}{\partial\operatorname{Re}z}\cdot\frac{\operatorname{Re}0'+i\operatorname{Im}0'}{0'}=\frac{\partial f(z)}{\partial\operatorname{Re}z}\cdot\frac{0'}{0'}=\frac{\partial f(z)}{\partial\operatorname{Re}z}\\
&=-i\frac{\partial f(z)}{\partial\operatorname{Im}z}\cdot\frac{\cos\arg 0'}{\operatorname{sgn}0'}+\frac{\partial f(z)}{\partial\operatorname{Im}z}\cdot\frac{\sin\arg 0'}{\operatorname{sgn}0'}\\
&=-i\frac{\partial f(z)}{\partial\operatorname{Im}z}\cdot\frac{\operatorname{Re}0'+i\operatorname{Im}0'}{0'}=-i\frac{\partial f(z)}{\partial\operatorname{Im}z}\cdot\frac{0'}{0'}=-i\frac{\partial f(z)}{\partial\operatorname{Im}z}.
\end{aligned}
$$

As an example of the general complex derivative, let $M(z)\equiv 3\operatorname{Re}z+2i\operatorname{Im}z$, which does not satisfy the Cauchy-Riemann equations and are thus not conventionally differentiable. We then have

$$\frac{{}^{0'}dM(z)}{{}^{0'}dz}=\frac{3\cos\arg 0'+2i\sin\arg 0'}{\operatorname{sgn}0'}.$$

Letting $\theta\equiv\arg 0'$, this becomes

$$
\begin{aligned}
\frac{{}^{0'}dM(z)}{{}^{0'}dz}&=\frac{3\cos\theta+2i\sin\theta}{e^{i\theta}}\\
&=[\cos^2\theta+2]-i[\cos\theta\sin\theta]
\end{aligned}
$$

Letting $x \equiv \operatorname{Re} \frac{dM}{dz}$ and $y \equiv \operatorname{Im} \frac{dM}{dz}$ gives

$$x = \frac{1}{2}\cos 2\theta + \frac{5}{2}$$
$$y = -\frac{1}{2}\sin 2\theta$$

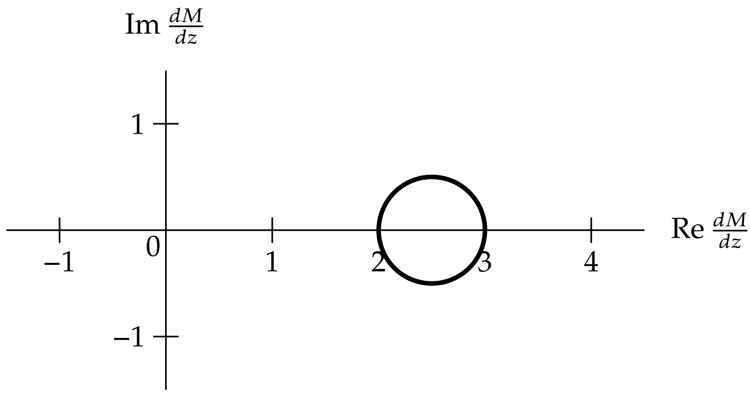

FIG. 85:
Derivative of $M(z) = 3 \operatorname{Re} z + 2i \operatorname{Im} z$

The derivative is the class of all points on a circle with radius $\frac{1}{2}$ and centered on $\frac{5}{2}$, shown in Figure 85. Since the partial derivatives are constant with respect to z, so is the total derivative. While the derivative does not depend on z, it does depend on $0'$. When $\arg 0' = 0$, the derivative is 3, but when $\arg 0' = \frac{\pi}{2}$, the derivative is 2.

Generalizing this example, let a and b be real coefficients, and

$$L(z) \equiv a \operatorname{Re} z + bi \operatorname{Im} z$$
$$\frac{{}^{0'}dL(z)}{{}^{0'}dz} = b + (a-b)\cos^2\theta + (b-a)i\sin^2\theta$$
$$\left(x - \frac{a+b}{2}\right)^2 + y^2 = \left(\frac{a-b}{2}\right)^2.$$

If $a = b$, this becomes

$$L(z) \equiv az$$
$$\frac{dL(x)}{dz} = a$$

and, since a is real, the circle shrinks to the point $z = a$.

The Cauchy integral formula

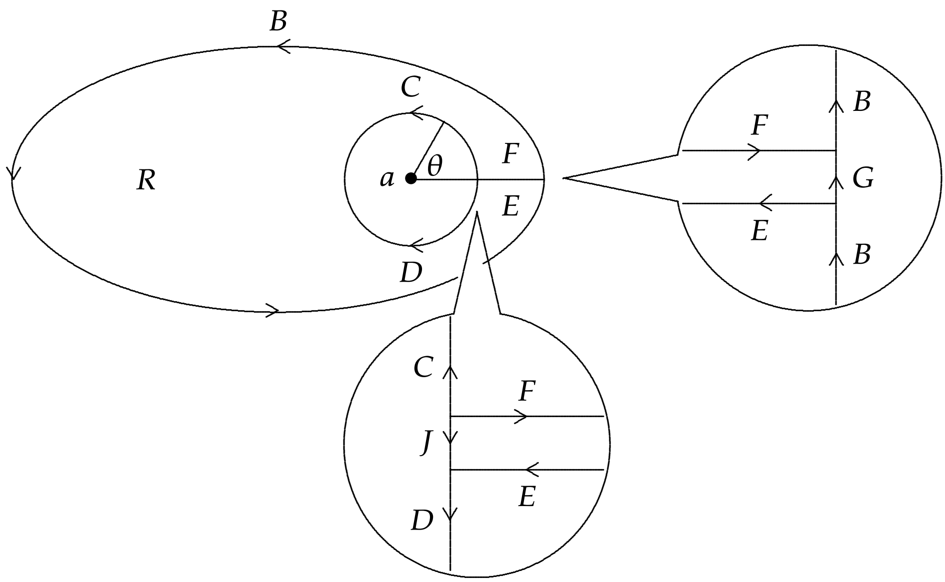

FIG. 86:
Contours for path independence
around a singularity

The Cauchy integral formula is a theorem of complex analysis that is conventionally proved with limits. Below is an equipoint proof. First we derive a preliminary theorem which is also used in the derivation of **Laurent series coefficients** (p. 241).

Given a function f that is analytic within a region R with boundary B, with the possible exception that f is not analytic at some point $a \in R$, and any closed path C within R that goes once around a, then $\int_B f(z)\,dz = \int_C f(z)\,dz$, assuming that we integrate along B and C in the same direction.

PROOF. We start by drawing the contours shown in Figure 86:

- Without loss of generality, assume B is directed counterclockwise.
- Draw a path D coincident with C but directed clockwise.
- Draw a directed line E from any point on B to any point on C, and line F, separated from line E by a distance of $0'$, directed out from C.
- Let G be the infinitesimally short portion of B between E and F, and let H be the portion of B with G removed. Then $H \equiv B \setminus G =' B$.
- Let J be the infinitesimally short portion of D between E and F, and let K be the portion of D with J removed. Then $K \equiv D \setminus J =' D$.
- Let L be the concatenation of, in order, H, E, K, and F. That is, start from the point where F meets B, go almost all the way around B to E, go in on E to D, go almost all the way around D to F, and go out on F to the starting point at B.

L is a closed contour which does not include a. By the Cauchy-Goursat integral theorem, we then have

$$\begin{aligned}
0 &= \int_L f(z)\,dz \\
&= \int_H f(z)\,dz + \int_E f(z)\,dz + \int_K f(z)\,dz + \int_F f(z)\,dz \\
&= \int_H f(z)\,dz + \int_K f(z)\,dz \\
&= \int_B f(z)\,dz + \int_D f(z)\,dz \\
&= \int_B f(z)\,dz - \int_C f(z)\,dz,
\end{aligned}$$

or

$$\int_B f(z)\,dz = \int_C f(z)\,dz. \blacksquare$$

We note that the curves B and C can be finite, infinite, or infinitesimal, with lengths of E and F to match, and the separators G and J infinitesimal compared to B and C.

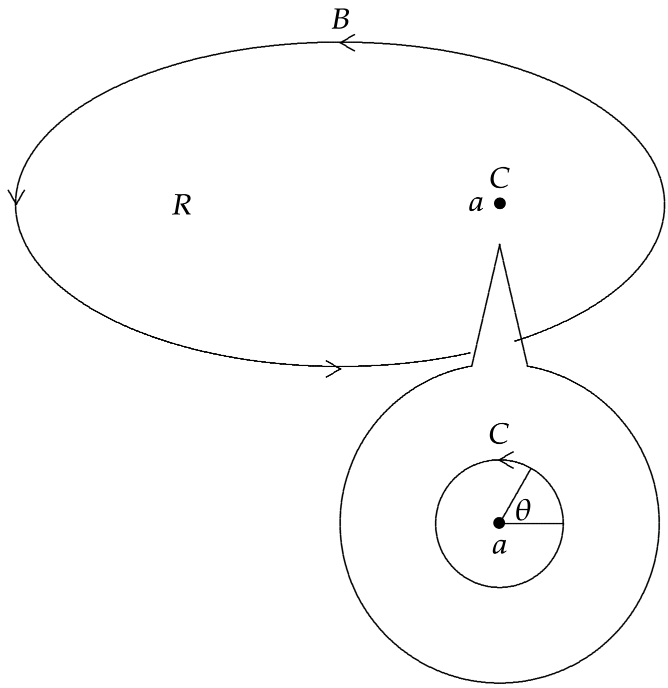

FIG. 87:
Contours for proof of the
Cauchy integral formula

CAUCHY INTEGRAL FORMULA: Given a function $f(z)$ that is analytic in a simply connected region R and on its boundary B, and given a point $a \in R$, then

$$f(a) = \frac{1}{2\pi i} \int_B \frac{f(z)\,dz}{z-a}.$$

PROOF. We draw contours as in Figure 87. Without loss of generality, we again assume B is directed counterclockwise. We draw an infinitesimal circle C, of diameter $0'$, also directed counterclockwise, around a. By the previous theorem, the integral around the boundary B equals the integral around the circle C. To integrate around C, since it is infinitesimal, we make the substitution

$$z = a + 0' e^{i\theta}$$
$$dz = 0' i e^{i\theta}$$

and compute

$$\begin{aligned}
\int_C \frac{f(z)\,dz}{z-a} &= \int_C \frac{f\left(a+0'ie^{i\theta}\right)0'ie^{i\theta}d\theta}{0'e^{i\theta}} \\
&= i\int_C f\left(a+0'ie^{i\theta}\right)d\theta \\
&=' if(a)\int_C d\theta \\
&= 2\pi i f(a)
\end{aligned}$$

or

$$f(a) = \frac{1}{2\pi i}\int_B \frac{f(z)\,dz}{z-a}. \blacksquare$$

We now make the point a variable and rewrite this theorem as

$$f(z) = \frac{1}{2\pi i}\int_B \frac{f(w)\,dw}{w-z}.$$

CAUCHY INTEGRAL FORMULA FOR DERIVATIVES: Given the same conditions as in the previous theorem,

$$\boldsymbol{f^{(n)}(z) = \frac{n!}{2\pi i}\int_B \frac{f(w)\,dw}{(w-z)^{n+1}}.}$$

PROOF. Since an integral is an infinite series and a derivative is a quotient difference, and since we establish in the numeristic theory of infinite series (see **Divergent Series** (p. 301–409)) that we can handle them much as we do finite series, e.g. they commute, associate, and distribute as finite series do, we can calculate simply:

$$\begin{aligned}
\frac{df(z)}{dz} &= \frac{d}{dz}\frac{1}{2\pi i}\int_B \frac{f(w)\,dw}{w-z} \\
&= \frac{1}{2\pi i0'}\int_B \frac{f(w)\,dw}{w-(z-0')} - \frac{1}{2\pi i0'}\int_B \frac{f(w)\,dw}{w-z} \\
&= \frac{1}{2\pi i0'}\int_B \left[\frac{f(w)}{w-(z-0')} - \frac{f(w)}{w-z}\right]dw \\
&= \frac{1}{2\pi i}\int_B \frac{d}{dz}\frac{f(w)}{w-z}\,dw \\
&=' \frac{1}{2\pi i}\int_B \frac{f(w)\,dw}{(w-z)^2}.
\end{aligned}$$

Further differentiations yield the theorem. ∎

Taylor and Laurent series

The numeristic theory of infinite series (see **Divergent Series** (p. 301–409)) also establishes that certain series, when summed through extended numersitic arithmetic, are valid not only where they converge, but also where they diverge. The following is one such series, which is valid everywhere in the complex plane, even though it converges only within the unit circle:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots = \sum_{k=0}^{\infty} x^k$$

We now derive two alternate forms of this series, which we will use in the proofs of the following two theorems:

$$\begin{aligned}
\frac{1}{w-z} &= \frac{1}{(w-a)-(z-a)} \\
&= \frac{\frac{1}{w-a}}{1-\frac{z-a}{w-a}} \\
&= \frac{1}{w-a}\left[1 + \frac{z-a}{w-a} + \left(\frac{z-a}{w-a}\right)^2 + \left(\frac{z-a}{w-a}\right)^3 + \ldots\right] \\
&= \frac{1}{w-a} + \frac{z-a}{(w-a)^2} + \frac{(z-a)^2}{(w-a)^3} + \frac{(z-a)^3}{(w-a)^4} + \ldots, \\
\frac{-1}{w-z} &= \frac{1}{(z-a)-(w-a)} \\
&= \frac{\frac{1}{z-a}}{1-\frac{w-a}{z-a}} \\
&= \frac{1}{z-a}\left[1 + \frac{w-a}{z-a} + \left(\frac{w-a}{z-a}\right)^2 + \left(\frac{w-a}{z-a}\right)^3 + \ldots\right] \\
&= \frac{1}{z-a} + \frac{w-a}{(z-a)^2} + \frac{(w-a)^2}{(z-a)^3} + \frac{(w-a)^3}{(z-a)^4} + \ldots.
\end{aligned}$$

TAYLOR SERIES COEFFICIENTS: Given a function $f(z)$ that is analytic within a simply connected region R and a point $a \in$

***R*, then, for any $z \in R$:**

$$\boldsymbol{f(z) = f(a) + (z-a)f'(a) + (z-a)^2\frac{f''(a)}{2!} + (z-a)^3\frac{f^{(3)}(a)}{3!} + \ldots}$$
$$\boldsymbol{= \sum_{k=0}^{\infty}(z-a)^k\frac{f^{(k)}(a)}{k!}.}$$

PROOF. Let C be any closed path around a. C could be infinitesimal. Using the first of the above identities and the n-th derivative Cauchy integral formula, we compute

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\int_C \frac{f(w)\,dw}{w-z} \\ &= \frac{1}{2\pi i}\int_C \frac{f(w)\,dw}{w-a} + \frac{z-a}{2\pi i}\int_C \frac{f(w)\,dw}{(w-a)^2} + \frac{(z-a)^2}{2\pi i}\int_C \frac{f(w)\,dw}{(w-a)^3} + \ldots \\ &= f(a) + (z-a)f'(a) + (z-a)^2\frac{f''(a)}{2!} + (z-a)^3\frac{f^{(3)}(a)}{3!} + \ldots \\ &= \sum_{k=0}^{\infty}(z-a)^k\frac{f^{(k)}(a)}{k!}. \ \blacksquare \end{aligned}$$

LAURENT SERIES COEFFICIENTS: **Given a function $f(z)$ that is analytic within a region R between an outer boundary B (which may be infinite) and an inner boundary C (which may be infinitesimal), and given a point a inside C (so that $a \notin R$), then, for any $z \in R$:**

$$\boldsymbol{f(z) = \sum_{k=-\infty}^{\infty}\frac{(z-a)^k}{2\pi i}\int_B \frac{f(w)\,dw}{(w-a)^{k+1}}.}$$

PROOF. Following Figure 86, let L be the concatenation of, in order, H, E, K, and F. Let z be any point in R and let $g(w) \equiv \frac{f(w)}{w-z}$. Then

$$\begin{aligned} \int_L g(w)\,dw &= \int_H g(w)\,dw + \int_E g(w)\,dw + \int_K g(w)\,dw + \int_F g(w)\,dw \\ &= \int_H g(w)\,dw + \int_K g(w)\,dw \\ &= \int_B g(w)\,dw + \int_D g(w)\,dw \end{aligned}$$

$$= \int_B g(w)\,dw - \int_C g(w)\,dw.$$

Since L encloses z, and using both of the above identies,

$$\begin{aligned}
f(z) &= \frac{1}{2\pi i}\int_L g(w)\,dw \\
&= \frac{1}{2\pi i}\int_B g(w)\,dw - \frac{1}{2\pi i}\int_C g(w)\,dw \\
&= \frac{1}{2\pi i}\int_B \frac{f(w)\,dw}{w-z} - \frac{1}{2\pi i}\int_C \frac{f(w)\,dw}{w-z} \\
&= \frac{1}{2\pi i}\int_B \frac{f(w)\,dw}{w-a} + \frac{z-a}{2\pi i}\int_B \frac{f(w)\,dw}{(w-a)^2} + \frac{(z-a)^2}{2\pi i}\int_B \frac{f(w)\,dw}{(w-a)^3} + \dots \\
&\quad + \frac{1}{2\pi i(z-a)}\int_C f(w)\,dw + \frac{1}{2\pi i(z-a)^2}\int_C f(w)(w-a)\,dw \\
&\quad + \frac{1}{2\pi i(z-a)^3}\int_C f(w)(w-a)^2\,dw + \dots \\
&= \sum_{k=0}^{\infty}\frac{(z-a)^k}{2\pi i}\int_B \frac{f(w)\,dw}{(w-a)^{k+1}} + \sum_{k=1}^{\infty}\frac{1}{2\pi i(z-a)^k}\int_C f(w)(w-a)^{k-1}\,dw.
\end{aligned}$$

The integrals in the second sum vanish since they do not enclose any singuarities. Hence

$$f(z) = \sum_{k=-\infty}^{\infty}\frac{(z-a)^k}{2\pi i}\int_B \frac{f(w)\,dw}{(w-a)^{k+1}}. \blacksquare$$

Complex poles

In **Types of singularity** (p. 207), we defined a pole of a function f as a point x such that $f(x) = \frac{g(x)}{h(x)}$, g and h are analytic, $h(x)$ has a root (zero) at x, and the multiplicity of the root is finite.

In numeristics, every elementary function is defined over the whole complex plane, even at its singularities. Since a function may be defined at a singularity, the domain of such a function may still be simply connected.

As in conventional analysis, we transform a contour by parameterizing it into a directed real interval.

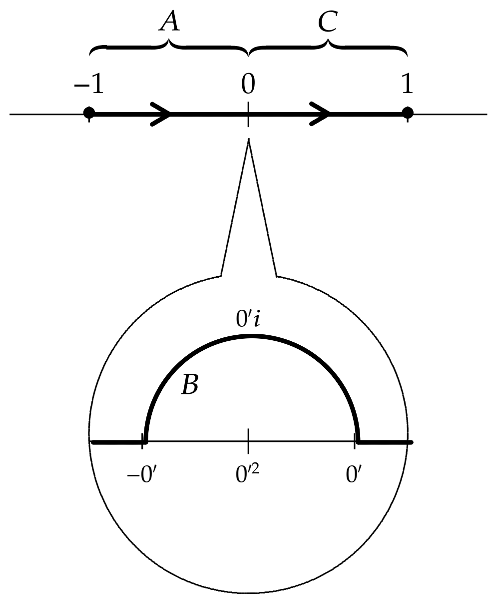

FIG. 88:
Contour G consisting of
three portions A, B, C

In real analysis, as described in **real poles** (p. 223), the effective antiderivative of $\frac{1}{x}$ is $\ln|x|$, which assumes that we integrate either completely on the negative side of the real axis or completely on the positive side.

In complex space, the antiderivative is $\ln x$, and the path of integration is connected. A path which includes an infinitesimal region is shown in Figure 88. This path, contour G, has three portions that link two points, -1 and $+1$:

A: A path along the real axis from -1 to $-0'$,

B: An infinitesimal semicircle around the origin from $-0'$ to $+0'$, and

C: A path along the real axis from $+0'$ to $+1$.

In real space, we must omit portion B and use the effective antiderivative $\ln|x|$. Although this path includes all but one point of the real interval

$[-1, +1]$, the resulting integral of $\frac{1}{x}$ differs from the complex version:

$$\int_{-1}^{+1} \frac{dx}{x} = \int_{A+C} \frac{dx}{x} = \ln|x|\Big|_{-1}^{-0'} + \ln|x|\Big|_{+0'}^{+1} = [-\infty' - 0] + [0 + \infty'] =' 0.$$

In complex space, we can include portion B and use the actual antiderivative $\ln x$:

$$\begin{aligned}\int_G \frac{dz}{z} &= \int_{A+B+C} \frac{dz}{z} = \int_{-1}^{-0'} \frac{dz}{z} + \int_{\pi}^{0} d\theta + \int_{1}^{0'} \frac{dz}{z} = \ln z\Big|_{-1}^{-0'} + \theta\Big|_{\pi}^{0} + \ln z\Big|_{+0'}^{+1} \\ &= \left[\ln(-0') - \ln(-1)\right] + [0 - \pi] + \left[\ln 1 - \ln 0'\right] \\ &= \ln 0' - \pi - \ln 0' = -\pi.\end{aligned}$$

Additional windings around the pole, inside this same infinitesimal complex space hidden within the real line, give the class of values $(2\mathbb{Z} + 1)\pi i$. This agrees with the Fundamental Theorems of Calculus:

$$\int_G \frac{dz}{z} = \ln z\Big|_{-1}^{+1} = \ln 1 - \ln(-1) = 2\mathbb{Z}\pi - (2\mathbb{Z} + 1)\pi = (2\mathbb{Z} + 1)\pi.$$

With other poles, the discrepancy between real and complex integrals happens whenever the integral of portion B is nonzero. Some examples of this integral:

$$\int_B \frac{dx}{x} = \frac{2\mathbb{Z} + 1}{2}$$

$$\int_B \frac{dx}{x^2} = \infty$$

$$\int_B \frac{dx}{x^3} = 0.$$

Complex axial function

The axial function is defined:

$$A(z) \equiv \begin{cases} 0 & \text{for } z \neq 0 \\ \not\infty & \text{for } z = 0 \end{cases}$$

Its name derives from the real version of this function, discussed in **Axial function** (p. 225). The graph of the real function coincides the coordinate axes. The complex version of this function coincides with the plane that contains the two axes of the domain, and the plane containing the two axes of the range.

In real space, an integral through the origin yields an indeterminacy, as it does with a pole. To integrate from one side of the origin to the other, we have to integrate piecewise on each side, which yields two independent constants of integration.

In complex space, we can integrate on a path around the origin, as with did with a **complex pole** (p. 243). This yields a single constant of integration.

Function $e^{\frac{1}{x}}$

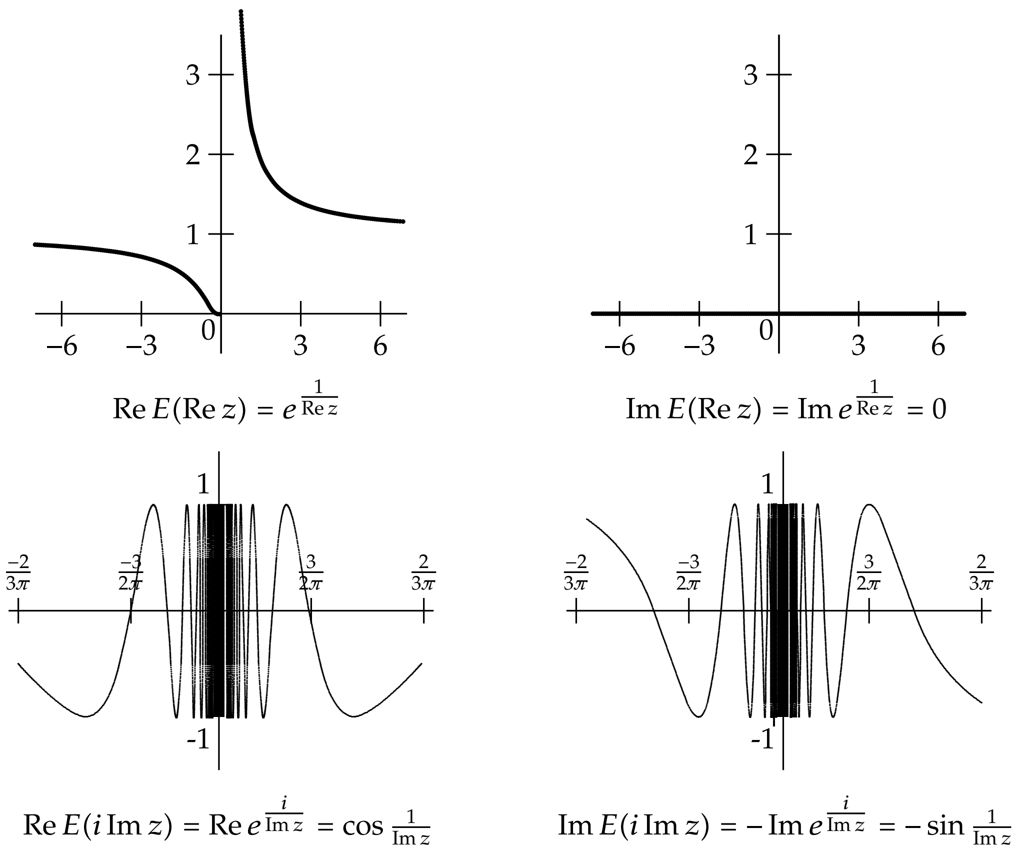

FIG. 89: Essential singularity of $E(z) \equiv e^{\frac{1}{z}}$

The function $E(z) \equiv e^{\frac{1}{z}}$ is graphed via real and imaginary parts in Figure 89. $E(x)$ is a complex version of the function $S(x)$ investigated in **Function $\sin \frac{1}{x}$** (p. 227).

As a real function, $E(z)$ has a jump discontinuity at $z = 0$, but as a complex function, this is not the case. The offset values $E(0')$ for imaginary $0'$ can be any value within $[-1, +1]$, so the offset values are not semiuniform, and the singularity is not a removable discontinuity or jump discontinuity. The following shows that it is not a pole.

$$\begin{aligned}
e^{\frac{1}{z}} &= 1 + \frac{1}{z} + \frac{1}{2!x^2} + \dots \\
&= \frac{x+1}{x} + \frac{1}{2!x^2} + \dots \\
&= \frac{2!x^2 + (2)_1 x + 1}{2!x^2} + \frac{1}{x^3} + \dots \\
&= \frac{3!x^3 + (3)_2 x^2 + (3)_1 x + 1}{3!x^3} + \frac{1}{x^4} + \dots \\
&= \sum_{k=0}^{\infty'} (\infty')_{\infty'-k} x^{\infty'-k} \infty'! x^{\infty'}
\end{aligned}$$

The denominator of the final fraction has a root at $z = 0$ of infinite multiplicity. Thus the singularity is an essential singularity.

PICARD'S THEOREM: If a complex function f has an essential singularity at x, then $f(x) = \not\infty$.

We will not prove the general case of this theorem, but we will show that it holds for $E(z)$, i.e. that $E(0) = e^{\frac{1}{0}} = \not\infty$. To do this, we will show that for given any z, we can find $0'$ such that $z = e^{\frac{1}{0'}}$. If we write $z = re^{i\theta} = e^{\frac{1}{0'}}$, then we are looking for r and θ such that $\ln r + i\theta$ is infinite.

For infinite or zero r, $\ln r$ is infinite, and θ can be any value. For perfinite r, θ must be infinite. Since, as we saw in **Function $\sin \frac{1}{x}$** (p. 227), $\sin \infty = [-1, +1]$, then for an unfolded infinite θ, $e^{i\theta}$ is on the unit circle as it is for finite θ. Thus, for any z, $\ln r + i\theta = \infty'$ for some complex infinite value ∞', and $0' =' \dfrac{1}{\infty'}$. ■

CALCULUS OF VARIATIONS

Definition of functional

The terminology, definitions, notation, and approaches of the calculus of variations are not yet fully standardized. The calculus of variations is also known as *variational calculus*. Older mathematical literature also called it *functional calculus*, but currently this term has a much different meaning.

Here we use terms, notation, and definitions which are common but not universal. The approach that we use for the main topic of the calculus of variations, the functional, is to define it as a type of infinite dimensional vector space. For definitions and theorems, first we state the finite dimensional vector case, and then the functional case.

For an entertaining account of the conventional vector space approach to functionals, see [W16 p. 341-370].

We define an ***n*-dimensional vector function** to be of the form

$$f(\{x_j\}) = h,$$

where $\{x_j\}$ is a sequence (not a class) of real or complex scalars, h is a real or complex scalar, and the index j ranges from 1 to n. This could be stated as $f(x_1, x_2, x_3, \ldots, x_n) = h$, but we are choosing the first form to emphasize its connection to functionals.

Example: Let

$$f(\{x_j\}) \equiv \sum_{j=1}^{n} x_j^2$$

then for $n = 3$, $f(\{1, 2, 3\}) = 14$.

A ***functional***

$$F[x(s)] = h$$

maps the real or complex function $x(s)$ to h, where s, $x(s)$, h are real or complex scalars.

Example: Let

$$F[x(s)] \equiv \int_0^1 x(s)^2 ds$$

then $F[2s] = \frac{4}{3}$.

Iteration

Some definitions in this chapter use iterations of operators which we denote with ***iteration notation***. This notation defines an indexed iteration, a simple type of recursion. Two or more relations are required to define a function recursively, but iteration notation can define an iterated function with a single expression.

Discrete iteration denotes a finite number of iterations at unit intervals of a function on an initial value, a ***seed***:

$\mathop{\mathrm{I}}\limits_{k=m}^{n} h(\langle a \rangle, k) \equiv r_n,$ the last term of the following sequence:

$$\begin{aligned} r_m &= h(a, m) \\ r_k &= h(r_{k-1}, k) \\ r_n &= h(r_{n-1}, n) \end{aligned}$$

Continuous iteration denotes an infinite number of iterations at infinitesimal intervals of a function on a seed:

${}^{0'}\mathscr{I}_{t=u}^{v} h(\langle a \rangle, t) \equiv s(t),$ the last term of the following sequence:

$$\begin{aligned} s(u) &= h(a, u) \\ s(t) &= h(s(t - 0'(v-u)), t) \\ s(v) &= h(s(v - 0'(v-u)), v) \end{aligned}$$

Sums, products, integrals, and prodegrals are easily expressed in iteration notation:

$$\begin{aligned} \sum_{k=m}^{n} f(k) &= \mathop{\mathrm{I}}\limits_{k=m}^{n} \langle 0 \rangle + f(k) \\ \prod_{k=m}^{n} f(k) &= \mathop{\mathrm{I}}\limits_{k=m}^{n} \langle 1 \rangle f(k) \\ \int_a^b f(x)dx &= {}^{dx}\mathscr{I}_{x=a}^{b} \langle 0 \rangle + f(x) \end{aligned}$$

$$\int_a^b f(x)^{dx} = {}^{dx}\mathscr{I}_{x=a}^{b}\langle 1\rangle \cdot f(x)$$

Functional differentials

For a vector function, applying the definition in **Differentials and integrants** (p. 192) gives the following definition of a ***partial differential*** of a vector function:

$$^{0'}\partial_{x_i} f(\{x_j\}) \equiv f(\{x_j + 0'\delta_{i,j}\}) - f(\{x_j\}).$$

Adapting this definition for a functional yields the following definition for the ***(partial) differential of a functional***:

$$^{0'}\delta_{t,x(s)} F[x(s)] \equiv F[x(s) + 0'\delta(s-t)] - F[x(s)]$$

The functional differential is sometimes called the *variation* of F, but the **corresponding derivative** (p. 252), defined in the next section, is also sometimes called the variation.

The ***total differential*** of a vector function is the sum of the partial differentials:

$$^{0'}d_{\{x_j\}} f(\{x_j\}) \equiv \sum_{i=1}^{n} {}^{0'}\partial_{x_i} f(\{x_j\}).$$

Similarly, the total differential of a functional is the integral of the partial differentials:

$$^{0'}\delta_{*,x(s)} F[x(s)] \equiv \int_{-\infty}^{+\infty} {}^{0'}\delta_{t,x(s)} F[x(s)]dt.$$

The ***omnivariate differential*** of a vector function is the repeated partial differential for all variables:

$$\partial^{n}_{x_1,\ldots,x_n} f(\{x_j\}) = \prod_{i=1}^{n} \partial_{x_i}\langle f(\{x_j\})\rangle.$$

Analogously, the ***omnivariate differential*** of a functional is the partial differential repeated for each point in the function space:

$$^{0'}\mathfrak{D}_{t,x(s)} F[x(s)] \equiv {}^{0'}\mathscr{I}_{t=-\infty}^{+\infty} {}^{0'}\delta_{t,x(s)}\langle F[x(s)]\rangle.$$

In the independent function space, for both vector and functional cases, the differentials are:

$$\partial x_1 \ldots \partial x_n = \prod_{i=1}^{n} \partial x_i$$

$${}^{0'}\delta_{x(t)} x(t) \equiv {}^{0'}d_{x(t)} x(t)$$

$${}^{0'}\mathfrak{D}_{x(t)} x(t) \equiv {}^{0'}\!\!\int_{t=-\infty}^{+\infty} {}^{0'}\delta_{x(t)} x(t)$$

Functional derivatives

Applying the **definition of derivative** (p. 163) and the **definition of partial differential** (p. 192) to a vector function, the ***partial derivative*** of a vector function is:

$$\frac{\partial f(\{x_j\})}{\partial x_i} \equiv \frac{{}^{0'}\partial_{x_i} f(\{x_j\})}{{}^{0'}d_{x_i} x_i} = \frac{f(\{x_j + 0'\delta_{i,j}\}) - f(\{x_j\})}{0'}.$$

Analogously, the ***functional derivative*** is defined as:

$$\frac{\delta F[x(s)]}{\delta x(t)} \equiv \frac{{}^{0'}\delta_{t,x(s)} F[x(s)]}{{}^{0'}\delta_{x(t)} x(t)} = \frac{F[x(s) + 0'\delta(s-t)] - F[x(s)]}{0'}.$$

As mentioned above, both the functional derivative and the functional differential are called the *variation*.

Quadratic example, vector function case: Let

$$f(\{x_j\}) \equiv \sum_{j=1}^{n} x_j^2.$$

Then

$$\frac{\partial f(\{x_j\})}{\partial x_i} = \frac{\partial \sum_{j=1}^{n} x_j^2}{\partial x_i} = \sum_{j=1}^{n} 2x_j \cdot \delta_{i,j} = 2x_i.$$

Quadratic example, functional case: Let

$$F[x(s)] \equiv \int_a^b [x(s)]^2 ds.$$

Then

$$\frac{\delta F[x(s)]}{\delta x(t)} = \frac{\int_a^b [x(s) + 0'\delta(s-t)]^2\, ds - \int_a^b x(s)^2 ds}{0'}$$
$$= \frac{\int_a^b [x(s)^2 + 2x(s)\, 0'\delta(s-t) + 0'^2\delta(s-t)^2]\, ds - \int_a^b x(s)^2 ds}{0'}$$
$$= \int_a^b [2x(s)\, \delta(s-t) + 0'\delta(s-t)^2]\, ds$$
$$= 2x(t) + \int_a^b 0'\delta(s-t)\, ds = 2x(t).$$

For further reading on the conventional theory and applications of the functional derivative, the author suggests [WC].

Higher functional derivatives

Higher derivatives of vector functions are, of course, iterations of the first derivative. The following is the form using binomial coefficients that was derived in **Higher order derivatives and integrals** (p. 197).

$$\frac{\partial^2 f(\{x_j\})}{\partial x_i^2} = \frac{f(\{x_j + 2 \cdot 0'\delta_{j,i}\}) - 2f(\{x_j + 0'\delta_{j,i}\}) + f(\{x_j\})}{0'^2}$$

$$\frac{\partial^n f(\{x_j\})}{\partial x_i^n} = \frac{\sum_{k=0}^{m} (-1)^{n-k} \binom{n}{k} f(\{x_j + 0'k\delta_{j,i}\})}{0'^n}$$

Higher functional derivatives are also iterations of the first derivative, which are derived analogously to the higher derivatives of vector and ordinary functions.

$$\frac{\delta^2 F[x(s)]}{\delta F[x(t)]^2} \equiv \frac{F[x(s) + 2 \cdot 0'\delta(s-t)] - 2F[x(s) + 0'\delta(s-t)] + F[x(s)]}{0'^2}$$

$$\frac{\delta^n F[x(s)]}{\delta F[x(t)]^n} \equiv \frac{\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} F[x(s) + 0' k\delta(s-t)]}{0'^n}$$

The second functional derivative also called the *second variation*, etc.

Functional product rule

PRODUCT RULE FOR VECTOR FUNCTIONS:

$$\boldsymbol{\frac{\partial}{\partial x_i} f(\{x_j\}) g(\{x_j\}) = f(\{x_j\}) \frac{\partial}{\partial x_i} g(\{x_j\}) + g(\{x_j\}) \frac{\partial}{\partial x_i} f(\{x_j\}).}$$

PROOF. The proof is essentially the same as for the **ordinary product rule** (p. 172).

$$\begin{aligned}
&\frac{\partial}{\partial x_i} f(\{x_j\}) g(\{x_j\}) \\
&= \frac{f(\{x_j + 0'\delta_{j,i}\}) g(f(\{x_j + 0'\delta_{j,i}\}) - f(\{x_j\}) g(\{x_j\})}{0'} \\
&= \frac{1}{0'} \Big[f(\{x_j\}) g(\{x_j\}) \\
&\qquad + f(\{x_j\}) \left[g(\{x_j + 0'\delta_{j,i}\}) - g(\{x_j\}) \right] \\
&\qquad + \left[f(\{x_j + 0'\delta_{j,i}\}) - f(\{x_j\}) \right] g(\{x_j\}) \\
&\qquad + \left[f(\{x_j + 0'\delta_{j,i}\}) - f(\{x_j\}) \right] \left[g(\{x_j + 0'\delta_{j,i}\}) - g(\{x_j\}) \right] \\
&\qquad - f(\{x_j\}) g(\{x_j\}) \Big] \\
&= \frac{1}{0'} \Big[f(\{x_j\}) g(\{x_j\}) \\
&\qquad + f(\{x_j\}) \left[g(\{x_j + 0'\delta_{j,i}\}) - g(\{x_j\}) \right] \\
&\qquad + \left[f(\{x_j + 0'\delta_{j,i}\}) - f(\{x_j\}) \right] g(\{x_j\}) \\
&\qquad - f(\{x_j\}) g(\{x_j\}) \Big] \\
&= \frac{f(\{x_j\}) \left[g(\{x_j + 0'\delta_{j,i}\}) - g(\{x_j\}) \right] + g(\{x_j\}) \left[f(\{x_j + 0'\delta_{j,i}\}) - f(\{x_j\}) \right]}{0'}
\end{aligned}$$

$$= f(\{x_j\})\frac{\partial}{\partial x_i}g(\{x_j\}) + g(\{x_j\})\frac{\partial}{\partial x_i}f(\{x_j\}). \blacksquare$$

PRODUCT RULE FOR FUNCTIONALS:

$$\boldsymbol{\frac{\delta}{\delta x(t)}F[x(s)]G[x(s)] = F[x(s)]\frac{\delta}{\delta x(t)}G[x(s)] + G[x(s)]\frac{\delta}{\delta x(t)}F[x(s)].}$$

PROOF.

$$\begin{aligned}
&\frac{\delta}{\delta x(t)}F[x(s)]G[x(s)] \\
&= \frac{F[x(s)+0'\delta(s-t)]G[x(s)+0'\delta(s-t)] - F[x(s)]G[x(s)]}{0'} \\
&= \frac{1}{0'}\Big[F[x(s)]G[x(s)] \\
&\quad + F[x(s)][G[x(s)+0'\delta(s-t)] - G[x(s)]] \\
&\quad + \left[F[x(s)+0'\delta(s-t)] - F[x(s)]\right]G[x(s)] \\
&\quad + \left[F[x(s)+0'\delta(s-t)] - F[x(s)]\right]\left[G[x(s)+0'\delta(s-t)] - G[x(s)]\right] \\
&\quad - F[x(s)]G[x(s)]\Big] \\
&= \frac{1}{0'}\Big[F[x(s)]G[x(s)] \\
&\quad + F[x(s)]\left[G[x(s)+0'\delta(s-t)] - G[x(s)]\right] \\
&\quad + \left[F[x(s)+0'\delta(s-t)] - F[x(s)]\right]G[x(s)] \\
&\quad - F[x(s)]G[x(s)]\Big] \\
&= \frac{1}{0'}\Big[F[x(s)]\left[G[x(s)+0'\delta(s-t)] - G[x(s)]\right] \\
&\quad + G[x(s)]\left[F[x(s)+0'\delta(s-t)] - F[x(s)]\right]\Big] \\
&= F[x(s)]\frac{\delta}{\delta x(t)}G[x(s)] + G[x(s)]\frac{\delta}{\delta x(t)}F[x(s)]. \blacksquare
\end{aligned}$$

Functional power rule

The power rule for a vector function follows easily from the **product rule** (p. 254).

POWER RULE FOR VECTOR FUNCTIONS:

$$\boldsymbol{\frac{\partial}{\partial x_i}\sum_j x_j^n = n x_i^{n-1}.}$$

PROOF.

$$\frac{\partial}{\partial x_i}\sum_j x_j^n = \frac{\sum_j (x_j + 0'\delta_{j,i})^n - \sum_j x_j^n}{0'}$$

$$= \frac{\sum_j \sum_{k=1}^{n} \binom{n}{k} x_j^{n-k} 0'^k \delta_{j,i}^k}{0'}$$

$$= \sum_j n x_j^{n-1}\delta_{j,i} + \sum_j \sum_{k=2}^{n} \binom{n}{k} x_j^{n-k} 0'^{k-1}\delta_{j,i}^k$$

$$=' \sum_j n x_j^{n-1}\delta_{j,i} = n x_i^{n-1}. \blacksquare$$

POWER RULE FOR FUNCTIONALS:

$$\boldsymbol{\frac{\delta}{\delta x(t)}\int x(s)^n ds = n x(t)^{n-1}.}$$

PROOF.

$$\frac{\delta}{\delta x(t)}\int x(s)^n ds = \frac{\int [x(s) + 0'\delta(s-t)^n] ds - \int x(s)^n ds}{0'}$$

$$= \frac{\int \sum_{k=1}^{n} \binom{n}{k} x(s)^{n-k} 0'^k \delta(s-t)^k}{0'}$$

$$= \int nx(s)^{n-1}\delta(s-t)ds + \int \sum_{k=2}^{n} \binom{n}{k} x_j^{n-k} 0'^{k-1} \delta(s-t)^k ds$$
$$=' \int nx(s)^{n-1}\delta(s-t)ds = nx(t)^{n-1}. \blacksquare$$

This proof applies only when n is a positive integer and is analogous to the proof of the ordinary **power rule** (p. 175) for n a positive integer. The other cases of n in that proof also have analogs for functional derivatives, but they are not presented here.

Functional transfer rule

The transfer rules show how the derivatives of certain vector functions and functionals simplify to an ordinary derivative. They make use of the **power rules** (p. 256).

TRANSFER RULE FOR VECTOR FUNCTIONS: **If f is analytic,**

$$\frac{\partial}{\partial x_i} \sum_j f(x_j) = f'(x_j).$$

PROOF.

$$\frac{\partial}{\partial x_i} \sum_j f(x_j) = \frac{\partial}{\partial x_i} \sum_j \sum_{k=0}^{\infty} a_k x_j^k = \frac{\partial}{\partial x_i} \sum_{k=0}^{\infty} \sum_j a_k x_j^k$$
$$= \sum_{k=0}^{\infty} a_k \frac{\partial}{\partial x_i} \sum_j x_j^k = \sum_{k=1}^{\infty} a_k k x_j^{k-1}$$
$$= \frac{d}{dx_j} f(x_j) = f'(x_j). \blacksquare$$

TRANSFER RULE FOR FUNCTIONALS: **If f is analytic,**

$$\frac{\delta}{\delta x(t)} \int f(x(s))ds = f'(x(t)).$$

PROOF.

$$\frac{\delta}{\delta x(t)}\int f(x(s))ds = \frac{\delta}{\delta x(t)}\int \sum_{k=0}^{\infty} a_k x(s)^k ds = \frac{\delta}{\delta x(t)}\sum_{k=0}^{\infty}\int a_k x(s)^k ds$$
$$= \sum_{k=0}^{\infty} a_k \frac{\delta}{\delta x(t)}\int x(s)^k ds = \sum_{k=1}^{\infty} a_k k x(t)^{k-1}$$
$$= \frac{d}{dx(t)} f(x(t)) = f'(x(t)). \blacksquare$$

Functional chain rule

The **chain rule** (p. 171) for ordinary functions has to be somewhat modified for vector functions and functionals. Since vector functions and functionals reduce the dimensions of their domains, chain rules involving them must start with domains of increased dimensions.

An ***indexed vector function*** maps a two-dimensional vector (a matrix) to a one-dimensional vector. The indexed vector function $f(\{x_{i,j}\})_j$ is evaluated on the matrix $\{x_{i,j}\}$ over all j, yielding a one-dimensional vector $\{y_i\}$. The indexed function $f(\{x_i\})_i$ is the same as the simple vector function $f(\{x_i\})$, which maps to a scalar.

An ***indexed functional*** maps a two-argument function to a one-argument function. The indexed functional $F[x(s,t)]_t$ is evaluated over all t, yielding a one-argument function $y(s)$. The indexed functional $F[x(s)]_s$ is the same as the simple functional $F[x(s)]$, which maps to a scalar.

The following are examples of indexed vector functions and functionals and how they can be composed with simple vector functions and functionals:

$$g(\{x_{j,k}\})_j \equiv \left\{\sum_{j=1}^{m} x_j^2 + y_k^2\right\}_{k=1}^{n}$$

$$f(\{z_k\}) \equiv \sum_{k=1}^{n} z_k$$

$$f(g(\{x_{j,k}\})_j) = \sum_{k=1}^{n}\sum_{j=1}^{m} x_j^2 + y_k^2$$

$$G[x(s,t)]_s \equiv \int_1^2 [x(s)^2 + y(t)^2]ds$$

$$F[z(t)] \equiv \int_0^1 z(t)dt$$

$$F[G[x(s,t)]_s] = \int_0^1 \int_1^2 [x(s)^2 + y(t)^2]\, ds\, dt$$

CHAIN RULE FOR VECTOR FUNCTIONS: Given a simple vector function f and an indexed vector function g,

$$\frac{\partial f(g(\{x_{j,k}\})_j)}{\partial x_i} = \sum_l \frac{\partial f(g(\{x_{j,k}\}_j)}{\partial g(\{x_{j,l}\})_j} \cdot \frac{\partial g(\{x_{j,l}\})}{\partial x_i}.$$

PROOF. Define the following:

$$\{y_n\} \equiv g(\{x_{j,n}\})_j$$
$$\{0_n\} \equiv g(\{x_{j,n} + 0'\delta_{i,j})_j - g(\{x_{j,n}\})_j.$$

Then

$$\begin{aligned}
\frac{\partial f(g(\{x_{j,k}\})_j)}{\partial x_i} &= \frac{f(g(\{x_{j,k} + 0'\delta_{i,j}\})_j) - f(g(\{x_{j,k}\})_j)}{0'} \\
&= \frac{f(\{y_k + 0_k\}) - f(\{y_k\})}{0'} \\
&= \frac{\sum_n [f(\{y_k + 0_k\delta_{k,n}\}) - f(\{y_k\})]}{0'} \\
&= \sum_n \frac{f(\{y_k + 0_k\delta_{k,n}\}) - f(\{y_k\})}{0_n} \cdot \frac{0_n}{0'} \\
&= \sum_n \frac{\partial f(\{y_k\})}{\partial y_n} \cdot \frac{\partial y_n}{\partial x_i} = \sum_n \frac{\partial f(g(\{x_{j,k}\}_j)}{\partial g(\{x_{j,n}\})_j} \cdot \frac{\partial g(\{x_{j,n}\})}{\partial x_i}. \blacksquare
\end{aligned}$$

CHAIN RULE FOR FUNCTIONALS: Given a simple functional F and an indexed functional G,

$$\frac{\delta F[G[x(s,u)]_s]}{\delta x(t)} = \int \frac{\delta F[G[x(s,v)]_s]}{\delta G[x(s,v)]_s} \cdot \frac{\delta G[x(s,v)]_s}{\delta x(t)} dv.$$

PROOF. Define the following:

$$y(v) \equiv G[x(s,v)]_s$$
$$0_v \equiv G[x(s,v) + 0'\delta(s-t)]_s - G[x(s,v)]_s.$$

Then

$$\begin{aligned}
\frac{\delta F[G[x(s,u)]_s]}{\delta x(t)} &= \frac{F[G[x(s,u)+0'\delta(s-t)]_s]-F[G[x(s,u)]_s]}{0'} \\
&= \frac{F[y(u)+0_u]-F[y(u)]}{0'} \\
&= \frac{\int (F[y(u)+0_u\delta(u-v)]-F[y(u)])\,dv}{0'} \\
&= \int \frac{F[y(u)+0_u\delta(u-v)]-F[y(u)]}{0_v}\cdot\frac{0_v}{0'}dv \\
&= \int \frac{\delta F[y(u)]}{\delta y(v)}\cdot\frac{\delta y(v)}{\delta x(t)}dv \\
&= \int \frac{\delta F[G[x(s,v)]_s]}{\delta G[x(s,v)]_s}\cdot\frac{\delta G[x(s,v)]_s}{\delta x(t)}dv. \blacksquare
\end{aligned}$$

Straight line theorem

A classic example of the application of functional derivatives is a proof that the shortest path between two points is a straight line. The equipoint proof of this theorem is not only simpler than conventional proofs but, since a numeristic function is unrestricted and can be multivalued, it covers all possible paths, including the case of a vertical line.

STRAIGHT LINE THEOREM: **In a plane, the shortest path between two points is a straight line.**

PROOF. We define an arc length functional L on the space of functions $f(X)$ and minimize L using its functional derivative.

$$\begin{aligned}
L[f(X)] &\equiv \int_a^b \sqrt{1+f'(X)^2}\,dX \\
\frac{\delta L[f(X)]}{\delta f(x)} &= \frac{\delta}{\delta f(x)}\int_a^b \sqrt{1+f'(X)^2}\,dX \\
&= \frac{d}{df(x)}\sqrt{1+f'(x)^2} \qquad \text{by transfer rule (p. 257)} \\
&= \frac{f'(x)}{\sqrt{1+f'(x)^2}}\,\frac{d^2 f(x)}{df(x)\,dx}
\end{aligned}$$

$$= \frac{f''(x)}{\sqrt{1 + f'(x)^2}}$$

The minimum occurs when this last expression is 0. This can happen in two ways:

- When $f''(x) = 0$. In this case, $f(x)$ is a horizontal or oblique straight line of the form $f(x) = mx + b$, where m is finite.
- When $f'(x)$ is infinite and $f''(x)$ is finite. In this case, $f(x)$ is a vertical straight line at a point $x = a$ and infinite valued elsewhere: $f(x) = \infty(x - a)$. ■

Functional integration

Like vector and functional derivatives, vector and functional integrals map a vector or functional (respectively) to a scalar. Unlike an ordinary integral, vector and functional integrals are not inverses of their respective derivatives.

To define these integrals, we first look at the special case of a two-dimensional vector. Since functional integrals are definite integrals that integrate over the entire real range, from $-\infty$ to $+\infty$, we examine only vector function integrals of this type. From the **definition of definite integral** (p. 165) and **infinite bounds on integrals and path integral** (p. 166), we have

$$\begin{aligned}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)\,dx\,dy &\equiv \sum_{m=1}^{\infty'^2}\sum_{k=1}^{\infty'^2} f\left(-\infty' + \frac{2\infty' k}{\infty'^2}, -\infty' + \frac{2\infty' m}{\infty'^2}\right)\frac{4}{\infty'^2} \\ &= \sum_{k_2=1}^{\infty'^2}\sum_{k_1=1}^{\infty'^2} f\left(\frac{2k_1}{\infty'} - \infty', \frac{2k_2}{\infty'} - \infty'\right)\frac{4}{\infty'^2}.\end{aligned}$$

The general ***vector function integral***, using **iteration notation** (p. 250), is given by:

$$\begin{aligned}\int \dots \int_{-\infty}^{+\infty} f(\{x_j\})\,dx_1 \dots dx_n &\equiv \underbrace{\int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty}}_{n} f(\{x_j\})\,dx_n \dots dx_2 dx_1 \\ &= \mathrm{I}_{i=1}^{n} \int_{-\infty}^{+\infty} \langle f(\{x_j\})\rangle\,dx_i\end{aligned}$$

$$\equiv \mathop{\mathrm{I}}_{i=1}^{n} \sum_{k_i=1}^{\infty'^2} \left\langle f\left(\left\{\frac{2k_i}{\infty'} - \infty'\right\}\right)\right\rangle \frac{2}{\infty'}.$$

Now we adapt this definition to functionals. Conventional analysis usually denotes and defines the ***functional integral*** as follows:

$$\int F[x(s)]\, \mathfrak{D}x(t) \equiv \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} F[x(s)] \prod_t dx(t),$$

which uses an ellipsis for iterated integrals and and the product symbol $\prod$ for a continuous product. The equipoint definition instead uses continuous iteration notation:

$$\int F[x(s)]\, \mathfrak{D}x(t) \equiv \mathscr{I}_{t=-\infty}^{+\infty} \int \langle F[x(s)]\rangle\, \delta x(t)$$

$$\equiv \mathscr{I}_{t=-\infty}^{+\infty} \sum_{k_t=1}^{\infty'^2} \left\langle F\left[x(t) + \frac{2k_t}{\infty'}\right]\right\rangle \frac{2}{\infty'}.$$

For further reading on the conventional theory and applications of the functional integral, the author suggests [K16].

Functional delta function

The functional delta function is the functional analog of the **Dirac delta function** (p. 217). The functional delta function is a functional, not an ordinary function. The integration property is expressed in terms of **functional integration** (p. 261).

First we examine the vector function case of $n = 2$.

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)\delta(x-a)\,\delta(y-b)\,dx\,dy = \int_{-\infty}^{+\infty} f(a,y)\,\delta(y-b)\,dy = f(a,b).$$

This enables us to easily define the general vector delta function in terms of ordinary delta function:

$$\delta(\{x_j\}) \equiv \prod_{j=1}^{n} \delta(x_j) = \prod_{j=1}^{n} \int_{-\infty}^{+\infty} e^{2\pi i x_j y_j}\, dy_j = \underbrace{\int \cdots \int_{-\infty}^{+\infty}}_{n} e^{2\pi i \sum_{k=1}^{n} x_k y_k} \prod_{i=1}^{n} dx_i,$$

which has the property

$$\underbrace{\int \cdots \int_{-\infty}^{+\infty}}_{n} f(\{x_j\})\,\delta(\{x_j\} - \{a_j\}) \prod_{i=1}^{n} dx_i = f(\{a_j\}).$$

The functional delta function is thus defined:

$$\delta[x(s)] \equiv \prod_{s=-\infty}^{+\infty} \delta(x(s)) = \prod_{s=-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{2\pi ist}\, dt = \int e^{2\pi i \int_{-\infty}^{+\infty} x(s)y(s)ds}\, \mathfrak{D}y(s),$$

and has the property

$$\int F[x(s)]\, \delta[x(s) - a(s)]\, \mathfrak{D}x(s) = F[a(s)].$$

PROJECTIVE SPACES

Folded projective spaces

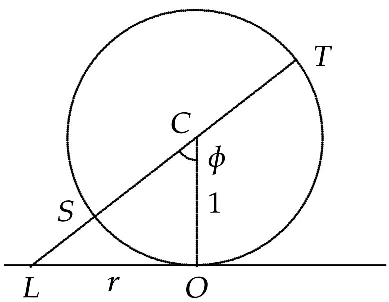

FIG. 90:
Projectively extended
real numbers,
mapping of finite number
$r = \tan\phi$

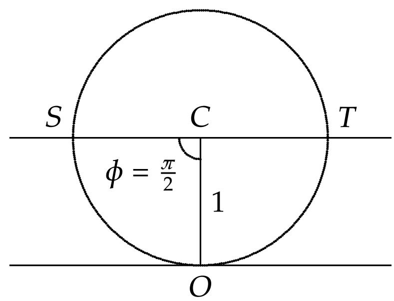

FIG. 91:
Projectively extended
real numbers,
mapping of
$\infty = \tan\frac{\pi}{2}$

The projectively extended real numbers, introduced in **Real infinite element extensions** (p. 68), naturally apply to projective spaces. This number system can be derived from coordinates of projective space, or, conversely, projective space can be derived by extending Euclidean space with projectively extended coordinates.

Figure 90 shows the mapping of an arbitrary point L on a projective line OL to a line ST through the point C. Figure 91 shows the special case when L is the point at infinity, which is mapped to the line ST that is parallel to OL.

As we saw in Unfolding infinite numbers, space is unfolded in the same way that numbers are. In folded 2-dimensional projective space, points at infinity have polar coordinates (∞, θ) with distinct points for $-\dfrac{\pi}{2} < \theta \leq +\dfrac{\pi}{2}$.

The line at infinity is a circle of infinite radius with center at any finite point, i.e. the curve with polar equation $r = \infty$. Euclidean lines are also circles of infinite radius with center at a point of infinity in the direction orthogonal to the line.

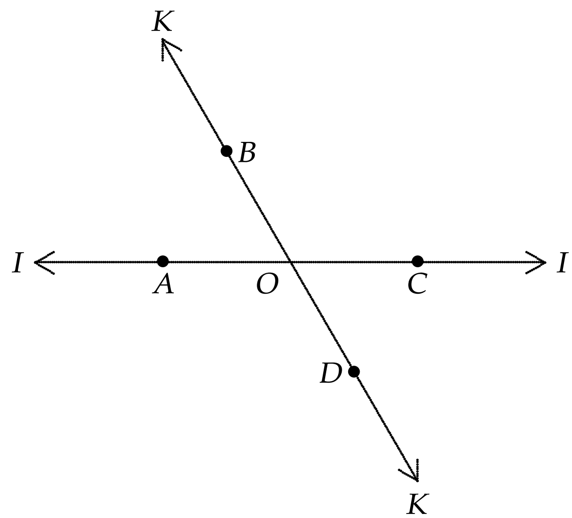

FIG. 92:
Infinite triangles
in projective plane

At the folded level, $+\infty = -\infty$, so "opposite" ends of an infinte line are actually a single point at infinity. This can lead to ambiguity when notating line segments. For example, in Figure 92, the notations OI and $\triangle IOK$ are ambiguous: OI may be the the half line that includes point A, or the half line that includes point B, and $\triangle IOK$ may be the infinite sided triangle that includes points A and B, or B and C, or C and D, or D and A. To eliminate this ambiguity, we use the notation $O\dot{A}I$ for the half line that includes A, and the notation $\triangle I\dot{A}O\dot{B}K$ for triangle that includes A and B.

Unfolded projective spaces

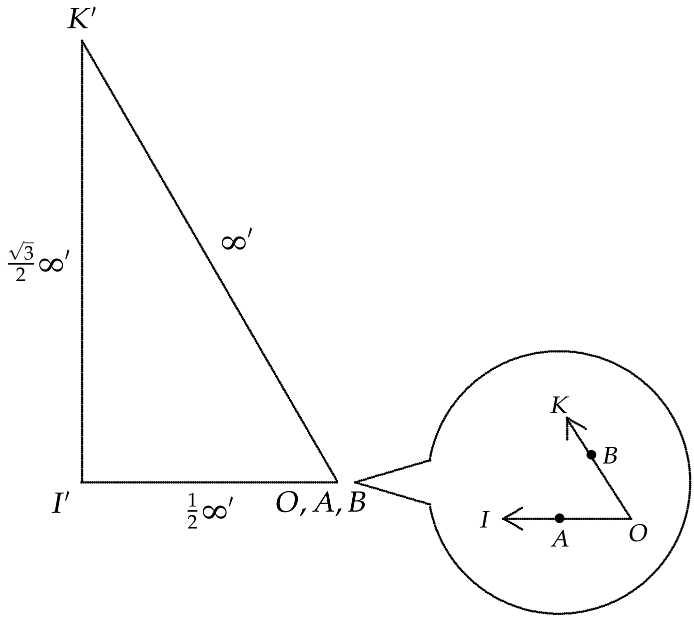

FIG. 93:
Infinite isoceles triangle in
folded and unfolded projective plane

As mentioned in the previous section, space is unfolded in the same way that numbers are. In unfolded projective space, we unfold a single point at infinity into multiple infinities.

Figure 93 shows an unfolding, $\triangle OI'K'$, of one of the infinite right triangles in Figure 92, $\triangle I\dot{A}O\dot{B}K$. In this unfolding, $I' \in' I$ and $J' \in' J$. The figure shows how we can use unfolded infinities to calculate lengths, angles, and other measures of infinite figures in projective space.

While we can use unfolded infinite points and lengths to calculate folded angles such as $\angle I\dot{A}O\dot{B}K$, we can also use folded points and lengths such as A, B, OA, and OB. For this reason, in coordinates for 2-dimensional projective space, folded arithmetic can handle polar coordinates with infinite radius, but to handle infinite rectangular coordinates, we usually must use unfolded lengths.

To convert between polar and rectangular coordinates in 2-dimensional projective space, we proceed as follows, denoting polar coordinates $[r, \theta]$ and rectangular coordinates (x, y). The notation $\sqrt{a}$ for real a means the principal square root of a, i.e. $|a^{\frac{1}{2}}|$.

A point at infinity has the form $[\infty, \theta]$, where $\theta = \tan^{-1} \frac{y}{x}$, so $[\infty, \theta] = [\infty, \theta + \mathbb{Z}\pi]$. Because of the quotient $\frac{y}{x}$, we must use unfolded arithmetic when both x and y are infinite to get a determinate θ.

For a general conversion between polar and rectangular, from $x = r \sin \theta$ and $y = r \cos \theta$, it follows that

$$r \in \pm\sqrt{x^2 + y^2}$$
$$\theta \in \tan^{-1} \frac{y}{x}$$

but

$$r = \sqrt{x^2 + y^2}$$
$$\theta = \tan^{-1} \frac{y}{x}$$

does not work, since $\tan^{-1} \frac{y}{x}$ returns too many values. For example, if $x = 1$ and $y = 1$, then $r = \sqrt{2}$ and $\theta = \tan^{-1} 1 = \frac{\pi}{4}, \frac{5\pi}{4}, \ldots$. These values for θ are in both the first and third quadrants, where they should be only in the first.

The formulas

$$r = \pm\sqrt{x^2 + y^2}$$
$$\theta = \tan^{-1} \frac{y}{x}$$

also do not work, as again there are too many values. In the example of $x = 1$ and $y = 1$, we have $r = \pm\sqrt{2}$ and $\theta = \tan^{-1} 1 = \frac{\pi}{4}, \frac{5\pi}{4}, \ldots$. The pairs $\left[+\sqrt{2}, \frac{\pi}{4}\right]$ and $\left[-\sqrt{2}, \theta = \frac{5\pi}{4}\right]$ are correct, but $\left[+\sqrt{2}, \frac{5\pi}{4}\right]$ and $\left[-\sqrt{2}, \frac{\pi}{4}\right]$ are not.

A solution is to use the atan2 function that is used in **Argument function** (p. 87) to define the argument of a complex number. This function accepts x and y as separate parameters and returns only the correct values of θ. While atan2 is defined in many programming languages as a single valued function, in numeristics it is multivalued:

$$\operatorname{atan2}(x, y) \equiv \cos^{-1} \frac{y}{\sqrt{x^2 + y^2}} \cap \sin^{-1} \frac{x}{\sqrt{x^2 + y^2}}.$$

We then have two methods for converting from rectangular to polar coordinates.

- ***r* allowed to be positive only:**

$$r = \sqrt{x^2 + y^2}$$
$$\theta = \begin{cases} \operatorname{atan2}(x, y) & \text{for finite } r \\ \tan^{-1} \frac{y}{x} & \text{for infinite } r \end{cases}$$

- ***r* allowed to be positive or negative:**

$$\begin{aligned} [r, \theta] &= \left[\underset{1}{\pm}\sqrt{x^2 + y^2}, \operatorname{atan2}(\underset{1}{\pm}x, \underset{1}{\pm}y) \right] \\ &= \left[\sqrt{x^2 + y^2}, \operatorname{atan2}(x, y) \right] \cup \left[-\sqrt{x^2 + y^2}, \operatorname{atan2}(-x, -y) \right] \end{aligned}$$

r is zero if and only if x and y are both zero, in which case $\theta = \not\infty$, since $\dfrac{y}{x} = \operatorname{atan2}(x, y) = \not\infty$.

The 0-90-90 triangle

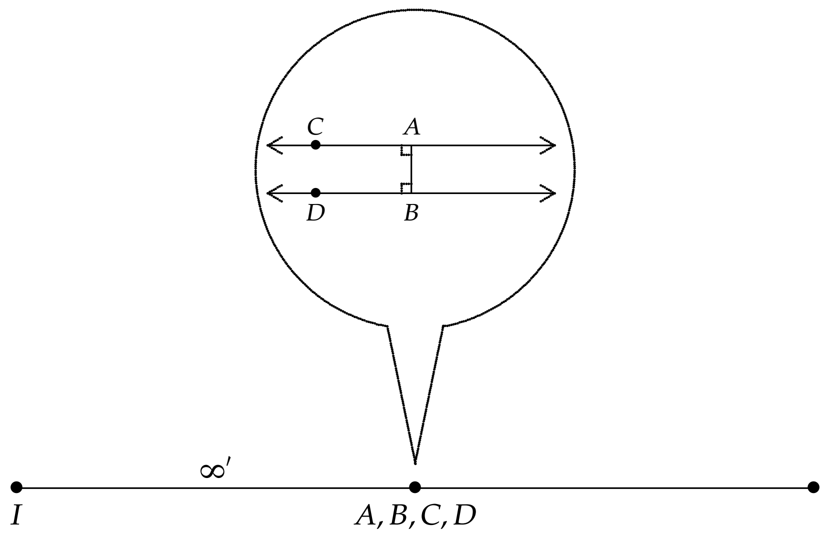

FIG. 94:
0-90-90 triangle in
folded and unfolded projective plane

Figure 94 shows $\triangle I\dot{C}AB\dot{D}$, consisting of two parallel half lines $A\dot{C}I$ and $B\dot{D}I$ of infinite length, and a third side AB of finite length, which is perpendicular to both $A\dot{C}I$ and $B\dot{D}I$. The half lines meet at I, a point at infinity.

We let

$$AB \equiv 1$$
$$\infty' \equiv B\dot{D}I$$

and apply several laws of Euclidean trigonometry to unfolded projective space. From

$$\angle ABD = \frac{\pi}{2},$$

we have

$$\begin{aligned} A\dot{C}I &= \sqrt{\infty'^2 - 1} =' \infty \\ \angle CAB &= \tan^{-1}\sqrt{\infty'^2 - 1} = \frac{\pi}{2} + 0' =' \frac{\pi}{2} \end{aligned}$$

where

$$0' \equiv \tan^{-1}\infty' - \tan^{-1}\sqrt{\infty'^2 - 1},$$

and by law of sines

$$\begin{aligned} \frac{\angle CAB}{B\dot{D}I} &= \frac{\angle CID}{AB} \\ \angle CID &= \frac{\frac{\pi}{2} - 0'}{\infty'} =' 0. \end{aligned}$$

Homogenous coordinates

In conventional projective geometry, the ***homogenous coordinates*** of a point on a projective line are of the form $x : y$, where x and y are finite. They have the following properties:

$$\begin{aligned} &x : 1 \text{ is the finite point at position } x \\ &x : 0 \text{ for any } x \neq 0 \text{ is the point at infinity} \\ &0 : 0 \text{ is unassigned} \\ &x : y = kx : ky \text{ for perfinite } k. \end{aligned}$$

This last property is called homogeneity, but there are other coordinate systems that also have this property.

From the numeristic perspective, $x : y$ is simply $\frac{x}{y}$, as is suggested by the colon, an obsolete notation for a ratio. Numeristics does not leave $0 : 0$ unassigned but assigns the value $0 : 0 = \frac{0}{0} = \not\infty$, which covers the whole projective line.

Conventional homogenous coordinates on a projective plane have 3 terms, $x : y : z$, with x, y, z finite and the following properties:

$$
\begin{aligned}
&x : y : 1 \text{ is the Euclidean point } (x, y)\\
&x : y : 0 \text{ for } x \neq 0 \text{ or } y \neq 0 \text{ is } \left[\infty, \tan^{-1}\frac{y}{x}\right],\\
&\qquad\qquad \text{the point at infinity in the direction } \tan^{-1}\frac{y}{x}\\
&0 : 0 : 0 \text{ is unassigned}\\
&x : y : z = kx : ky : kz \text{ for perfinite } k.
\end{aligned}
$$

From the numeristic perspective, $x : y : z = \left(\frac{x}{z}, \frac{y}{z}\right)$, and $0 : 0 : 0 = (\not\infty, \not\infty)$, the whole projective plane.

For coabfinite x and y, we can use unfolded arithmetic to make $\frac{y}{x}$ and thus $\tan^{-1}\frac{y}{x}$ single valued. For example:

$$
\begin{aligned}
[\infty, 0] &= (\infty, 0) && = 1 : 0 : 0 && = \infty : 0 : 1\\
\left[\infty', \frac{\pi}{4}\right] &= \left(\frac{\infty'}{\sqrt{2}}, \frac{\infty'}{\sqrt{2}}\right) && = 1 : 1 : \frac{\sqrt{2}}{\infty'} && = \frac{\infty'}{\sqrt{2}} : \frac{\infty'}{\sqrt{2}} : 1\\
\left[\infty', \frac{\pi}{3}\right] &= \left(\frac{\infty'}{2}, \frac{\sqrt{3}\infty'}{2}\right) && = 1 : \sqrt{3} : \frac{2}{\infty'} && = \frac{\infty'}{2} : \frac{\sqrt{3}\infty'}{2} : 1\\
\left[\infty, \frac{\pi}{2}\right] &= (0, \infty) && = 0 : 1 : 0 && = 0 : \infty : 1.
\end{aligned}
$$

Conventional two-dimensional homogenous coordinates and polar coordinates cannot distinguish between points that are expressed in rectangular coordinates in which one coordinate is finite and the other infinite. For example, $(1, \infty)$ is in the graph of $y = \frac{1}{1-x}$, but $(2, \infty)$ is not. These two points are folded within $\left[\infty, \pm\frac{\pi}{2}\right] = 0 : 1 : 0 = (\mathbb{R}, \infty)$. To distinguish the first one using unfolded arithmetic:

$$
\begin{aligned}
(1, \infty) \ni' (1, \infty') &= \left[\sqrt{1 + \infty'^2}, \tan^{-1}\infty'\right]\\
&= 1 : \infty' : 1 = \frac{1}{\infty'} : 1 : \frac{1}{\infty'} \in' 0 : 1 : 0 = \left[\infty, \frac{\pi}{2}\right].
\end{aligned}
$$

Trilinear coordinates

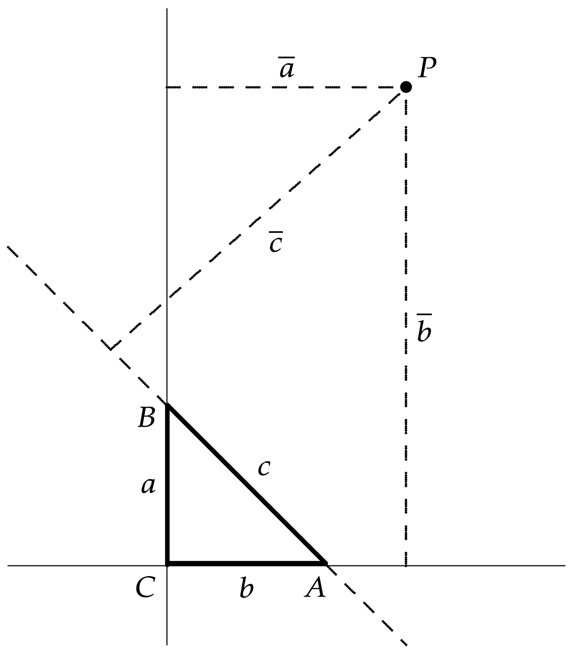

FIG. 95:
Trilinear coordinates
in projective plane

On a projective plane, the ***trilinear coordinates*** of a point are a triple $(x : y : z)$ in which each term is proportional to directed distances from the point to the extended sides of a reference triangle.

An example in shown in Figure 95. We use $\triangle ABC$ as the reference triangle, where $A = (1, 0)$, $B = (0, 1)$, $C = (0, 0)$, and the lengths of the sides are $a = b = 1$ and $c = \sqrt{2}$. To compute the trilinear coordinates of a point $P = (x, y)$, we extend the sides of $\triangle ABC$ and compute the lengths

$$\begin{aligned}\overline{a} &= x\\ \overline{b} &= y\\ \overline{c} &= -\frac{\sqrt{2}}{2}(x+y-1).\end{aligned}$$

The trilinear coordinates of P are then $(\overline{a}:\overline{b}:\overline{c}) = \left(x:y:-\dfrac{\sqrt{2}}{2}(x+y-1)\right)$.

Since each pair of coordinates need only be proportional to a distance, then $(\alpha:\beta:\gamma) = (k\alpha:k\beta:k\gamma)$ for perfinite k, i.e. trilinear coordinates have the property of homogeneity. If $k=1$, i.e. the coordinates are the actual distances to the extended sides, then the coordinates are called ***exact trilinear coordinates*** and are denoted (α,β,γ).

Conventional trilinear coordinates, those with finite terms, can represent points at infinity, but they are not exact trilinear coordinates. Such points can be represented with exact trilinear coordinates with unfolded infinite terms. For example:

$$\begin{aligned}
\left[\infty',0\right] &= (\infty',0) &&= \left(\infty',0,-\frac{\sqrt{2}}{2}(\infty'-1)\right) &&=' \left(1:0:-\frac{\sqrt{2}}{2}\right)\\
\left[\infty',\frac{\pi}{2}\right] &= (0,\infty') &&= \left(0,\infty',-\frac{\sqrt{2}}{2}(\infty'-1)\right) &&=' \left(0:1:-\frac{\sqrt{2}}{2}\right)\\
\left[\infty',\frac{\pi}{4}\right] &= (\infty',\infty') &&= \left(\infty',\infty',-\sqrt{2}(\infty'-\tfrac{1}{2})\right) &&=' \left(1:1:-\sqrt{2}\right)\\
\left[\infty',-\frac{\pi}{4}\right] &= (\infty',-\infty') &&= \left(\infty',-\infty',-\sqrt{2}(-\tfrac{1}{2})\right) &&=' (1:-1:0)
\end{aligned}$$

CLASS COUNTS

Class count comparisons

In set theory, the notation $\#C$ means the cardinality of the set C. Since numeristics does not use the concept of cardinality, we use this notation to mean simply the number of elements in the class C, which we call a ***class count***.

We will need to address finite and infinite counts separately. We will avoid saying that a class "is finite" or "is infinite," because such characterizations do not distinguish between the elements and the count. A class could have a finite number of infinite elements, or an infinite number of finite elements, or both finite, or both infinite.

Finite counts

A finite count can obviously be established through a bijection: $\# \pm 1 = 2$, for example, can be established through a bijection from $\mathbb{Z}_2$ to ± 1, such as $n \mapsto (-1)^n$ or $n \mapsto 2n - 1$. For all finite counts, the count is independent of the bijection.

Infinite counts

Of course, the counts of all infinite classes are the same value:

$$\#\mathbb{N} = \#\mathbb{Q} = \#\mathbb{R} = \#\mathbb{C} = \infty.$$

On the other hand, as with other infinite quantities, ratios and other operations between two infinite values may be finite. Much as the value of $\dfrac{dy}{dx}$ depends on the relation between x and y in unfolded space, $\dfrac{\#C}{\#D}$, so $\#C - \#D$, and other such expressions may depend on the relation between the counts of C and D in unfolded space, which in turn may depend on the way that C and D are mapped to each other.

Comparing $\mathbb{N}^*$ and $\mathbb{Z}^*$

As an example, consider $\mathbb{N}$ and $\mathbb{Z}^*$. If we map $\mathbb{Z}^*$ to $\mathbb{N}^*$ with a function f that takes n to $|n|$, then f maps two elements of $\mathbb{Z}^*$ to each element of $\mathbb{N}^*$. If $\infty' \equiv \#\mathbb{N}^*$, then $\#\mathbb{Z}^*$ established through this map is $2\infty'$, and $\dfrac{\#\mathbb{Z}^*}{\#\mathbb{N}^*} = 2$.

But if we map $\mathbb{Z}^*$ to $\mathbb{N}^*$ with a function g that takes n to $2n$ for positive n and $-2n-1$ for negative n, which maps $\ldots -3, -2, -1, 1, 2, 3 \ldots$ to $5, 3, 1, 2, 4, 6 \ldots$, then $\#\mathbb{Z}^*$ established through this map is ∞', and $\dfrac{\#\mathbb{Z}^*}{\#\mathbb{N}^*} = 1$.

Principles of infinite counts

As the above example shows, if the count of *folded* elements in a class is infinite, then comparing the count to other class counts in *unfolded* space may depend on the map or maps that are used to connect the classes.

Given $f : A \rightarrow B$, then $\#A$ is the number of mappings in f, and the relationship of $\#A$ to $\#B$ is determined from the way f maps A to B. We call this a ***comparison map*** and use the notation $(\# : f)$ before an expression containing a comparison to show that the classes are connected with the map f. In the above example, $(\# : f)\dfrac{\#\mathbb{Z}^*}{\#\mathbb{N}^*} = 2$, and $(\# : g)\dfrac{\#\mathbb{Z}^*}{\#\mathbb{N}^*} = 1$.

General rules for comparison maps

In general, for a class count comparison map f:

- If $f : A \rightarrow \mathbb{Z}_n$ is bijective, then $(\# : f)\ \#A = n$. In this case, A has a finite count, which is independent of f.
- If $f : A \rightarrow B$ is bijective, then $(\# : f)\ \#A = \#B$.
- If $f : A \rightarrow B \cup C$ is bijective and $B \cap C = \emptyset$, then $(\# : f)\ \#A = \#B + \#C$.
- If $f : A \rightarrow (B, C)$ is bijective, then $(\# : f)\ \#A = \#B\#C$.

The notation (B, C) means the Cartesian product of B and C, $\{(b, c) \mid b \in B \wedge c \in C\}$. Introduced in **Standard numeric classes** (p. 60), this notation is used to avoid confusion with $B \times C$, which means a product distributed over the elements of B and C, $\{bc \mid b \in B \wedge c \in C\}$.

- If $f : A \rightarrow B^{\swarrow C}$ is bijective, then $(\# : f)\ \#A = \#B^{\#C}$.

  The notation $B^{\swarrow C}$ means the class of functions from C to B, $\{f \mid f : C \rightarrow B\}$. Introduced in **Standard numeric classes** (p. 60), this notation is used to avoid confusion with B^C, which means the elements of C distributing powers over the elements of B, $\{b^c \mid b \in B \wedge c \in C\}$.

From these, we can derive the following:

- If $f : A \rightarrow B$ is surjective and maps n elements of A to one element of B, i.e. $\#f(a) = n$ for each $a \in A$, then $\#A = n\#B$.

- If $f : A \rightarrow B^{\times n}$ is bijective, then $(\# : f)\#A = \#B^n$.

  The notation $A^{\times n}$ means A extended to n dimensions: $(\underset{1}{A}, \underset{2}{A}, \ldots, \underset{n}{A})$. Introduced in **Multiple distribution and threads** (p. 59), this notation is used to avoid confusion with A^n, which means a power distributed over the elements of A, $\{a^n \mid a \in A\}$.

- If $f : A \rightarrow \mathbb{Z}_2^{\swarrow C}$ is bijective, then $(\# : f)\#A = 2^{\#C}$. This is the number of subclasses of C, as each subclass corresponds to a selection function that takes C to a class of two logical elements, true and false, which decides whether each element is in the subclass.

Comparing $\mathbb{Z}$ and $\mathbb{Q}$

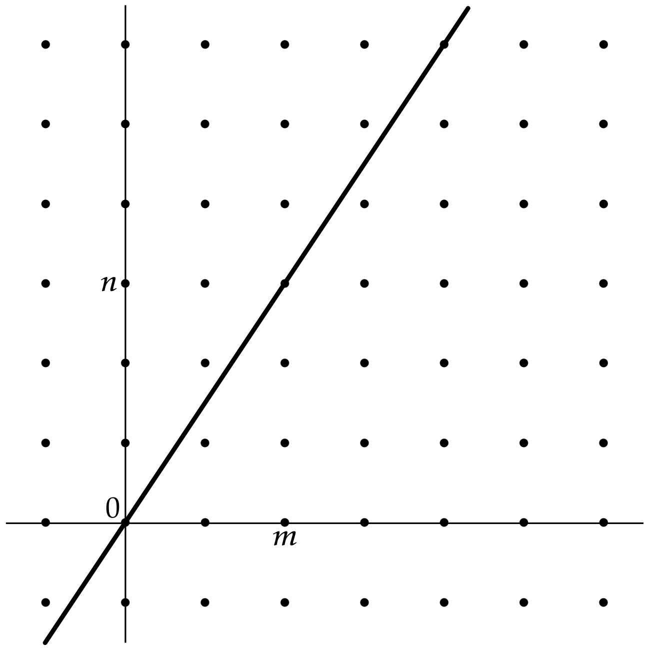

FIG. 96:
Map s from (m, n) to line of slope $\frac{m}{n}$

A class x is ***integrous*** if there is a bijection between the elements of x and some subset of the integers. Classes with a finite number of elements, $\mathbb{N}$, and $\mathbb{Z}$, are all integrous. In set theory, such classes are called ***countable*** or ***denumerable***, but from a numeristic point of view, these terms are misleading, since we can count and compare the number of elements of any class, including the nonintegrous classes $\mathbb{R}$ and $\mathbb{C}$.

Through a well known diagonal technique, it is possible to construct a bijection d^+ between $\mathbb{N}$ and $\mathbb{Q}^+$, and a similar bijection d between $\mathbb{Z}$ and $\mathbb{Q}$. This means that

$$(\# : d^+)\ \#\mathbb{N} = \mathbb{Q}^+$$

$$(\# : d)\ \#\mathbb{Z} = \mathbb{Q}$$

and that $\mathbb{Q}^+$ and $\mathbb{Q}$ are integrous. But d is not a very natural map, since among other things, it does not preserve order.

For a more natural map, we define $s : \mathbb{N}^{\times 2} \to \mathbb{Q}^+$. Figure 96 shows this map geometrically. s takes $(m, n) \in \mathbb{N}^{\times 2}$ to the line through (m, n) and the origin, which has slope $\dfrac{n}{m} \in \mathbb{Q}^+$. This is a many-to-one map with duplicates whenever m and n are not coprime (relatively prime).

Cesàro and others (see [H75, thm. 332, p. 269]) have shown that the probability of two random integers being coprime is $\dfrac{1}{\zeta(2)} = \dfrac{6}{\pi^2}$, where ζ is the Riemann zeta function. Therefore

$$\begin{aligned}
(\# : s)\ \frac{\#\mathbb{Q}^+}{\#\mathbb{N}^{\times 2}} &= \frac{6}{\pi^2} \\
(\# : s_2)\ \frac{\#\mathbb{Q}}{\#\mathbb{N}^{\times 2}} &= \frac{3}{\pi^2} \\
&\text{where } s_2 : \mathbb{N}^{\times 2} \to \mathbb{Q} \\
(\# : s_3)\ \frac{\#\mathbb{Q}}{\#\mathbb{Z}^{\times 2}} &= \frac{3}{2\pi^2} \\
&\text{where } s_3 : \mathbb{Z}^{\times 2} \to \mathbb{Q}.
\end{aligned}$$

Comparing $\mathbb{N}$ and $\mathbb{R}$

We first map $\mathbb{N}$ to the half-open unit interval $I = [0, 1)$ through base two radix representations (base two decimals). The expansion of $r \in I$ consists of a radix (decimal) point followed by an infinite string of binary digits. Such a string can be considered a map $k : \mathbb{N} \to \mathbb{Z}_2$. We define a class K of all possible such k, and then we define a map $j : K \to I, k \mapsto r$.

K has $2^{\#\mathbb{N}}$ elements, each of which maps to a unique r, except for duplicates of the form $0.(digits)0111\ldots = 0.(digits)1000\ldots$, which appear at $\#\mathbb{N}$ unique positions. Hence

$$(\# : j)\ \#I = 2^{\#\mathbb{N}} - \#\mathbb{N}.$$

In order to cover the real line, we make $\#\mathbb{Z}$ copies of I, using the map $y : \mathbb{R} \to I, r \mapsto r - \lfloor r \rfloor$. Hence

$$\begin{aligned}
(\# : k, y)\ \#\mathbb{R} &= \#\mathbb{Z}\#I \\
(\# : k, j, y)\ \#\mathbb{R} &= \#\mathbb{Z}\left[2^{\#\mathbb{N}} - \#\mathbb{N}\right].
\end{aligned}$$

Letting $\infty' \equiv \#\mathbb{N}$, we have

$$(\# : k, j, y)\ \sqrt[\#\mathbb{N}]{\#\mathbb{R}} = \sqrt[\#\mathbb{N}]{\#\mathbb{Z}}\ \sqrt[\#\mathbb{N}]{2^{\#\mathbb{N}} - \#\mathbb{N}}$$

$$
\begin{aligned}
&= \left(2\infty'\right)^{\frac{1}{\infty'}} \left(2^{\infty'} - \infty'\right)^{\frac{1}{\infty'}} \\
&= e^{\frac{\ln 2\infty'}{\infty'}} e^{\frac{\ln 2^{\infty'} - \infty'}{\infty'}} = e^{\frac{2}{2\infty'}} e^{\frac{(\ln 2)2^{\infty'} - 1}{2^{\infty'} - \infty'}} \\
&= e^{0'} e^{\frac{(\ln 2)^2 2^{\infty'}}{(\ln 2)2^{\infty'} - 1}} = e^{\frac{(\ln 2)^3 2^{\infty'}}{(\ln 2)^2 2^{\infty'}}} =' 2.
\end{aligned}
$$

A similar result holds for any radix. For a general j_b which uses a radix b instead of 2, we obtain

$$(\# : k, j_b, y) \quad \sqrt[\#\mathbb{N}]{\#\mathbb{R}} = b.$$

Comparing $\mathbb{R}$ and $\mathbb{C}$

If we define $p : \mathbb{C} \longrightarrow \mathbb{R}^{\times 2}, a + bi \mapsto (a, b)$, then

$$(\# : p) \ \frac{\#\mathbb{C}}{\#\mathbb{R}^{\times 2}} = 1.$$

Using class counts in derivatives and integrals

We now use class counts and other equipoint arguments to calculate the derivative and integral of the ***indicator*** or ***characteristic function of the rational numbers***:

$$[\mathbb{Q}](x) \equiv \begin{cases} 1 & \text{for } x \in \mathbb{Q} \\ 0 & \text{for } x \notin \mathbb{Q}. \end{cases}$$

First we calculate $\frac{\#\mathbb{Q}}{\#\mathbb{R}}$, using the maps s, k, j, and y from the previous section:

$$
\begin{aligned}
(\# : s, k, j, y) \ \frac{\#\mathbb{Q}}{\#\mathbb{R}} &= \frac{\frac{3}{\pi^2}\#\mathbb{N}^{\times 2}}{2^{\#\mathbb{N}}} \\
&= \frac{\frac{3}{\pi^2}\#\mathbb{N}^{\times 2}}{1 + \#\mathbb{N}\ln 2 + \frac{(\#\mathbb{N}\ln 2)^2}{2!} + \frac{(\#\mathbb{N}\ln 2)^3}{3!} + \frac{(\#\mathbb{N}\ln 2)^4}{4!} + \dots} \\
&= \frac{\frac{3}{\pi^2}}{\frac{1}{\#\mathbb{N}^{\times 2}} + \frac{\ln 2}{\#\mathbb{N}} + \frac{(\ln 2)^2}{2!} + \frac{\#\mathbb{N}(\ln 2)^3}{3!} + \frac{\#\mathbb{N}^{\times 2}(\ln 2)^4}{4!} + \dots} \\
&= \frac{\frac{3}{\pi^2}}{0'^2 + 0'\ln 2 + \frac{(\ln 2)^2}{2!} + \frac{\infty'(\ln 2)^3}{3!} + \frac{\infty'^2(\ln 2)^4}{4!} + \dots} =' 0
\end{aligned}
$$

Next we investigate the continuity of $[\mathbb{Q}](x)$ using the definition above, namely that $[\mathbb{Q}](x)$ is continuous at x when $[\mathbb{Q}](x+0') =' [\mathbb{Q}](x)$.

To compute $[\mathbb{Q}](x+0')$, pick an unfolded integer ∞' and let $0' \equiv 10^{-\infty'}$. The ∞'-th digit of the decimal representation of x is the origin of the space unfolded with $0'$. Call this digit d. The unit in this place has the value $0'$.

If x is rational, the decimal preserves the repetend of x, even in the ∞'-th place. Adding the unit $0'$ to x changes d to $d+1$ for $d < 9$, and 9 to 0 with a finite number of carries, with one exception noted later. With this change of at least one digit, the repetend is broken, and the number is no longer rational. Hence $[\mathbb{Q}](x+0') = 0 \neq [\mathbb{Q}](x)$, and $[\mathbb{Q}](x)$ is discontinuous at x.

The exception to the above process occurs when the repetend is 9, in which case there are an infinite number of carries. The 9s to the left of the ∞'-th place change to 0, but the digits to the right of this place remain 9. In this case also, the repetend is broken, and $[\mathbb{Q}](x)$ is discontinuous at x.

If x is irrational, the same thing occurs, except that there is never a repetend of 9, because there is never any repetend. So there are at most a finite number of carries, the folded digits are never affected, and $[\mathbb{Q}](x+0')$ is also irrational at the folded level. Hence $[\mathbb{Q}](x+0') = 0 = [\mathbb{Q}](x)$, and $[\mathbb{Q}](x)$ is continuous at x.

This differs from the conclusion of conventional analysis, which says that the function is discontinuous everywhere because the limit $\lim_{x\to a}[\mathbb{Q}](x)$ does not exist at any point. In equipoint, while there are an infinite number of discontinuities in any finite interval, the function is continuous at most points, since

$$(\# : s,k,j,y)\ \frac{\#\mathbb{Q}}{\#\mathbb{R}} = 0$$

$$(\# : s,k,j,y)\ \frac{\#(\mathbb{R}\setminus\mathbb{Q})}{\#\mathbb{R}} = 1.$$

We can now compute the derivative of $[\mathbb{Q}](x)$:

$$\begin{aligned}(\# : s,k,j,y)\ \frac{{}^{0'}d[\mathbb{Q}](x)}{{}^{0'}dx} &= \begin{cases}\delta'_{x+0',x} & \text{for } x\in\mathbb{Q}\\ 0 & \text{for } x\notin\mathbb{Q}.\end{cases}\\ &= \sum_{x\in\mathbb{Q}}\delta'_{\frac{1}{0'^2}}(x).\end{aligned}$$

$\int_0^a [\mathbb{Q}](x)\,dx$ is simply the ratio $\frac{\#\mathbb{Q}}{\#\mathbb{R}}$, which we have already seen is 0.

STRUCTURE OF UNFOLDED REAL NUMBERS

Decimal expansions

Folded real numbers

A folded real number r, identified with a decimal expansion, has the form

$$r = \sum_{m=-\infty}^{+\infty} d_m 10^m, \text{where } d_m \in \mathbb{Z}_{10} = 0, 1, \ldots, 9.$$

For example, for

$$\sqrt{2} = 1.414\ldots$$

we have

$$\begin{aligned} d_n &= 0 \text{ for } n > 0. \\ d_0 &= 1 \\ d_{-1} &= 4 \\ d_{-2} &= 1 \\ d_{-3} &= 4 \\ &\ldots \end{aligned}$$

Conventional analysis allows only a finite number of nonzero digits to the left of the decimal point, i.e. it requires that there be an n such that $d_m = 0$ for all $m > n$. But here we allow an infinite number of such digits and call such a digital representation an ***infinite left decimal***. An example of a repeating infinite left decimal is $\ldots 333 = \overline{3}$, which means an infinite number of 3's to the left of the decimal point, just as a repeating right decimal $0.333\ldots = 0.\overline{3}$ means an infinite number of 3's to the right.

Both left and right infinite decimals lead to duplicates, but we do not need uniqueness for this discussion, and allowing both makes the discussion easier.

$\overline{9} = \ldots 999$ and $0.\overline{9} = 0.999\ldots$ are examples of such duplicates, as we will now see. We start by recognizing them as infinite geometric series:

$$0.\overline{9} = 0.999\ldots = \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \ldots$$
$$\overline{9} = \ldots 999 = 9 + 90 + 900 + \ldots.$$

SUM OF AN INFINITE GEOMETRIC SERIES:

$$\boldsymbol{\sum_{k=n}^{\infty} a^k = a^n + a^{n+1} + a^{n+2} + \ldots = \frac{a^n}{1-a}.}$$

PROOF. Let

$$x \equiv \sum_{k=n}^{\infty} a^k = a^n + a^{n+1} + a^{n+2} + \ldots$$

then

$$ax = \sum_{k=n+1}^{\infty} a^k = a^{n+1} + a^{n+2} + a^{n+3} + \ldots$$
$$x - ax = a^n$$
$$x = \frac{a^n}{1-a} \quad \blacksquare$$

Unlike conventional theories of convergent and divergent series, the numeristic theory of infinite series, ***equipoint summation***, developed in the third part of this book, **Divergent Series** (p. 301–409), allows the use of ordinary commutative, associative, and distributive properties of addition and multiplication of both convergent and divergent infinite series without any known inconsistencies. Equipoint summation allows the above result for both the convergent ($|a| < 1$) and divergent ($|a| \geq 1$) cases.

Using this result for the above examples, we have:

$$0.\overline{9} = 0.999\ldots = 9\sum_{k=1}^{\infty} 10^{-k} = \frac{\frac{9}{10}}{1-\frac{1}{10}} = 1$$
$$\overline{9} = \ldots 999 = 9\sum_{k=0}^{\infty} 10^{k} = \frac{9}{1-10} = -1$$
$$0.\overline{3} = 0.333\ldots = 3\sum_{k=1}^{\infty} 10^{-k} = \frac{\frac{3}{10}}{1-\frac{1}{10}} = \frac{1}{3}$$

$$\overline{3} = \ldots 333 = 3\sum_{k=0}^{\infty} 10^k = \frac{3}{1-10} = -\frac{1}{3}.$$

The numeristic theory of repeating decimals, including infinite left decimals, is developed in detail in **Repeating Decimals** (p. 409–457).

Unfolded real numbers

In general, an unfolded real number is the sum of one or more of: (1) an infinite number, (2) a perfinite number, and (3) an infinitesimal number.

In the first unfolding, an arbitrary unfolded real number r takes this form:

$$\begin{aligned} r &= r_{+1}\,\infty' + r_0 + r_{-1}\,0' \\ &= \sum_{m=-\infty''}^{+\infty''-1} d_{+1,m}10^m\,\infty' + \sum_{m=-\infty''}^{+\infty''-1} d_{0,m}10^m + \sum_{m=-\infty''}^{+\infty''-1} d_{-1,m}10^m\,0', \end{aligned}$$

where $\infty' \equiv \frac{1}{0'}$ and r_{+1}, r_0, r_{-1} are folded real numbers, and we choose ∞'' so that this becomes a single sequence from smallest infinitesimal to largest infinite:

$$10^{-\infty''}\,\infty' = 10^{+\infty''} \quad \text{and} \quad 10^{-\infty''} = 10^{+\infty''}\,0' = 10^{+\infty''}\,\infty'^{-1}$$

$$\begin{aligned} \infty' &= 10^{2\infty''} \\ \infty'' &= \frac{1}{2}\log_{10}\infty' = -\frac{1}{2}\log_{10}0'. \end{aligned}$$

Then

$$\begin{aligned} r &= \sum_{m=-\infty''}^{+\infty''-1} d_{+1,m}10^{2\infty''+m} + d_{0,m}10^m + d_{-1,m}10^{-2\infty''+m} \\ &= \sum_{m=-3\infty''}^{3\infty''-1} d_m 10^m \end{aligned}$$

$$\text{where } d_m = \begin{cases} d_{+1,m-2\infty''} & \text{for } +\infty'' \le m \le +3\infty'' - 1 \\ d_{0,m} & \text{for } -\infty'' \le m \le +\infty'' - 1 \\ d_{-1,m+2\infty''} & \text{for } -3\infty'' \le m \le -\infty'' - 1 \end{cases}$$

$$\text{or } d_m = d_{k,m-2k\infty''} \text{ for } (2k-1)\infty'' \le m \le (2k+1)\infty'' - 1 \text{ and } k = +1, 0, -1.$$

In the second unfolding,

$$\begin{aligned} r &= r_{+2}\,\infty'^2 + r_{+1}\,\infty' + r_0 + r_{-1}\,0' + r_{-2}\,0'^2 \\ &= \sum_{m=-\infty''}^{+\infty''-1} d_{+2,m}10^m\,\infty'^2 + d_{+1,m}10^m\,\infty' + d_{0,m}10^m + d_{-1,m}10^m\,0' + d_{-2,m}10^m\,0'^2 \end{aligned}$$

$$= \sum_{m=-\infty''}^{+\infty''-1} d_{+2,m}10^{4\infty''+m} + d_{+1,m}10^{2\infty''+m} + d_{0,m}10^{m} + d_{-1,m}10^{-2\infty''+m} + d_{-2,m}10^{-4\infty''+m}$$

$$= \sum_{m=-\infty''}^{+\infty''-1} \sum_{k=-2}^{+2} d_{k,m}10^{2k\infty''+m}$$

$$= \sum_{m=-5\infty''}^{+5\infty''-1} d_m 10^m.$$

In the n-th unfolding,

$$\begin{aligned} r &= \sum_{m=-\infty''}^{+\infty''-1} \sum_{k=-n}^{+n} d_{k,m}10^{2k\infty''+m} \\ &= \sum_{m=-(2n+1)\infty''}^{+(2n+1)\infty''-1} d_m 10^m. \end{aligned}$$

Unfolded successor operation

The folded real numbers $\mathbb{R}$ are defined from the rational numbers $\mathbb{Q}$, which in turn are defined from the integers $\mathbb{Z}$, and these are defined from the natural numbers $\mathbb{N}$. The natural numbers are often defined with the ***successor function*** $S(n) \equiv n+1$. The whole structure of the real numbers can thus be built on the foundation of the successor function.

The unfolded real numbers can be built on nearly the same foundation, namely an unfolded successor function that distinguishes n from $n+1$ for both finite and infinite n. This leads to the unfolded infinite numbers, which then lead to the unfolded infinitesimals, and then to the first unfolding. As we have seen above, all unfoldings follow from the first.

Limitation of microscopes

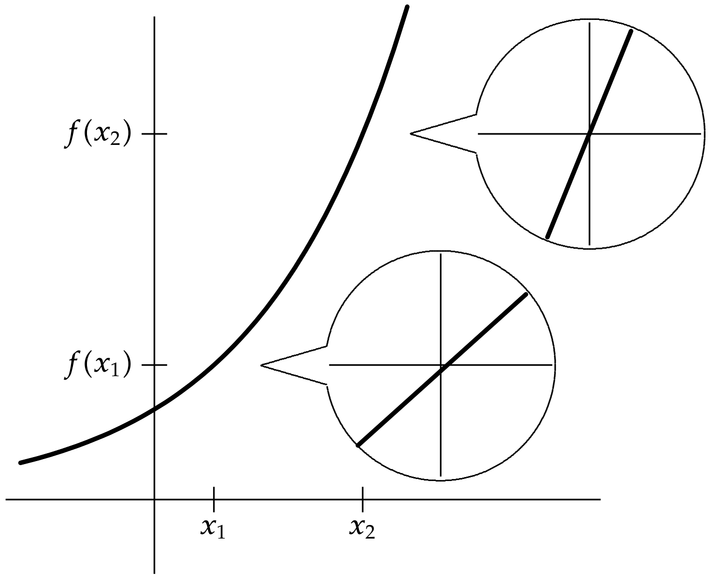

FIG. 97:
Microscopes of unfolded space
at two points in the graph of $f(x)$

Figure 97 shows typical microscope views of unfolded space within two points in the graph of $f(x)$. We have used this type of microscope to **calculate the first derivative** (p. 163). As long as we are not looking at some kind of singularity, a function $f(x)$ appears curved outside the microscope but straight within it. We used the fact of straighness within an unfolded space to draw an analogy to the slope of a straight line in folded space and thus calculate the derivative.

The straightness of this line can be misleading. In folded space, the derivative of a straight line is constant and the second derivative is zero. This could lead us to think that the second derivative of a curve such as the one in Figure 97 is constant and the third derivative is zero. In fact, the curve deviates infinitesimally from a straight line, but this is not visible at infinite magnification. By the time the curve moves from x_1 to x_2, for instance, this infinitesmial deviation adds up to a clearly finitesimal change in slope.

CONVERGENT SERIES PARADOX

In equipoint analysis, infinite series and definite integrals are both simple sums with an infinite number of terms. In a definite integral, all the terms are zero, while in an infinte series, an infinite number of terms are nonzero. When the terms can be directly compared, this may lead to a paradoxical condition wherein both a series and an integral yield a finite result. For example, consider that

$$\sum_{n=1}^{\infty} 2^{-n} = 1 < \int_0^1 2\,dx = \sum_{n=1}^{\infty'} \frac{2}{\infty'} = 2,$$

even though, if we look at individual terms,

$$2^{-n} \geq \frac{2}{\infty}$$

for all n, with equality holding only for infinite n.

In the numerisitic theory of infinite series, we find that convergent series such as the one above actually have *two* values, one finite and one infinite. The infinite value arises when we consider the series to be an infinite sum of strictly positive values, and the finite value comes from the identification of $+\infty$ and $-\infty$ in the projectively extended real numbers. This is explained in detail in the third and fourth parts of this book, **Divergent Series** (p. 301–409) and **Repeating Decimals** (p. 409–457).

QUANTUM RENORMALIZATION

Renormalization is a procedure used in quantum physics to "tame" infinities that occur in quantum formulas. The correctness of values derived through renormalization is well verified experimentally, but the mathematics of this procedure is poorly understood, and therefore its theoretical validity is controversial.

Equipoint analysis may be able to improve the understanding of quantum renormalization. The following example may show this. Although realistic examples of quantum renormalization usually involve very difficult formulas, we use here a very simplified example given by Klauber [K14, p. 305]. The problem is to evaluate

$$\int_{-\infty}^{\infty} x^2\,dx.$$

In conventional analysis, this evaluation is done in two main steps, regularization and renormalization. In this example, the regularized form is

$$\lim_{\Lambda\to\infty} \int_{-\Lambda}^{\Lambda} x^2\,dx.$$

This still diverges, so we renormalize by multiplying by the factor $\frac{1}{\Lambda^3}$. Then we have

$$\lim_{\Lambda\to\infty} \frac{1}{\Lambda^3} \int_{-\Lambda}^{\Lambda} x^2\,dx = \frac{2}{3}.$$

In equipoint analysis, we avoid limits and directly write

$$\frac{1}{\infty'^3} \int_{-\infty'}^{\infty'} x^2\,dx = \frac{2}{3}.$$

We can also separate the factor $\frac{1}{\infty'^3}$ from the integral and evaluate these expressions separately as infinite quantities.

See also [T99, ch. 18] and [D04] for elementary introductions to quantum renormalization.

APPENDIX: COMPARISON OF EQUIPOINT WITH OTHER THEORIES OF ANALYSIS

Comparison of equipoint and conventional analysis

Differences

The differences between equipoint analysis and conventional limit-based analysis have been observed throughout this part. Here we summarize these differences.

Conventional analysis	Equipoint analysis
• There are no infinite values.	• There are one or more infinite numeric values at the folded level.
• Every expression is single valued.	• An expression can represent a single value or a multivalued numeristic class.
• As a result of the previous two points, some operations are left undefined or regarded as meaningless.	• As a result of the previous two points, all operations are defined. An expression which is syntactically correct has a value and is never regarded as meaningless.
• Infinite and infinitesimal quantities are handled indirectly through limits.	• Infinite and infinitesimal quantities are handled directly through an extended multiple-level number system.
• Many proofs of simple results are difficult.	• Many proofs are short and easy.

- The Leibnitz derivative and Riemann integral cannot be used in some cases, giving rise to the need for constructs such as the Lebesgue integral.

- The Leibnitz derivative and Riemann integral suffice for any real or complex function.

Examples

Below are definitions and examples of the derivative and definite integral in coventional limit-based analysis. For equipoint equivalents, see **Definition of derivative** (p. 163) and **Definition of definite integral** (p. 165) above.

Conventional definition of derivative:

$$\frac{df(x)}{dx} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

Sample application of this definition:

$$\begin{aligned}\frac{dx^2}{dx} &= \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} \\ &= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} \\ &= \lim_{h\to 0} \frac{2xh + h^2}{h} \\ &= \lim_{h\to 0} (2x+h) \\ &= 2x.\end{aligned}$$

Conventional definition of definite integral:

$$\int_a^b f(x)\,dx = \lim_{N\to\infty} \sum_{k=1}^{N} f\left(a + \frac{k(b-a)}{N}\right) \frac{b-a}{N}.$$

Sample application of this definition:

$$\begin{aligned}\int_0^u 2x\,dx &= \lim_{N\to\infty} \sum_{k=1}^{N} 2\frac{ku}{N}\frac{u}{N} \\ &= \lim_{N\to\infty} \frac{2u^2}{N^2} \sum_{k=1}^{N} k \\ &= \lim_{N\to\infty} \frac{2u^2}{N^2} N \frac{N+1}{2}\end{aligned}$$

$$
\begin{aligned}
&= \lim_{N\to\infty} u^2\left(1+\frac{1}{N}\right) \\
&= u^2.
\end{aligned}
$$

Comparison of equipoint and nonstandard analysis

Similarities

Equipoint analysis has much in common with nonstandard analysis and has borrowed many of its concepts.

Nonstandard analysis

- Two levels of numbers: standard and nonstandard.
- Nonstandard values around a standard point.
- Nonstandard infinite values which are reciprocals of nonstandard infinitesimal values.
- An infinite or infinitesimal value is a numeric constant, often denoted H or ε respectively.
- Microscope diagram of nonstandard values.
- Approximate equality relation $\approx$.

Equipoint analysis

- Two levels of numbers: folded and unfolded.
- Unfolded values within a folded point.
- Unfolded infinite values which are reciprocals of unfolded infinitesimal values.
- An infinite or infinitesimal value is a numeric constant, often denoted ∞' or $0'$ respectively.
- Microscope diagram of unfolded values.
- Folded equality relation $='$.

Differences

Nonstandard analysis	Equipoint analysis
• Set theoretic foundation.	• Number based foundation.
• Every expression is single valued.	• An expression can represent a single value or a multivalued numeristic class.
• Nonstandard values all at the same level.	• Unfolded values form an infinite number of levels.
• There are no standard infinite values.	• There are one or more folded infinite numeric values.
• Division by zero is not allowed.	• One or more folded infinite values and multivalued expressions allow division by zero.
• Some operations are left undefined or regarded as meaningless.	• All operations are defined.
• The standard part of infinite values is undefined.	• Unfolded infinite values are folded into an infinite value.
• Two infinite values are approximately equal only if their difference is infinitesimal. Thus for an infinite H, $H \not\approx H+1$.	• Two infinite values are primed equal if they fold to the same infinite element. Thus, for an infinite a, $a =' a+1$.
• The function $\frac{1}{x}$ is not continuous at 0.	• The function $\frac{1}{x}$ is continuous at 0 on at least one side.
• The function $2x$ is not continuous at infinite values.	• The function $2x$ is continuous at infinite values.

For source material on nonstandard analysis, see [R74], [KE] and [KF].

Examples

Below are definitions and examples of the derivative and definite integral in nonstandard analysis. For equipoint equivalents, see **Definition of derivative** (p. 163) and **Definition of definite integral** (p. 165) above.

Nonstandard definition of derivative:

$$\frac{df(x)}{dx} = \mathrm{st}\left(\frac{f(x+\varepsilon)-f(x)}{\varepsilon}\right),$$

where ε is an infinitesimal, which in nonstandard analysis is nonzero but smaller than all nonzero reals, and st() is the standard part function, which maps a number of the form $a+\varepsilon$ to a, where a is real.

Sample application of this definition:

$$\begin{aligned}
\frac{dx^2}{dx} &= \mathrm{st}\left(\frac{(x+\varepsilon)^2-x^2}{\varepsilon}\right)\\
&= \mathrm{st}\left(\frac{x^2+2x\varepsilon+\varepsilon^2-x^2}{\varepsilon}\right)\\
&= \mathrm{st}\left(\frac{2x\varepsilon+\varepsilon^2}{\varepsilon}\right)\\
&= \mathrm{st}(2x+\varepsilon)\\
&= 2x.
\end{aligned}$$

Nonstandard definition of definite integral:

$$\int_a^b f(x)\,dx = \mathrm{st}\left(\sum_{k=1}^{H} f\left(a+\frac{k(b-a)}{H}\right)\frac{b-a}{H}\right),$$

where H is an infinite number, the reciprocal of an infinitesimal.

Sample application of this definition:

$$\begin{aligned}
\int_0^u 2x\,dx &= \mathrm{st}\left(\sum_{k=1}^{H} 2\frac{ku}{H}\frac{u}{H}\right)\\
&= \mathrm{st}\left(\frac{2u^2}{H^2}\sum_{k=1}^{H} k\right)\\
&= \mathrm{st}\left(\frac{2u^2}{H^2}H\frac{H+1}{2}\right)
\end{aligned}$$

$$\begin{aligned}
&= \mathrm{st}\left(u^2\left(1+\frac{1}{H}\right)\right)\\
&= \mathrm{st}\left(u^2(1+\varepsilon)\right)\\
&= u^2.
\end{aligned}$$

Stroyan's uniform derivative

Stroyan's system of analysis is a variant of nonstandard analysis. One of its main features is the ***uniform derivative***, which is defined over an interval and contrasts with the usual nonstandard derivative, which Stroyan calls the ***pointwise derivative***.

Stroyan gives several equivalent definitions of the uniform derivative, the first of which is the following. A real function $f(x)$ has a derivative $f'(x)$ on the interval (a,b) iff for every hyperreal x such that $a < x < b$, $x \not\approx a$, $x \not\approx b$, and $\delta x \approx 0$,

$$f(x+\delta x) - f(x) = f'(x)\delta x + \varepsilon \cdot \delta x$$

for some $\varepsilon \approx 0$ [S97, p. 54].

Comparison of equipoint and varipoint analysis

Varipoint analysis is an alternative to equipoint analysis conceived by the author. It starts with numeristics but does unfolding differently. In varipoint analysis, unfolding adds values *around* a point instead of ***within*** the point and thus avoids expanding a point into a space.

In varipoint analysis, all numbers, except folded infinities, are points that do not expand. In this way, it is similar to nonstandard analysis. For instance, instead of $0' \subset 0$, we have $0' \neq 0$, with the two values both being points that are separated by an infinitesimal distance.

In varipoint analysis, instead of defining $='$ and other primed relations with inclusion, we must use a function similar to the st() function in nonstandard analysis, which in varipoint we call fold().

$$\mathrm{fold}(a) \equiv \begin{cases} b \mid b \text{ is folded and } a-b \text{ is infinitesimal} & \text{for finite } a \\ b \mid b \text{ is folded and } a-b \text{ is finite} & \text{for infinite } a \end{cases}$$

Then we can define

$$a =' b \quad \Leftrightarrow \quad \text{fold}(a) = \text{fold}(b).$$

A significant difference between varipoint and nonstandard analysis is that varipoint analysis inherits folded infinite values and classes from numeristics, so division by zero and fold() of infinite numbers are both defined.

Comparison of equipoint and Fermat's adequality

Fermat's adequality is one of several 17th century antecedents to calculus. Other systems were developed by Cavalieri and Wallis, but Fermat's was the first known general method for determining extrema.

Fermat described his method in a manuscript written about 1636 and published in 1679 [Fe79]. Both Newton and Leibnitz acknowledged Fermat's adequality as an antecedent of their own work.

Fermat said that adequality derived from a technique used by the Greek mathematician Diophantus, who called it παρισότης *parisotēs*, but Diophantus used this word only to mean approximation.

Similarities

Fermat adequality	Equipoint analysis
• Uses the adequality relation to convert a finite difference into an instantaneous difference.	• Uses the equality relation to convert an unfolded quantity into a folded quantity.
• Uses a term e which the adequality relation discards when added to another quantity.	• Uses an unfolded element $0'$ which the equality relation discards when added to another quantity.

Differences

Fermat adequality	Equipoint analysis
• Used only for computing maxima and minima.	• Used for a wide variety of applications.
• Suitable only for polynomial functions.	• Suitable for any type of function.
• Does not define a derivative or integral but uses a procedure which is algebraically equivalent to computing a derivative and setting it to zero.	• Defines a derivative and an integral which are used in many different ways.
• Adquality is stated in formal terms with little justification. The nature of adequality and the reason it discards the special term e are not made clear.	• Unfolding and multilevel equality are defined and explained in both formal and informal terms which make its nature and application clear.

Outline and example

Fermat's method for finding a maximum or minimum of $f(x)$:

1. Ad-equate $f(x)$ and $f(x+e)$.
2. Simplify as you would with equality, including dividing both sides by e, until there is at least one term that does not contain e.
3. Discard terms containing e.
4. Convert the adequation to an equation.

Fermat gave the following example to find the maximum of $bx - x^2$. The adequality relation is denoted $\backsim$.

$$
\begin{aligned}
bx - x^2 &\backsim b(x+e) - (x+e)^2 = bx + be - x^2 - 2xe + e^2 \\
0 &\backsim be - 2xe + e^2 \\
be &\backsim 2xe - e^2 \\
b &\backsim 2x - e \\
b &= 2x.
\end{aligned}
$$

Comparison of equipoint and ultrasmall/relative analysis

Similarities

Relative analysis	Equipoint analysis
• Multiple layers of numbers.	• Multiple levels of unfolding.
• Ultrasmall and ultralarge numbers.	• Unfolded infinitesimal and infinite numbers.
• Equality is relative to a given level, a proper class often denoted V, and the equality denoted $\simeq_V$.	• Equality is relative to an unfolding unit, an infinitesimal often denoted $0'$, and the equality denoted $='$.
• Variables "appear" at certain levels.	• Values become distinguishable at certain levels.

Differences

Relative analysis	Equipoint analysis
• Set theoretic foundation, to which is added a new set theoretic relation which is used to build set theoretic proper classes of numbers, which are formed into layers.	• Number based foundation, to which is added the principle of unfolding a point into layers of unfolding.

- Every expression is single valued.
- There are no real infinite values.
- Division by zero is not allowed.
- Some operations are left undefined or regarded as meaningless.

- An expression can represent a single value or a multivalued numeristic class.
- There are one or more infinite real values at the folded level.
- One or more folded infinite values and multivalued expressions allow division by zero.
- All operations are defined.

For source material on relative analysis, see [H10], [OD09], and [OD11].

Examples

Relative analysis definition of derivative:

$$f'(x) \equiv n\left(\frac{f(x+h)-f(x)}{h}\right),$$

where $n(x)$ or $n_V(x)$ denotes the ***neighbor*** of x, the unique real number ultraclose to x at level V.

Relative analysis definition of definite integral:

$$\int_a^b f(x)\,dx \equiv n\left(\sum_{i=0}^{N-1} f(x_i)h\right),$$

for h ultrasmall and N ultralarge, $h \equiv \frac{b-a}{N}$ and $x_i \equiv a + ih$.

For equipoint equivalents, see **Equipoint definition of derivative** (p. 163) and **Equipoint definition of definite integral** (p. 165).

Comparison of equipoint and smooth infinitesimal analysis

Similarities

Smooth infinitesimal analysis	Equipoint analysis
• Simple algebra of infinitesmial operations.	• Simple algebra of infinitesmial operations.

Differences

Smooth infinitesimal analysis	Equipoint analysis
• Intuitionistic logic.	• Classical logic.
• A single level of equality.	• Multiple levels of equality.
• Every expression is single valued.	• An expression can represent a single value or a multivalued numeristic class.
• There are no infinite values.	• There are one or more infinite values at folded and unfolded levels.
• Division by zero or infinitesimals is not allowed.	• Division by zero and infinitesimals is allowed.
• Some operations are left undefined or regarded as meaningless.	• All operations are defined.
• Indirect definition of derivative and integral.	• Direct definition of derivative and integral.
• The square of an infinitesimal ε is 0.	• An infinitesimal $0'$ has an infinite number of powers which are distinguishable at various levels of unfolding.

For source material on smooth infinitesimal analysis, see [BI], [BP], and [La].

Smooth infinitesimal definitions

Smooth infinitesimal analysis has two postulates for infinitesimals:

$$(\exists! D)(\forall \varepsilon) f(\varepsilon) = f(0) + D$$
$$[(\forall \varepsilon)\varepsilon a = \varepsilon b] \Rightarrow a = b$$

Using intuitionistic logic, these postulates imply $\varepsilon \neq 0$ and $\neg\ (\varepsilon \neq 0)$, but the second is not equivalent to $\varepsilon = 0$, i.e. ε equality does not obey the law of excluded middle. These postulates also imply $\varepsilon^2 = 0$, i.e. infinitesimals are nilpotent.

This approach leads to formulae such as

$$f(x + \varepsilon) - f(x) = \varepsilon f'(x)$$

as the definition of derivative, which is only implicit and has to be proved to exist. Since there is no division by infinitesimals, we cannot use notation such as $\dfrac{dy}{dx}$ or $f'(x) = \dfrac{f(x+\varepsilon) - f(x)}{\varepsilon}$.

Integration is even less direct, being given not by a formula but by an ***Integration Principle***: Given a smooth function $f : [0, 1] \rightarrow \mathbb{R}$, there exists a unique smooth function $g : [0, 1] \rightarrow \mathbb{R}$, such that $g' = f$ and $g(0) = 0$.

The equipoint equivalents of these definitions are explicit formulas given in **Definition of derivative** (p. 163) and **Definition of definite integral** (p. 165).