EQUIPOINT ANALYSIS

A NUMERISTIC APPROACH TO CALCULUS

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Editions before the sixth used the term ***equinfinitesimal*** instead of ***equipoint***.

CONTENTS

SUMMARY

This monograph extends the concepts of numeristics to analysis. Numeristics is introduced in a separate monograph. Here a theory of analysis is developed, based on infinitesimals which are all exactly equal to zero, and infinite values that are their reciprocals.

Fundamental concepts derive from Maharishi Mahesh Yogi's Vedic Mathematics, Charles Musès's analysis of zero and infinity, and Abraham Robinson's non-standard analysis. This theory uses multiple levels of sensitivity to extend real and complex arithmetic and evaluate equality. It then defines derivatives and integrals solely in terms of elementary arithmetic operations in this extended arithmetic.

Topics include:

- Levels of sensitivity, including multilevel numbers, functions, and relations.
- The fundamental theorems of calculus.
- Chain rule, product rule, derivatives of trigonometric and exponential functions.
- Limit defined in terms of sensitivity levels, and continuity in terms of these limits.
- The natural logarithm developed as a polynomial in the extended arithmetic.
- Singularities: jump singularities, removable singularities, poles, essential singularities.
- Complex analysis: complex derivative, Cauchy integral formula, Taylor and Laurent series, complex poles, complex essential singularities.

An appendix compares equipoint analysis to other theories of analysis: conventional analysis, nonstandard analysis, relative analysis, and smooth infinitesimal analysis.

Non-standard analysis frequently simplifies substantially the proofs, not only of elementary theorems, but also of deep results. This is true, e.g., also for the proof of the existence of invariant subspaces for compact operators, disregarding the improvement of the result; and it is true in an even higher degree in other cases. This state of affairs should prevent a rather common misinterpretation of non-standard analysis, namely the idea that it is some kind of extravagance or fad of mathematical logicians. Nothing could be farther from the truth. Rather, there are good reasons to believe that non-standard analysis, in some version or other, will be the analysis of the future.—Kurt Gödel [G]

And there is every reason to believe that the codification of intuitive concepts and the reinterpretation of accepted principles will continue also in future and will bring new advances, into territory still uncharted.—Abraham Robinson [R68]

[Srinivasa Ramanujan] sometimes spoke of ‘zero’ as the symbol of the Absolute (Nirguna Brahman) of the extreme monistic school of Hindu Philosophy, that is, the reality to which no qualities can be attributed, which cannot be defined or described by words and is completely beyond the reach of the human mind; according to Ramanujan, the appropriate symbol was the number ‘zero’, which is the absolute negation of all attributes. He looked on the number ‘infinity’ as the totality of all possibilities which was capable of becoming manifest in reality and which was inexhaustible. According to Ramanujan, the product of infinity and zero would supply the whole set of finite numbers. Each act of creation, as far as I could understand, could be symbolized as a particular product of infinity and zero, and from each such product would emerge a particular individual of which the appropriate symbol was a particular finite number. . . . He spoke with such enthusiasm about the philosophical questions that sometimes I felt he would have been better pleased to have succeeded in establishing his philosophical theories than in supplying rigorous proofs of his mathematical conjectures.—P. C. Mahalanobis [Mn]

अणोरणीयान् महतो महीयान् आत्मास्य जन्तोर्निहितो गुहायाम् ॥
Aṇoraṇīyān mahato mahīyān ātmāsya jantornihito guhāyām.
The Self is smaller than the smallest, bigger than the biggest, and is hidden in a secret place of all creatures.—Katha Upanishad 2.20

यथा पिण्डे तथा ब्रह्माण्डे ॥
Yathā piṇḍe tathā brahmāṇḍe.
As is the point, so is the infinite.—Charaka Samhita

DEFINITION AND SCOPE

Equipoint analysis is an application of numeristic principles to analysis. Numeristics is introduced in [CN]; the present monograph is a sequel.

Broad conceptualizations for numeristics come from Maharishi Mahesh Yogi in his Vedic Mathematics. These sources are summarized in [CS] and include especially [MM]. Equipoint analysis extends these concepts into mathematical methods, using ideas from non-standard analysis and Charles A. Musès. Non-standard analysis is a modern theory of infinitesmals and is described in more detail below. Musès supplied some key insignts for refining the ideas of non-standard analysis in [Mu72]. The present monograph aims to be self-sufficient and not require familiarity with any of these sources.

The term *equipoint analysis* means that the infinitesimals it uses are exactly equal to zero. This concept (but not the term) is briefly introduced by Musès in [Mu65, p. 178–179] and [Mu72].

NON-CONVENTIONAL THEORIES OF ANALYSIS

The conventional theory of analysis, based on set theory and limits, was first developed in the 19th century. Since 1960, the the following theories of analysis have emerged as alternatives to classical analysis. Here we briefly describe the history of these theories. See the appendix for a more detailed description and comparison to equipoint analysis.

Non-Standard Analysis

Non-standard analysis has its roots in the original development of calculus in terms of infinitesimals by Leibnitz in the 17th century. In the intervening centuries, calculus was found to be very useful, but the explanation of it in terms of infinitesimals did not satisfy very many mathematicians. With the increasing demand for rigor in the 19th century, the theory of infinitesimals was replaced by classical limits-based analysis.

In 1960, Abraham Robinson resurrected the theory of infinitesimals by developing it as a modern set theoretic system he called ***non-standard analysis*** [R74]. Jerome Keisler used the principles of non-standard analysis in his elementary calculus textbook [KE] and undergraduate analysis text [KF]. Non-standard analysis is widely considered to be a significantly simpler and more elegant system than classical analysis, yet in the more than 50 years since its introduction, it has not achieved widespread use, either in teaching or in research.

Relative Analysis

More recently, ***relative analysis*** was developed by Karel Hrbacek, Oliver Lessman, and Richard O'Donovan [H10], and used by O'Donovan in high school instruction [OD09]. This theory uses the terms *ultrasmall* and *ultralarge*, whereas *infinitesimal* and *infinite* are used in non-standard analysis. Like non-standard analysis, relative analysis has not achieved widespread use.

Smooth Infinitesimal Analysis

Smooth infinitesimal analysis was developed by John L. Bell, in [BI] and [BP], as a branch of synthetic differetial geometry. It was originally developed by F. William Lawvere from category theory starting in 1967, but it remained obscure until Lawvere's 1998 article [L]. Like other alternatives, smooth infinitesimal analysis has not achieved widespread use.

ORIGIN OF EQUIPOINT ARITHMETIC

As explained in [CN], numeristics is based on the infinite and the experience of the silent, unmanifest point of infinity, *samādhi* or zero. In numeristics this is conceptualized to give an arithmetic of 0 and ∞, including unrestricted multiplication and division by these quanities, such as $\frac{1}{0}$, $\frac{0}{0}$, and $\infty + 1$.

Some of these unrestricted operations, including $\frac{0}{0}$, $\infty - \infty$, and $\infty \cdot 0$, give rise to indeterminate expressions. Numeristics gives a value to these expressions. This value is called the ***full class*** ($\not\infty$), a class which includes all numeric values.

In some cases, numeristic arithmetic yields an inderminate expression in response to a question which clearly has a determinate result. One example is the calculation of the slope of the tangent to a curve $y = f(x)$ at a point a. Numeristic arithmetic alone yields the result $\frac{f(x) - f(x)}{0} = \frac{0}{0} = \not\infty$. In such cases, numeristic arithmetic needs to be refined to yield a determinate result. This need is called the ***principle of determinacy***, and it is implemented through equipoint arithmetic.

In this and similar works, the principle of determinacy is used in the following:

- Derivatives.
- Integrals.
- Offset derivatives.
- Class count comparisons.
- Divergent series in [CD].
- Infinite left decimals in [CR].

Numeristics starts with the experience of infinity and zero as the point of infinity. Equipoint arithmetic extends with the experience of the point opening up into a vast inner space distinct from and much richer than ordinary space, and the contraction of this space back into ordinary space.

Equipoint analysis conceptualizes some aspects of the expanded space by considering it as a space of points which all have the ordinary single value of zero from the perspective of ordinary space, but which form a class of distinct zeros from the perspective of the expanded space.

This allows us to do calculus with infinitesimals that are exactly zero, and with numeric infinites that are reciprocals of these zeros. In the next chapter, we formalize the multiple perspectives as ***levels of sensitivity***.

SENSITIVITY

Unfolding zero

In equipoint analysis, every number has multiple ***levels of sensitivity***. At the default level of sensitivity, we distinguish real numbers, integers, rational numbers, etc. as elements. When, as described in the previous chapter, zero opens up into a space of zeros, these zeros are all exactly equal to zero at the default level of sensitivity, but at a greater level of sensitivity, zero is multivalued, and the individual zeros are distinct elements. This is shown pictorially below.

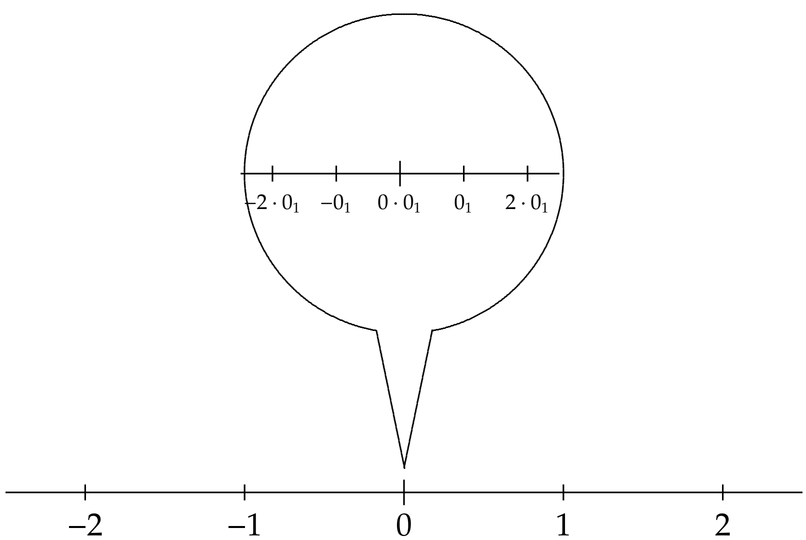

FIG. 1:
Real number line with
microscope view of unfolded 0

In Figure 1, we have the ordinary real number line with the real number 0 expanded into a space. When 0 is expanded into a space, we find multiple distinct zeros in that space. We call the ordinary number line the ***folded*** space and the expanded space the ***unfolded*** space around 0. The bubble showing the infinitely expanded space is called a ***microscope***, and the original graph is called a ***macroscope***.

The figure shows one of the unfolded zeros denoted as 0_1, and it also shows some multiples of 0_1. We could denote one of these other points, say $3 \cdot 0_1$, as 0_2. At the default level of sensitivity, $0_1 = 0_2 = 0$, but at the level of sensitivity of the unfolded space, $0_1 \neq 0_2$ and $0_1, 0_2 \in 0$.

In the unfolded space of 0, each of the individual values of 0 in that space has a well defined ratio with every other point in that space. For instance, if $0_2 = 3 \cdot 0_1$, then $\dfrac{0_2}{0_1} = 3$. This also means that the unfolded space is ordered analogously to real space, e.g. $0_2 > 0_1$ at the unfolded level of sensitivity.

Finite multiples of each value in the unfolded space are distinct, but squares and higher powers of any value in this space end up at the origin of the unfolded space: $0_1^2 = 0_1^n = 0 \cdot 0_1$ for any 0_1 in the space and any $n > 1$.

We characterize a level of sensitivity by a ***sensitivity unit***, a number which does not lie at the origin. The sensitivity unit of the folded space is 1, while the sensitivity unit of the unfolded space in Figure 1 is 0_1.

Since equality may depend on the sensitivity unit, we use the notation $\sim u$, usually as a subscript, to indicate a sensitivity unit u, and refer to it as ***sense u***. The default sense is 1: $a = b$ means $a =_{\sim 1} b$, and $a =_{\sim u} b$ is equivalent to $\dfrac{a}{u} = \dfrac{b}{u}$.

These unfolded zeros are the infinitesimals of this system of analysis. The name ***equipoint*** reflects the fact that these infinitesimals are all equal at sense 1.

To summarize, at the folded level, sense 1, there is only one 0:

$$0_1 = 0_2 = 0$$
$$0_1 =_{\sim 1} 0_2 =_{\sim 1} 0$$
$$0 = 0^2 = 2 \cdot 0$$
$$0_1 = 0_1^2 = 2 \cdot 0_1$$
$$0^2 = 0^3 = 2 \cdot 0^2$$
$$0_1^2 = 0_1^3 = 2 \cdot 0_1^2$$
$$0^3 = 0^4 = 2 \cdot 0^3$$
$$0_1^3 = 0_1^4 = 2 \cdot 0_1^3$$

At the unfolded level of sense 0_1, there are multiple distinct values of 0:

$$0_1 \neq_{\sim 0_1} 0_1^2$$

$$0_1 \neq_{\sim 0_1} 2 \cdot 0_1$$

$$0_1^2 =_{\sim 0_1} 0_1^3 =_{\sim 0_1} 2 \cdot 0_1^2$$

$$0_1^3 =_{\sim 0_1} 0_1^4 =_{\sim 0_1} 2 \cdot 0_1^3$$

Unfolding perfinite numbers

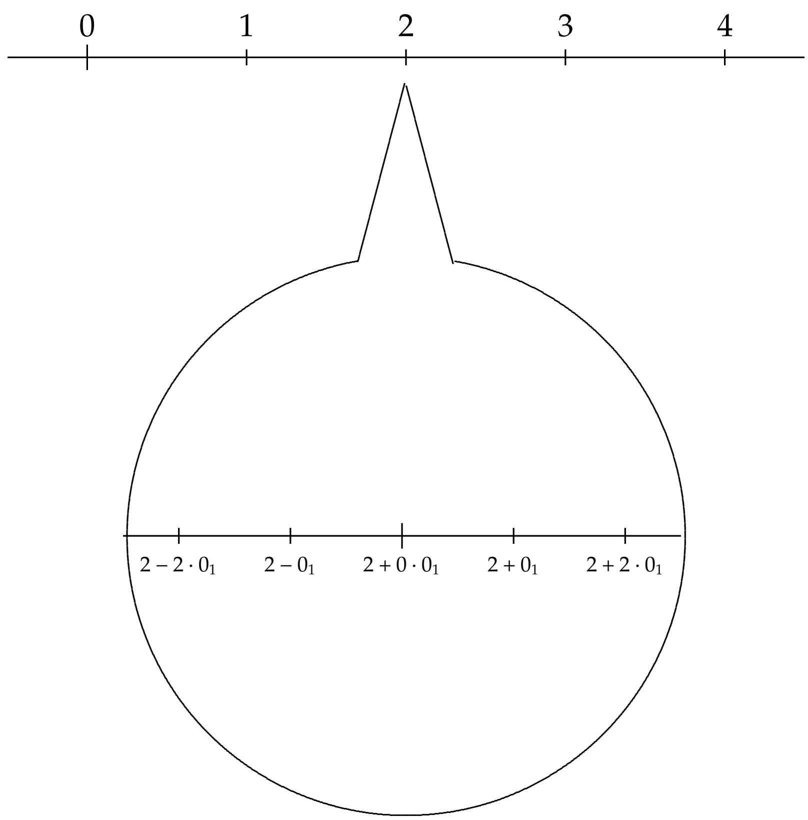

FIG. 2:
Real number line with
microscope view of unfolded +2

Since $a + 0 = a$ for any real a, every real number a can open up into a space consisting of a plus zeros. Figure 2 shows this for $a = 2$. Just as we locate 0_1 within 0 and use it to perform a sensitive arithmetic within 0, we can also locate a_1 within a and use it as the base of a sensitive arithmetic within a.

Following are examples of arithmetic within a finite real a.

$$
\begin{aligned}
a - a &=_{\sim 1} 0,\\
a + 0_1 &=_{\sim 1} a,\\
a_1 &=_{\sim 1} a,\\
a + 0_1 &\neq_{\sim 0_1} a,\\
a_1 - a_1 &=_{\sim 0_1} 0_1,\\
a_1 &\subset_{\sim 0_1} a,\\
a_1 &\in_{\sim 0_1} a.
\end{aligned}
$$

As with the equality relation, the interpretation of a number may also depend on a sensitivity unit. The ordinary, folded real number 2 we designate as $2_{\sim 1}$, while the origin of the unfolded 2, shown as $2 + 0 \cdot 0_1$ in Figure 2, we more properly designate as $2_{\sim 0_1}$.

As with equality, the default sensitivity unit of a number is 1: $2 + 1 = 3$ means $2_{\sim 1} + 1_{\sim 1} =_{\sim 1} 3_{\sim 1}$.

At sense 0_1, the folded 2 is a multivalued class consisting of the unfolded 2 plus the real multiples of 0_1; in symbols $2_{\sim 1} =_{\sim 0_1} 2_{\sim 0_1} + \mathbb{R} \cdot 0_1$.

Unfolding infinite numbers

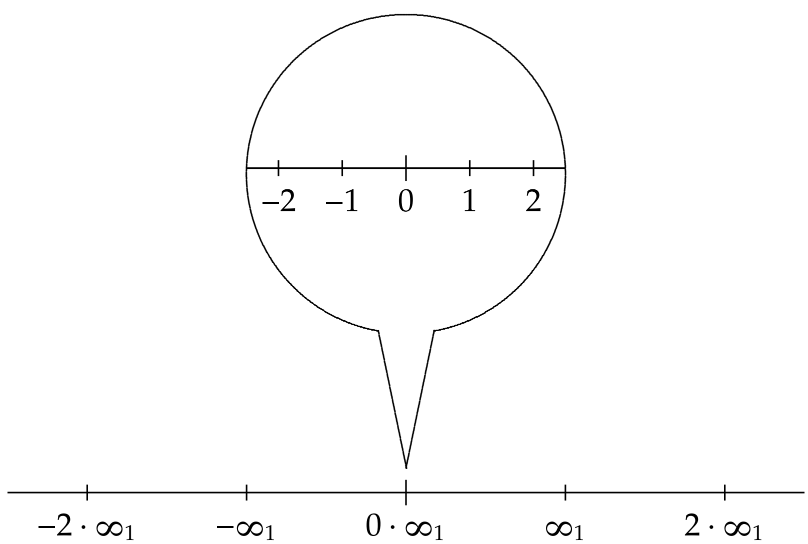

FIG. 3:
Line of infinities with microscope view of
real number line within $0 \cdot \infty_1$

Inversely to the expansion of a point of real space, the real number line can collapse into a point within a space of infinities. This is shown in Figure 3.

When an infinite number is unfolded, the roles of microscope and macroscope are reversed: The microscope shows folded finite space, and the macroscope shows unfolded infinite space.

Within this space of unfolded infinity, we may have, for example, $\infty_2 = 3 \cdot \infty_1$, $\dfrac{\infty_2}{\infty_1} = 3$, $\infty_2 > \infty_1$, at a higher level of sensitivity.

The class of folded infinite values, and the unfoldings derived from them, depend on the type of infinite element extension. In the projectively extended real numbers, we defined $\infty := \dfrac{1}{0}$, while in the affinely extended reals, we defined $\infty := \left|\dfrac{1}{0}\right|$.

$$
\begin{aligned}
\infty_1 &= \infty_2 = \infty_1 \\
\infty &= \infty^2 = 2\infty \\
\infty^2 &= \infty^3 = 2\infty^2 \\
\infty^3 &= \infty^4 = 2\infty^3
\end{aligned}
$$

$$
\begin{aligned}
&\infty_1 \neq_{\sim 0_1} \infty_1^2 \\
&\infty_1 \neq_{\sim 0_1} 2\infty_1 \\
&\infty_1^2 =_{\sim 0_1} \infty_1^3 =_{\sim 0_1} 2\infty_1^2 \\
&\infty_1^3 =_{\sim 0_1} \infty_1^4 =_{\sim 0_1} 2\infty_1^3
\end{aligned}
$$

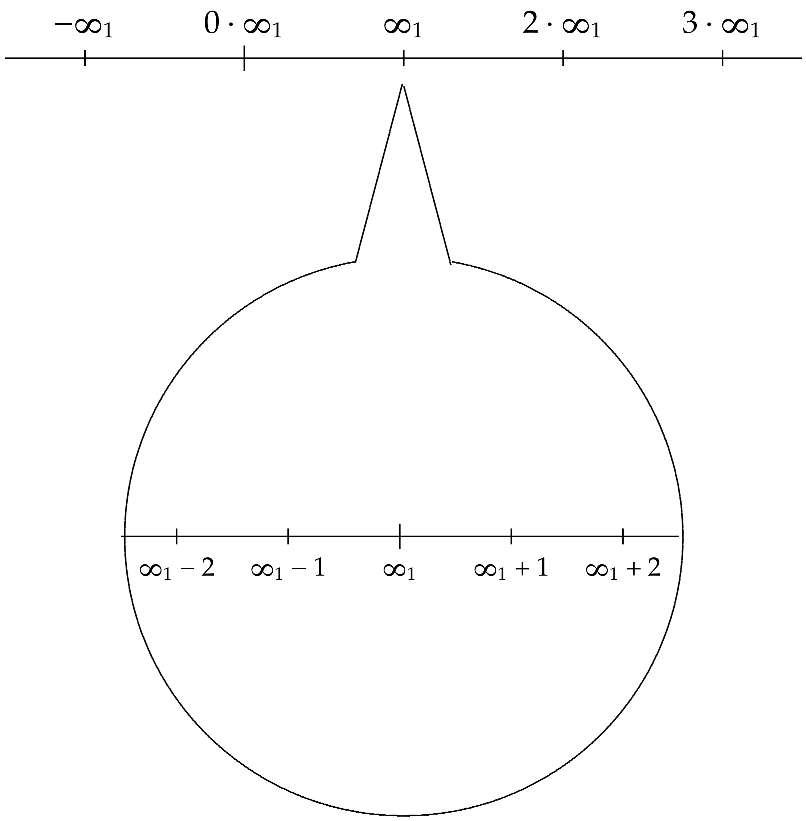

FIG. 4:
Line of infinities with
perfinite unfolding of ∞_1

Figure 4 shows that each infinite element in the space of infinities unfolds into a space in which finite numbers added to the infinite element are distinct points.

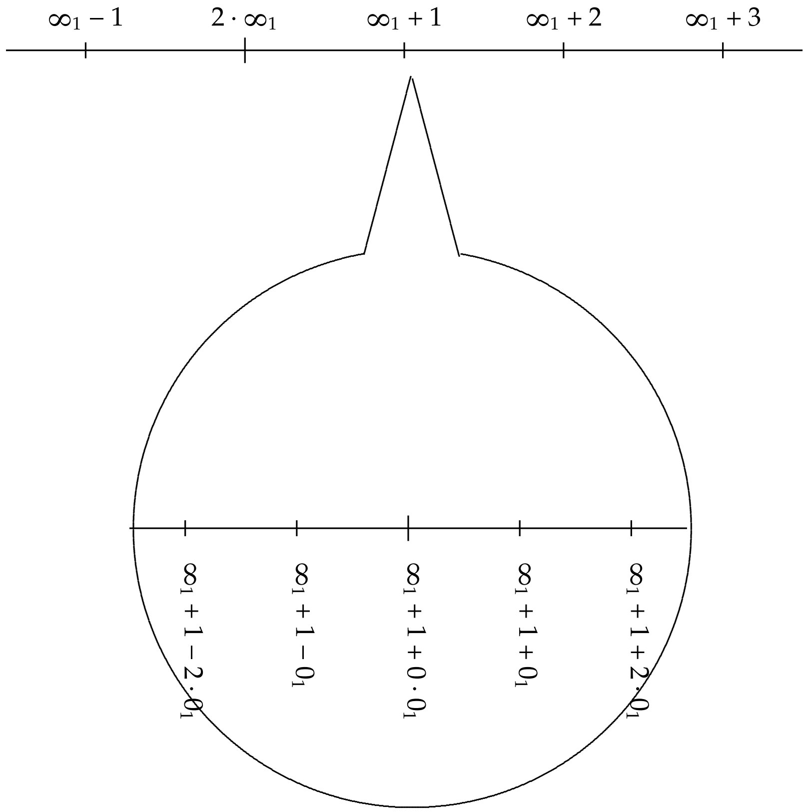

FIG. 5:
Line of infinities with
infinitesimal unfolding of $\infty_1 + 1$

Figure 5 shows that each point of infinite plus finite unfolds into a space with infinitesimals added.

Superunfolding

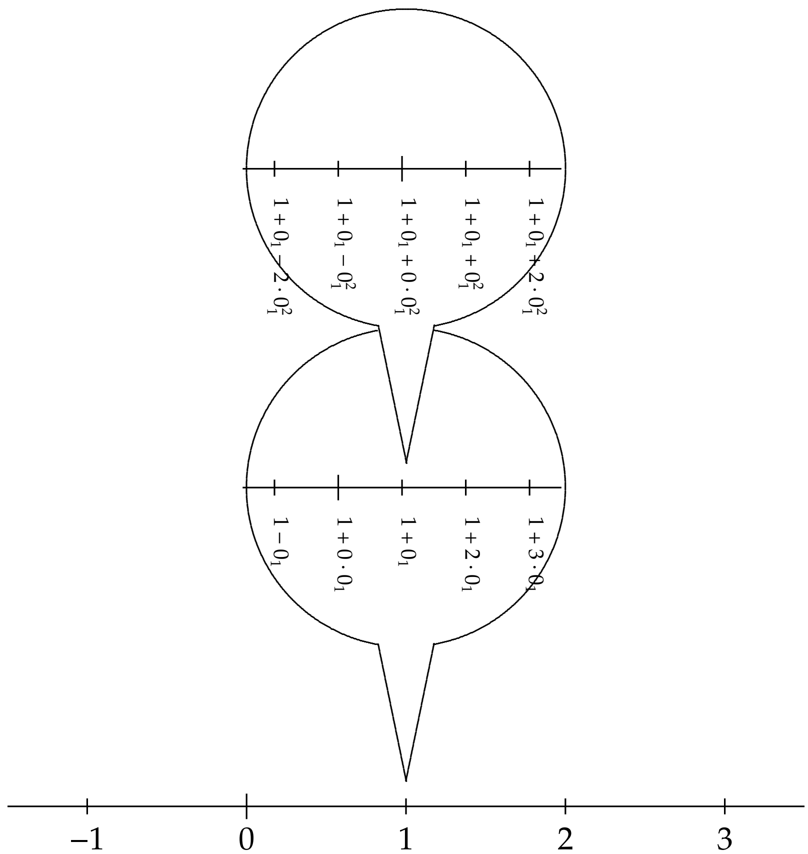

FIG. 6:
Unfolding of $1 + 0_1$,
superunfolding of 1

Within an unfolded space, any point can be unfolded again. This ***second unfolding*** uses a sensitivity unit of 0_1^2 instead of 0_1. We can also call this an unfolding with respect to 0_1^2.

For each positive integral power 0_1^n, we can make an ***n-th unfolding*** or unfolding with respect to 0_1^n. This includes $n = \infty$, which is called the ***ultimate unfolding***. The ***0-th unfolding*** is the folded numbers. Any unfolding beyond the first is called a ***superunfolding***.

Figure 6 shows the superunfolding of $1 + 0_1$, which is itself an element of the unfolded 1.

$$0_1 = 0_1^2 = 2 \cdot 0_1$$
$$0_1^2 = 0_1^3 = 2 \cdot 0_1^2$$
$$0_1^3 = 0_1^4 = 2 \cdot 0_1^3$$
$$\infty = \infty^2 = 2\infty$$
$$\infty^2 = \infty^3 = 2\infty^2$$
$$\infty^3 = \infty^4 = 2\infty^3$$

$$0_1 \neq_{\sim 0_1} 0_1^2 \neq_{\sim 0_1} 2 \cdot 0_1$$
$$0_1^2 =_{\sim 0_1} 0_1^3 =_{\sim 0_1} 2 \cdot 0_1^2$$
$$0_1^3 =_{\sim 0_1} 0_1^4 =_{\sim 0_1} 2 \cdot 0_1^3$$
$$\infty_1 \neq_{\sim 0_1} \infty_1^2 \neq_{\sim 0_1} 2\infty_1$$
$$\infty_1^2 =_{\sim 0_1} \infty_1^3 =_{\sim 0_1} 2\infty_1^2$$
$$\infty_1^3 =_{\sim 0_1} \infty_1^4 =_{\sim 0_1} 2\infty_1^3$$

$$0_1 \neq_{\sim 0_1^2} 0_1^2 \neq_{\sim 0_1^2} 2 \cdot 0_1$$
$$0_1^2 \neq_{\sim 0_1^2} 0_1^3 \neq_{\sim 0_1^2} 2 \cdot 0_1^2$$
$$0_1^3 =_{\sim 0_1^2} 0_1^4 =_{\sim 0_1^2} 2 \cdot 0_1^3$$
$$\infty_1 \neq_{\sim 0_1^2} \infty_1^2 \neq_{\sim 0_1^2} 2\infty_1$$
$$\infty_1^2 \neq_{\sim 0_1^2} \infty_1^3 \neq_{\sim 0_1^2} 2\infty_1^2$$
$$\infty_1^3 =_{\sim 0_1^2} \infty_1^4 =_{\sim 0_1^2} 2\infty_1^3$$

Equality at the ultimate unfolding, indicated with the symbol $=_{\sim 0_1^\infty}$, can also be indicated with $\equiv$ and called ***equivalence***. Power series enable us to evaluate various unfoldings of non-polynomial functions:

$$
\begin{aligned}
e^{0_1 x} &=_{\sim 1} 1 \\
&=_{\sim 0_1} 1 + 0_1 x \\
&=_{\sim 0_1^2} 1 + 0_1 x + \frac{0_1^2 x^2}{2} \\
&\ \ \ldots \\
&\equiv \sum_{k=1}^{\infty} \frac{0_1^k x^k}{k!}.
\end{aligned}
$$

Subscriptless notation

Since any perfinite real multiple of 0_1 can function as a sensitivity unit at the same level as 0_1, we will usually need a symbol for only the unit in an unfolded space. Thus, when referring to the numbers of unfolded space, we will henceforth drop the subscripts and use a prime instead: $0'$ and ∞' instead of 0_1 and ∞_1.

Another alternative is to use the letters ρ and ω, which somewhat resemble the symbols 0 and ∞. The disadvantage of this approach is that letters may appear to denote variables, when in fact these units are constants.

A third alternative is to use the symbols 0 and ∞ themselves, being careful to note that they represent the *units* of their respective unfolded spaces and not their *origins* $0_{\sim 0_1}$ and $\infty_{\sim 0_1}$. Since this notation easily creates confusion, we will not use it.

We will also use prime notation on relations to denote sense $0'$: instead of $a =_{\sim 0'} b$, we write $a =' b$ ("is equal prime to" or "is unfolded equal to"), and likewise $a <' b$, $a \in' b$, etc.

We do not use prime notation on functions, because it conflicts with prime as a derivative notation, and we seldom need to specify a function's sensitivity level.

With the prime convention, we write:

$$
\begin{aligned}
0 &= 0' = 2 \cdot 0' \\
0' &\neq' 2 \cdot 0' \\
0' &\in' 0 \\
\infty &= \infty' = 2 \cdot \infty' \\
\infty' &\neq' 2 \cdot \infty' \\
\infty' &\in' \infty
\end{aligned}
$$

Sensitivity extensions of standard classes

Starting with the projectively extended real numbers $\widehat{\mathbb{R}}$, or the affinely extended real numbers $\overline{\mathbb{R}}$, the above sections have described the unfolding of this class to the $0'$ sensitivity level. We denote this unfolded real number class $\widehat{\mathbb{R}}_{\sim 0'}$ or $\overline{\mathbb{R}}_{\sim 0'}$. The unfolding was described in four steps:

1. Unfold $0_{\sim 1}$ into the space of infinitesimals, $0'\mathbb{R}$ (Fig. 1).

2. Unfold every perfinite number by adding the infinitesimals to the perfinite (Fig. 2).

3. Unfold $\infty_{\sim 1}$ into the space of infinities, $\infty'\mathbb{R}$, the reciprocals of the infinitesimals (Fig. 3).

4. Unfold each element in the space of infinities by adding the finites, $\mathbb{R}$, to each element (Fig. 4).

5. Unfold each element in the space of infinites plus finites by adding the infinitesimals to each element (Fig. 5).

The result is

$$\mathbb{R}_{\sim 0'} =' \left(\mathbb{R}_{:1} + 0'\mathbb{R}_{:2}\right) \cup \left(\infty'\mathbb{R}_{:1} + \mathbb{R}_{:2} + 0'\mathbb{R}_{:3}\right).$$

Entirely analogous procedures can be used to unfold $\mathbb{Q}$, $\mathbb{C}$, or higher dimensional classes.

Unfolding relations and functions

We have unfolded real numbers and the equality and membership relations at various sensitivity units. We have also implicitly unfolded addition and multiplication. We now unfold relations and functions more formally. For every folded relation or function, and every unfolding of number space, we postulate that there exists a unique extension of the relation or function in the unfolded number space, which we call the ***unfolded relation*** or ***unfolded function***.

We further postulate that unfolded relations and functions follow the ***transfer principle***: Any statement or expression using functions and relations in the folded space is equivalent to the corresponding statement or expression in the unfolded space.

In other words, an unfolded function or relation inherits its behavior from its folded original. For example, $a_{\sim 0'} +_{\sim 0'} b_{\sim 0'} =_{\sim 0'} c_{\sim 0'}$ if and only if $a_{\sim 1} +_{\sim 1} b_{\sim 1} =_{\sim 1} c_{\sim 1}$. The transfer principle applies only to points at the origin of the unfolded space: $a_{\sim 0'}$, for example, is the origin of the unfolded space around a. The transfer principle does not apply to other points in the unfolded space, even though the unfolded function or relation exists at those points.

A function or relation may be defined in an unfolded space but not be the unfolding of any folded function or relation; in other words, it cannot be defined solely at the folded level. In this case, we call it a ***proper unfolded*** function or relation; otherwise it is ***ordinary unfolded***. A proper unfolded function or relation can be folded, but some information will be lost. An ordinary unfolded function or relation does not lose any information when folded. The Dirac delta function is a proper unfolded function.

The n-th unfolding of a relation distributes that unfolding to its arguments, i.e. the arguments are all taken at the n-th unfolding. The default unfolding number of a relation is 0, i.e. if no unfolding is indicated, its arguments are presumed to be folded. We have seen how this works with equality, which by default is folded, and which in turn assumes its two arguments are folded.

The n-th unfolding of a function is the maximum unfolding used by its arguments and results. Exactly how this works depends on the function. With addition and subtraction, all the arguments must be at the same unfolding, so the unfolding of the function is the maximum unfolding of the arguments. For multiplication and division, the arguments and result can be at a variety of unfolding levels: the unfolding number n of the function is the maximum of the unfolding numbers of the two arguments and the result.

If a number's unfolding needs to be increased to be compatible with other function or relation arguments, it can be unfolded to that level and remain equivalent. The unfolded number is at the origin of the unfolded space.

If the unfolding needs to be decreased, it can be folded into the unique element it belongs to at the lower unfolding, but equivalence can only be guaranteed if the unfolded number is at the origin of the unfolded space.

We now examine some examples of these principles.

EXAMPLE 1. $2+1 \equiv 2_{\sim 1} +_{\sim 1} 1_{\sim 1} = 3_{\sim 1}$. Everything in these expressions is in folded arithmetic.

EXAMPLE 2. $2_{\sim 0'} +_{\sim 0'} 0_{\sim 0'} \equiv 2_{\sim 1} +_{\sim 1} 0_{\sim 1} \equiv 2$. All items in the first expression are at the same level of unfolding, so the transfer principle allows us to fold them all in the same way and maintain equivalence.

EXAMPLE 3. $2+0' \equiv 2_{\sim 0'} +_{\sim 0'} 0'_{\sim 0'} = 2_{\sim 1} +_{\sim 1} 0_{\sim 1} = 2$. The first equal sign defaults to folded arithmetic, so the addition also changes to folded at this step.

EXAMPLE 4. $2+0' >' 2$. The unfolded order relations are defined in much the same way as the equality relation.

EXAMPLE 5. $0'^2 =' 0 \cdot 0'$, but $0'^2 \in_{\sim 0'^2} 0 \cdot 0'$. The unit in the superunfolded infinitesimals is $0'^2$.

EXAMPLE 6. $\infty + 1 \equiv \infty_{\sim 1} +_{\sim 1} 1_{\sim 1} = \infty$. The arithmetic of ∞, like other numbers, defaults to folded.

EXAMPLE 7. $\dfrac{0}{0} \equiv \dfrac{0_{\sim 1}}{0_{\sim 1}}{\sim 1} = \not\infty$. Again, the default arithmetic is folded.

EXAMPLE 8. $\dfrac{0'}{0'} \equiv \dfrac{0'_{\sim 0'}}{0'_{\sim 0'}}{\sim 0'} \equiv 1_{\sim 1} = 1$. The arguments and value of division can be at any levels of unfolding. In this case, the maximum unfolding is the first unfolding, and the division is the inverse of $1_{\sim 1} \cdot 0'_{\sim 0'} =' 0'_{\sim 0'}$.

EXAMPLE 9. For finite x, $x - x \equiv x_{\sim 1} -_{\sim 1} x_{\sim 1} = 0$. For folded perfinite p and unfolded infinite x such that $x =' p\infty'$, $x - x \equiv x_{\sim 0'} -_{\sim 0'} x_{\sim 0'} \equiv 0_{\sim 0'} = 0$. For folded infinite $x = \infty$, $x - x \equiv x_{\sim 1} -_{\sim 1} x_{\sim 1} = \not\infty$. The operations that work in folded arithmetic work identically in unfolded arithmetics, provided every item is unfolded. Therefore, expressions such as these can be evaluated in unfolded arithmetic whether x is folded or unfolded.

EXAMPLE 10. Similarly, for perfinite x, $\dfrac{x}{x} \equiv \dfrac{x_{\sim 1}}{x_{\sim 1}}{\sim 1} = 1$, and for folded afinite a and unfolded afinite x such that $x =' a \cdot \{0', \infty'\}$, $\dfrac{x}{x} \equiv \dfrac{x_{\sim 0'}}{x_{\sim 0'}}{\sim 0'} = 1$, but for folded afinite $x = a \cdot \{0, \infty\}$, $\dfrac{x}{x} \equiv \dfrac{x_{\sim 1}}{x_{\sim 1}}{\sim 1} = \not\infty$. As in the previous example, this expression can be evaluated in unfolded arithemetic whether x is folded or unfolded.

EXAMPLE 11. In the unfolded arithmetic, if we use $x =' 0$ instead of $x =' 0'$, then $\dfrac{x}{x} \equiv \dfrac{x_{\sim 0'}}{x_{\sim 0'}}{\sim 0'} = \not\infty$, since $0_{\sim 0'}$ is at the origin of the unfolded infinitesimals, and $\mathbb{R}0_{\sim 0'} =_{\sim 0'} 0_{\sim 0'}$.

EXAMPLE 12. For finite folded r,

$$\begin{aligned}
\frac{0'r + 0'^2}{0'} &\equiv \frac{0'_{\sim 0'} \times_{\sim 0'^2} r_{\sim 1} +_{\sim 0'^2} 0'^2_{\sim 0'^2}}{0'_{\sim 0'}}{}_{\sim 0'^2} \\
&\equiv \frac{0'_{\sim 0'} \times_{\sim 0'^2} r_{\sim 1}}{0'_{\sim 0'^2}}{}_{\sim 0'^2} +_{\sim 0'^2} \frac{0'^2_{\sim 0'^2}}{0'_{\sim 0'^2}}{}_{\sim 0'^2} \\
&\equiv r_{\sim 1} +_{\sim 0'^2} 0'_{\sim 0'} \\
&\equiv r_{\sim 0'} +_{\sim 0'} 0'_{\sim 0'} \\
&= r_{\sim 1} +_{\sim 1} 0_{\sim 1} \\
&= r.
\end{aligned}$$

In what follows, we will abbreviate such derivations as follows:

$$\begin{aligned}
\frac{0'r + 0'^2}{0'} &\equiv \frac{0'r}{0'} + \frac{0'^2}{0'} \\
&\equiv r + 0' \\
&= r.
\end{aligned}$$

Nonintegral superunfoldings

A superunfolding normally uses a sensitivity unit of $0'^p$, where p is an integer. The microscope of such a superunfolded space magnifies unfolded space by a factor of ∞'^p, where $\infty' :\equiv \frac{1}{0'}$.

Any positive perfinite p, including noninteger p, results in a magnification by an infinite amount, since ∞'^p is infinite. For any positive perfininte $q < p$, the unfolding with $0'^p$ is a superunfolding of the unfolding with $0'q$, since $p - q$ is also positive perfinite and ∞'^{p-q} is infinite. Thus there is a continuum of superunfoldings corresponding to the real continuum, each with its own sensitivity level.

A positive infinitesimal p, such as $p :\equiv 0'$, results in a finite magnification, since ∞'^p is finite. In this case, we do not obtain an unfolding or a separate sensitivity level.

DEFINITIONS OF DERIVATIVE AND INTEGRAL

Defintion of derivative

The equipoint derivative directly calculates the rate of change at a point using sensitivity levels and the transfer principle.

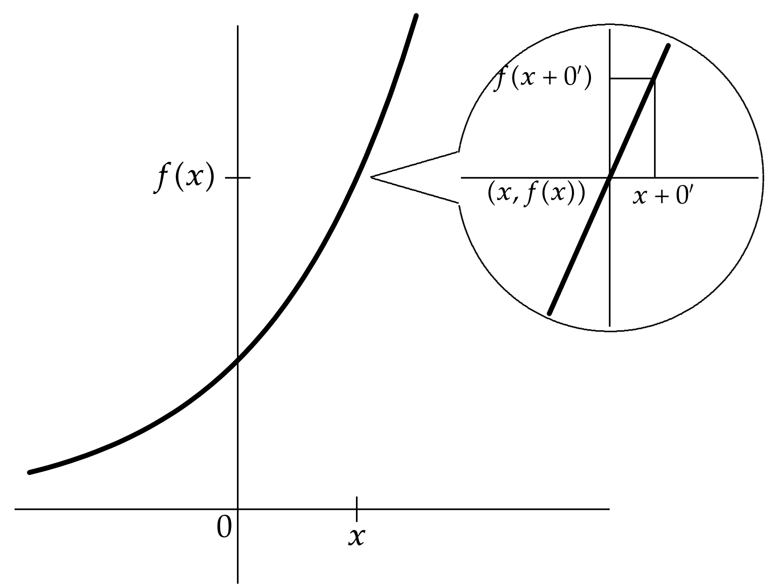

FIG. 7:
Calculation of derivative
as slope within a point

Figure 7 shows a curve $y = f(x)$ and a microscope view of the point $(x, f(x))$. Within the point, the curve is infinitely magnified and becomes a straight line. The Δx of this line is an infinitesimal $0'$, the sensitivity level of the microscope, and the Δy of this line is $f(x+0') - f(x)$. In unfolded space, we denote Δx and Δy as dx and dy. The slope of the line, and the ***derivative*** of $f(x)$ at x, is

$$f'(x) \equiv \frac{df(x)}{dx} \equiv \frac{f(x+0') - f(x)}{0'}.$$

As an example of this calculation:

$$
\begin{aligned}
f(x) &= x^2 \\
\frac{df(x)}{dx} &\equiv \frac{(x+0')^2 - x^2}{0'} \\
&\equiv \frac{x^2 + 2\cdot 0'x + 0'^2 - x^2}{0'} \\
&\equiv \frac{2\cdot 0'x}{0'} \\
&= 2x.
\end{aligned}
$$

Since $\frac{0'^2}{0'} \equiv 0'$ and $\frac{x^2 - x^2}{0'} \equiv \frac{0'^2}{0'} \equiv 0'$, these terms vanish from the final result.

For a comparison of this definition of the derivative with that in other systems of analysis, see the Appendix.

Defintion of definite integral

The equipoint integral directly calculates an area as an infinite sum of zero width rectangles.

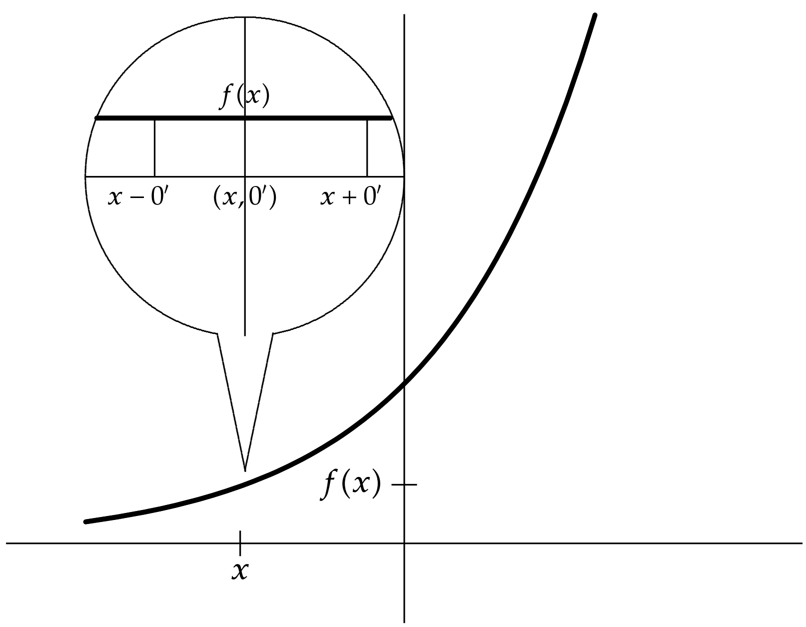

FIG. 8:
Calculation of integral
as sum of zero width rectangles

Figure 8 shows a curve $y = f(x)$ and a microscope view of the sliver at x an infinitely thin area under the curve $f()$ at the point x. The microscope expands the sliver in the x direction but not in the y direction. Within the sliver, the curve becomes a flat line, and sliver is a rectangle with height $f(x)$ and width $x + 0' - x = 0'$.

The total area under the curve from $x = a$ to $x = b$ is the sum of the areas of these slivers, and the number of these slivers is $\dfrac{b-a}{0'} \equiv (b-a)\infty'$. The total area from a to b, and the ***definite integral*** of $f(x)$ from a to b, is

$$\int_a^b f(x)\,dx \equiv \sum_{k=1}^{\infty'} f\left(a + \frac{k(b-a)}{\infty'}\right)\frac{b-a}{\infty'}.$$

As an example of this calculation:

$$\begin{aligned}
\int_0^u 2x\,dx &\equiv \sum_{k=1}^{\infty'} 2\frac{ku}{\infty'}\frac{u}{\infty'} \\
&\equiv \frac{2u^2}{\infty'^2}\sum_{k=1}^{\infty'} k \\
&\equiv \frac{2u^2}{\infty'^2}\infty'\frac{\infty'+1}{2} \\
&\equiv u^2\left(1 + \frac{1}{\infty'}\right) \\
&\equiv u^2(1+0') \\
&= u^2.
\end{aligned}$$

For a comparison of this definition of the definite integral with that in other systems of analysis, see the Appendix.

Infinite bounds on integrals and path integral

In the equipoint definition of integral, $b - a$ may be infinite if either of the limits a or b is infinite. In this case, we simply choose a $0'$ of a high enough sensitivity that $\dfrac{b-a}{0'}$ is infinitesimal, e.g. $\dfrac{1}{(b-a)^2}$.

Equipoint analysis can be used with any infinite element extension discussed in [CN]. With a projectively extended system, bounds of integration may appear ambigous, since $+\infty$ and $-\infty$ may be identical. From the numeristic point of view, bounds of integration implicitly establish a *path* of integration: integrating from $-\infty$ to $+\infty$ integrates through 0 and all the finite values, integrating from 0 to $+\infty$ integrates through all the positive finite values, etc. The equipoint integral along a path $x = P(t)$, where t runs from a to b, is given by

$$\int_P f(x)\,dx = \int_{t=a}^{b} f(P(t))\,dP(t) = \int_a^b f(P(t))\frac{dP(t)}{dt}\,dt.$$

Differentiability and integrability

The Singularities chapter discusses several types of singularity which may present difficulties using the above definitions of derivative and integral.

Briefly, at a jump discontinuity, the derivative is infinite, and the integral can be calculated straighforwardly through the singularity. See the discussions of the absolute value function, the Kronecker delta function, and the Dirac delta function.

At punctured functions, poles, and essential singularities, it is necessary to use an offset derivative, and attempts to integrate through these singularities may be incorrect. See the discussions of the punctured constant function, the axial function, poles, and the function $\sin\frac{1}{x}$.

THE FUNDAMENTAL THEOREMS OF CALCULUS

For the following equipoint proof of the first and second fundamental theorems of calculus, we assume the following:

1. The splitting property $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ for all c, which is easily proved from the definition of the definite integral.

2. A corollary, the zero property $\int_a^a f(x)\,dx = 0$ for all a.

3. Another corollary, the reversal property $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$.

We do *not* assume the mean value theorem.

We recall that the definite integral is defined as

$$\int_a^b f(x)\,dx \equiv \sum_{k=1}^{\infty'} f\left(a + \frac{k(b-a)}{\infty'}\right)\frac{b-a}{\infty'},$$

and the derivative as

$$\frac{df(x)}{dx} \equiv \frac{f(x+0') - f(x)}{0'}.$$

THE FIRST FUNDAMENTAL THEOREM OF CALCULUS:

$$\boldsymbol{\int_a^b \frac{df(x)}{dx}\,dx = f(b) - f(a).}$$

PROOF.

$$\begin{aligned}
\int_a^b \frac{df(x)}{dx}\,dx &\equiv \int_a^b \frac{f\left(x-\frac{b-a}{\infty'}\right)-f(x)}{\frac{b-a}{\infty'}}\,dx \\
&\equiv \frac{b-a}{\infty'}\frac{\infty'}{b-a}\sum_{k=1}^{\infty'} f\left(a+\frac{k(b-a)}{\infty'}+\frac{b-a}{\infty'}\right)-f\left(a+\frac{k(b-a)}{\infty'}\right) \\
&\equiv \sum_{k=1}^{\infty'} f\left(a+(k+1)\frac{b-a}{\infty'}\right)-f\left(a+k\frac{b-a}{\infty'}\right) \\
&\equiv f\left(a+(\infty'+1)\frac{b-a}{\infty'}\right)-f\left(a+\frac{b-a}{\infty'}\right) \\
&\equiv f\left(a+(b-a)+0'\right)-f\left(a+0'\right) \\
&= f(b)-f(a). \;\square
\end{aligned}$$

THE SECOND FUNDAMENTAL THEOREM OF CALCULUS:

$$\boldsymbol{\frac{d}{dx}\int_c^x f(u)\,du = f(x).}$$

PROOF.

$$\begin{aligned}
\frac{d}{dx}\int_c^x f(u)\,du &\equiv \frac{\int_c^{x+0'} f(u)\,du-\int_c^x f(u)\,du}{0'} \\
&\equiv \frac{\int_c^x f(u)\,du+\int_{x+0'}^{x+0'} f(u)\,du-\int_c^x f(u)\,du}{0'} \\
&\equiv \frac{\int_{x+0'}^{x+0'} f(u)\,du}{0'} \\
&\equiv \frac{f(x+0')0'}{0'} \\
&\equiv f(x+0') \\
&= f(x). \;\square
\end{aligned}$$

We then have

$$\int_c^x f(u)\,du = F(x)+k,$$

where $F(x)$ is any function such that

$$\frac{dF(x)}{dx} = f(x),$$

and k is a constant that depends on c. Then we have

$$\begin{aligned}\int_a^b f(x)\,dx &= \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \\ &= \int_c^b f(x)\,dx - \int_c^a f(x)\,dx \\ &= F(b) - F(a).\end{aligned}$$

Numeristics and equipoint analysis allow us to apply these definitions and theorems to a very wide range of functions. A function that is conventionally considered discontinuous has an infinite equipoint derivative at the point of discontinuity. A similarly wide net is cast for integration. Abscissas and ordinates may be finite or infinite, single valued or multivalued.

There are few types of singularity where these theorems do not apply completely, since the derivative as given here is not determinate. In these cases, it may be necessary to use an offset derivative and restrict the range of integration. Singularities where this occurs include poles and essential singularities. This consideration is discussed in detail in the Singularities chapter.

DERIVATIVE THEOREMS

Chain rule

$$\boldsymbol{\frac{d}{dx}f(g(x)) = \left[\frac{d}{d\ g(x)}\right]\left[\frac{d}{dx}g(x)\right]}$$

PROOF. Because equipoint derivatives are arithmetic fractions of differentials, the chain rule is almost trivial.

$$\begin{aligned}
\frac{d}{dx}f(g(x)) &\equiv \frac{f(g(x+0')) - f(g(x))}{0'} \\
&\equiv \frac{\left[f(g(x+0')) - f(g(x))\right]\left[g(x+0') - g(x)\right]}{0'\left[g(x+0') - g(x)\right]} \\
&\equiv \frac{f(g(x+0')) - f(g(x))}{g(x+0') - g(x)}\,\frac{g(x+0') - g(x)}{0'} \\
&\equiv \left[\frac{d}{d\ g(x)}\right]\left[\frac{d}{dx}g(x)\right]. \ \square
\end{aligned}$$

Product rule

$$\frac{d}{dx}f(x)g(x) = f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x)$$

PROOF. To derive the product rule, we use the fact that f and g at the infinitesimal level are straight lines, and $f(x+0')g(x+0')$ can be considered the area of a rectangle with sides $f(x+0') \equiv f(x) + [f(x+0') - f(x)]$ and $g(x+0') \equiv g(x) + [g(x+0') - g(x)]$. We also use the fact that $[f(x+0') - f(x)][g(x+0') - g(x)] =' 0'^2$, which vanishes from the result.

$$\begin{aligned}
\frac{d}{dx}f(x)g(x) &\equiv \frac{f(x+0')g(x+0') - f(x)g(x)}{0'} \\
&\equiv \frac{1}{0'}\Big[f(x)g(x) \\
&\quad + f(x)[g(x+0') - g(x)] \\
&\quad + [f(x+0') - f(x)]g(x) \\
&\quad + [f(x+0') - f(x)][g(x+0') - g(x)] \\
&\quad - f(x)g(x)\Big] \\
&\equiv \frac{f(x)[g(x+0') - g(x)] + g(x)[f(x+0') - f(x)]}{0'} \\
&\equiv f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x). \ \square
\end{aligned}$$

Derivatives of sine and cosine

$$\frac{d\cos\theta}{d\theta} = -\sin\theta$$
$$\frac{d\sin\theta}{d\theta} = \cos\theta$$

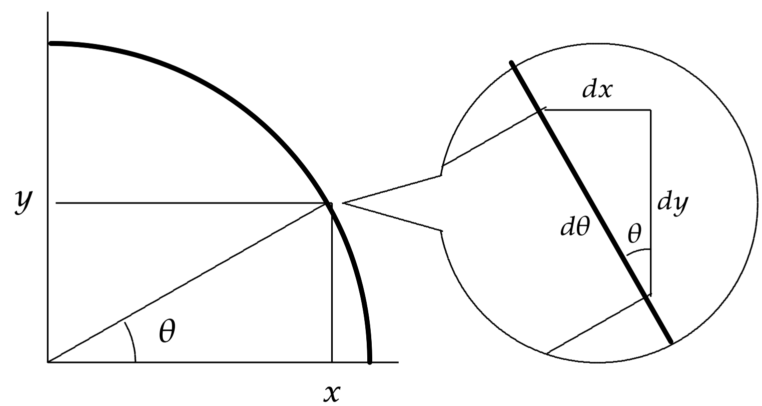

FIG. 9:
Calculation of derivatives
of sine and cosine

PROOF. Figure 9 depicts the calculation of the sine and cosine functions. In this figure, we have

$$x = \cos\theta$$
$$y = \sin\theta$$

and a microscope picture of the point (x, y).

In the microscope, the circle has become a straight line, coincident with the tangent to the circle at (x, y). Outside the microscope, the radius is a single line, but within the microscope, the radius is the class of all lines normal to the tangent. We show two such radius lines that are separated by the distance $d\theta$. The units of this distance must match the units in the tangent and radius, so we must measure $d\theta$, and thus θ itself, in radians.

The line segment along the tangent bounded by the two radii forms a triangle with legs dx and dy and hypotenuse $d\theta$. We then have

$$\begin{aligned}
\frac{y}{x} &= -\frac{dx}{dy} \\
\frac{dx}{d\theta} &= -\sin\theta \\
\frac{dy}{d\theta} &= \cos\theta. \;\square
\end{aligned}$$

Derivative of exponential function

$$\boldsymbol{\frac{d}{dx}e^x = e^x}$$

PROOF. We start with an equipoint definition of e and compute e^x.

$$\begin{aligned}
e &= (1+0')^{\infty'} \\
&= \left(1+\frac{1}{\infty'}\right)^{\infty'}, \\
e^x &= \left(1+\frac{1}{\infty'}\right)^{\infty' x} \\
&= \left(1+\frac{x}{\infty''}\right)^{\infty''}.
\end{aligned}$$

The last line comes from the substitution $\infty' x \to \infty''$, or $\infty' \to \dfrac{\infty''}{x}$. We then have

$$\begin{aligned}
e^{0'' x} &\equiv (e^x)^{0''} \\
&\equiv \left(1+\frac{x}{\infty''}\right)^{\infty'' 0''} \\
&\equiv \left(1+\frac{x}{\infty''}\right)^{1} \\
&\equiv \left(1+\frac{x}{\infty''}\right) \\
&\equiv \left(1+0'' x\right).
\end{aligned}$$

Solving for x we have

$$
\begin{aligned}
x &\equiv \frac{e^{0''x}-1}{0''}, \\
1 &\equiv \frac{e^{0''}-1}{0''}, \\
e^x &\equiv e^x \frac{e^{0''}-1}{0''} \\
&\equiv \frac{e^{x+0''}-e^x}{0''} \\
&= \frac{d}{dx}e^x.
\end{aligned}
$$

Natural logarithm as a polynomial

$$\mathbf{\ln \boldsymbol{t} = \frac{\boldsymbol{t}^{0'}-1}{0'}}$$

PROOF. The following derivation uses equipoint analysis to show that $\int_1^t t^{-1}\,dt = \ln t$ is more or less an instance of the general law $\int_0^t t^n\,dt = \frac{t^{n+1}}{n+1}$, and not merely an exception. We start with a result from the previous section and substitute $x = \ln y$.

$$
\begin{aligned}
x &= \frac{e^{0'x}-1}{0'} \\
\ln y &= \frac{y^{0'}-1}{0'} \\
&= \frac{y^{0'}-y^{0'^2}}{0'}.
\end{aligned}
$$

We then integrate t^{-1} with $\int_0^t t^n\,dt = \frac{t^{n+1}}{n+1}$ and obtain

$$
\begin{aligned}
\int_1^t t^{-1}\,dt &= \frac{t^{0'}}{0'} - \frac{1}{0'} \\
&= \frac{t^{0'}-1}{0'} \\
&= \ln t. \ \square
\end{aligned}
$$

This result can be verified with L'Hôpital's rule, which is proved below. For real t, this result is also verified by the following.

$$\int_1^x t^{-1}\,dt = \ln|x|$$
$$\frac{d}{dx}\ln|x| = \frac{\operatorname{sgn} x}{x} = \frac{x}{|x|^2} = \frac{1}{x}.$$

$\dfrac{y^{0'}-1}{0'}$ is a zeroth-order polynomial in unfolded arithmetic, or more accurately a polynomial of order $0'$. Its integrals are unfolded polynomals of higher degrees:

$$\begin{aligned}\int \ln x\,dx &= x\ln x - x\\ &= \frac{x^{1+0'}-x}{0'} - x\\ \int (x\ln x - x)\,dx &= \frac{x^2}{2}\ln x - \frac{3x^2}{4}\\ &= \frac{x^{2+0'}-x^2}{2\cdot 0'} - \frac{3x^2}{4}\end{aligned}$$

LIMITS AND CONTINUITY

Limits

A limit can be defined as an unfolded expression which gives results similar those given by conventional definitions. In many cases, these expressions can be evaluated where a conventional limit fails to exist. Any syntactically correct statement is meaningful, and so these expressions always have a meaning, which may include multivalued classes or the empty class. We will later see several examples of this.

$$\lim_{x\to a} f(x) := f(a+0'), \text{where } a \text{ is finite and } 0' \neq' 0'^2$$
$$\lim_{x\to a+} f(x) := f(a+0'), \text{where } a \text{ is finite and } 0' >' 0'^2$$
$$\lim_{x\to a-} f(x) := f(a+0'), \text{where } a \text{ is finite and } 0' <' 0'^2$$
$$\lim_{x\to \infty} f(x) := f(\infty'), \text{where } \infty' \neq' \infty'^2$$
$$\lim_{x\to +\infty} f(x) := f(\infty'), \text{where } \infty' <' \infty'^2$$
$$\lim_{x\to -\infty} f(x) := f(\infty'), \text{where } \infty' >' \infty'^2$$

Thus for example

$$\lim_{x\to 0} \frac{e^x - 1}{x} := \frac{e^{0'} - 1}{0'} = e$$

A limit in the form $f(a+0')$ or $f(\infty')$ will also be called an ***offset value***, since it is not the result of a process but simply a value of a function at an unfolded point. When compared to an offset, a value in the form of $f(a)$ or $f(\infty)$, which is at the origin of the unfolding, will be called an ***original value***.

Any of the above expressions may be multivalued and/or depend on $0'$ or ∞'. In such cases, we may wish to restrict our attention to those cases in which the expression is single valued and/or independent of $0'$ or ∞'.

An offset value $f(a+0')$ is ***uniform*** if it has the same value for all $0'$, i.e. if $f(a+0') = f(a+0'')$ for any $0', 0'' \in' 0$, even when $0'$ and $0''$ have different signs. If f is single valued and $f(a+0')$ is uniform, then the class $f(a+\mathbb{R}0')$ or $f(a+\mathbb{C}0')$ is single valued. A derivative $f'_{0'}(a) \equiv \dfrac{f(a+0') - f(a)}{0'}$ is uniform if it has the same value for all $0'$.

An offset value $f(a+0')$ is ***semiuniform*** if it is the same for every $0'$ of the same sign. If f is single valued and $f(a+0')$ is semiuniform, then the class $f(a+|\mathbb{R}|0')$ or $f(a+|\mathbb{C}|0')$ is single valued.

An offset value $f(a+0')$ is ***disuniform*** if it is neither uniform nor semiuniform.

Continuity

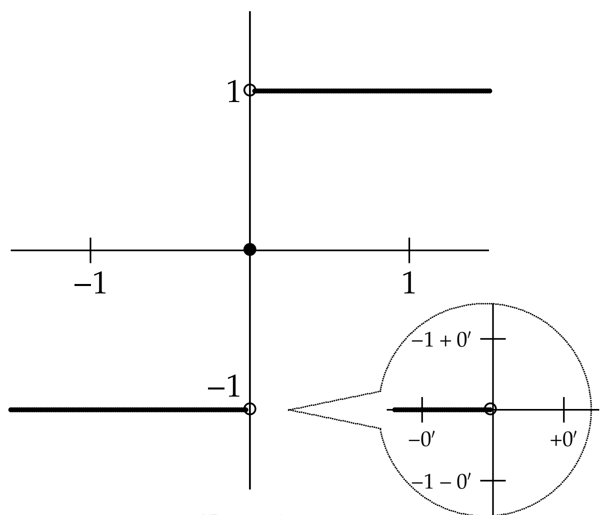

FIG. 10:
Discontinuity of signum function
sgn x at $x = 0$

We define continuity as follows. A function f is ***continuous*** at x if the offset values are uniform and equal to the original value, i.e. $f(x + a0') = f(x)$ for every $a \in \mathbb{R}$, or $f(x + \mathbb{R}0') = f(x)$.

Figure 10 shows an example of a discontinuity in the signum function

$$\operatorname{sgn} x := \begin{cases} -1 & \text{for } x < 0 \\ 0 & \text{for } x = 0 \\ +1 & \text{for } x > 0. \end{cases}$$

This function is discontinuous at $x = 0$ because $\operatorname{sgn}(x) = 0$ while $\operatorname{sgn}(x - 0') = -1$ and $\operatorname{sgn}(x + 0') = +1$ for $0' >' 0^2$.

If f is continuous at x, then f is locally linear: $f(x+0') - f(x) =' f(x + 0'(k+1)) - f(x + 0'k)$ for real k.

If f has a finite derivative at x, then it is continuous at x:

$$\begin{aligned} f'(x) &:= \frac{f(x+0')-f(x)}{0'} \\ f(x+0') &=' 0' f'(x) + f(x) \\ \text{Since } f'(x) \text{ is finite, } 0' f'(x) &=' 0 \\ f(x+0') &= f(x). \end{aligned}$$

Continuity involving infinite values may depend on the choice of infinite element extension. Consider the reciprocal function $f(x) :\equiv \frac{1}{x}$. At $x = 0$, $f(x + a0') \equiv \frac{1}{0 + a0'} = \frac{1}{a0'}$. In the projectively extended real numbers, $\frac{1}{a0'} = \infty$ for any finite real a, so $f(x)$ is continuous at $x = 0$. But in the affinely extended real numbers,

$$\frac{1}{a0'} = \begin{cases} +\infty \neq -\infty & \text{for } a > 0 \\ -\infty \neq +\infty & \text{for } a < 0, \end{cases}$$

so $f(x)$ is not continuous at $x = 0$.

Even though $\frac{1}{x}$ is continuous at $x = 0$ in the projectively extended real numbers, $e^{\frac{1}{x}}$ is not. At $x = 0$, this function has two values:

$$e^{\frac{1}{a0'}} = \begin{cases} +\infty & \text{for } a > 0 \\ 0 & \text{for } a < 0. \end{cases}$$

Multivalued functions may have partial continuity:

- ***Classwise continuity***: f is classwise continuous at a if $f(a+0') \overset{\{\}}{=} f(a)$, i.e. the two sides are the same class.
- ***Conjunctive continuity***: f is conjunctively continuous at a if $f(a+0') \overset{\wedge}{=} f(a)$, i.e. every element of $f(a+0')$ maps bijectively to $f(a)$.
- ***Disjunctive continuity***: f is disjunctively continuous at a if $f(a+0') \overset{\vee}{=} f(a)$, i.e. there exists at least one element of either the left or right side which is equal to an element on the other side.

DIFFERENTIAL AND INTEGRAL OPERATORS

Differentials and integrants

As the Leibniz notation $\frac{dy}{dx}$ indicates, a derivative is an arithmetic quotient of differentials. The differential of an independent variable is an infinitesimal, as is the differential of a dependent variable when the derivative is finite. Infinitesimals are unfolded members of folded zero, which are exactly equal to zero in folded aritmetic but distinct in unfolded arithmetic.

A ***differential*** is an operator on a function with respect to a member of zero. We define

$$^{0'}d_a f(x) :\equiv f(a + 0') - f(a),$$

from which follows

$$^{0'}d_a x :\equiv a + 0' - a \equiv 0'.$$

A derivative with respect to an infinitesimal $0'$ can therefore be defined as:

$$f'_{0'}(a) :\equiv \frac{^{0'}d_a f(x)}{^{0'}d_a x} \equiv \frac{f(a + 0') - f(a)}{0'}.$$

If the derivative is independent of the infinitesimal, we write:

$$f'(a) :\equiv \frac{d_a f(x)}{d_a x} \equiv \frac{f(a + 0') - f(a)}{0'}.$$

This occurs when $f(x)$ is analytic, since, for $f(x) = x^n$,

$$^{0'}d_a f(x) \equiv nx^{n-1}0' + \sum_{k=2}^{n} \binom{n}{k} a^{n-k} 0'^k =' nx^{n-1}0'.$$

We also define an ***integrant*** as an operator on a function:

$$^{0'}\!\!\int^a f(x) :\equiv \sum_{k=1}^{\frac{a}{0'}} f(0'k).$$

An integrant is infinite whenever the corresponding integral is nonzero.

The definite integral can be defined in terms of an integrant and a differential:

$$\overset{0'}{\int_a^b} f(x)\,dx :\equiv \overset{0'}{\int^b} f(x)0' - \overset{0'}{\int^a} f(x)0' \equiv \sum_{k=1}^{\frac{b}{0'}} f(0'k)0' - \sum_{k=1}^{\frac{a}{0'}} f(0'k)0'.$$

Again, if the integral is independent of the infinitesimal, we write:

$$\int_a^b f(x)\,dx :\equiv \sum_{k=1}^{\frac{b}{0'}} f(0'k)0' - \sum_{k=1}^{\frac{a}{0'}} f(0'k)0'.$$

We can define the indefinite integral operator in terms of the definite integral in two ways. The first way is as a definite integral plus an arbitrary constant:

$$\overset{0'}{\int} f(x)\,dx :\equiv \overset{0'}{\int^x} f(t)\,dt + \mathbb{R} \equiv \left\{ \overset{0'}{\int^x} f(t)\,dt + a \mid a \in \mathbb{R} \right\}$$

or

$$\int f(x)\,dx :\equiv \int^x f(t)\,dt + \mathbb{R} \equiv \left\{ \int^x f(t)\,dt + a \mid a \in \mathbb{R} \right\}.$$

The second way to define the indefinite integral is as a class of definite integrals with an arbitrary lower limit:

$$\overset{0'}{\int} f(x)\,dx :\equiv \overset{0'}{\int_{\mathbb{R}}^x} f(t)\,dt \equiv \left\{ \overset{0'}{\int_a^x} f(t)\,dt \mid a \in \mathbb{R} \right\}$$

or

$$\int f(x)\,dx :\equiv \int_{\mathbb{R}}^x f(t)\,dt \equiv \left\{ \int_a^x f(t)\,dt \mid a \in \mathbb{R} \right\}.$$

Either of these is a class of functions. If we denote the first A and the second as B, then given any two $F_1, F_2 \in A$, we have $F_2(x) = F_1(x) + c$, where c is a constant, and conversely. Similarly, given any two $F_1, F_2 \in B$ and their corresponding a_1, a_2, we have $F_2(x) = F_1(x) - F(a_1) + F(a_2)$. Thus $A \supseteq B$, with equality holding if all the members of A are surjective.

The integrant is the left inverse of the differential, which is essentially the first fundamental theorem of calculus:

$$\begin{aligned}
\int df(x) &\equiv \sum_{k\equiv 1}^{\frac{x}{0'}} f(0'k + 0') - f(0'k) \\
&\equiv \sum_{k\equiv 1}^{\frac{x}{0'}} f(0'(k+1)) - f(0'k) \\
&\equiv f(0'(\frac{x}{0'} + 1)) - f(0') \\
&\equiv f(x + 0') - f(0') \\
&= f(x) - f(0').
\end{aligned}$$

The integrant is also the right inverse of the differential, which is essentially the second fundamental theorem of calculus:

$$
\begin{aligned}
d\int f(x) &\equiv \int f(x+0') - f(x) \\
&\equiv \sum_{k\equiv 1}^{\frac{x+0'}{0'}} f(0'k) - \sum k \equiv 1^{\frac{x}{0'}} f(0'k) \\
&\equiv \sum_{k\equiv 1}^{\frac{x}{0'}+1} f(0'k) - \sum k \equiv 1^{\frac{x}{0'}} f(0'k) \\
&\equiv f\left(0'\left[\frac{x}{0'}+1\right]\right) \\
&\equiv f(x+0') \\
&= f(x).
\end{aligned}
$$

The ***partial differential*** is defined analogously to the differential. Here we define a partial differential on a function of two independent variables:

$$^{0'}\partial_x f(x,y) :\equiv f(x+0',y) - f(x,y),$$

or, if the result is independent of $0'$:

$$\partial_x f(x,y) :\equiv f(x+0',y) - f(x,y).$$

The total differential is then easily seen to be the sum of partial differentials:

$$
\begin{aligned}
df(x,y) &= d_{x,y} f(x,y) \\
&= f(x+0',y+0') - f(x,y) \\
&= \left[f(x+0',y+0') - f(x,y+0')\right] + \left[f(x,y+0') - f(x,y)\right] \\
&= \left[f(x+0',y) - f(x,y)\right] + \left[f(x,y+0') - f(x,y)\right] \\
&= \partial_x f(x,y) + \partial_y f(x,y) \\
&= (\partial_x + \partial_y) f(x,y)
\end{aligned}
$$

Quotiential and prodegrant operators

Closely related to the differential is its multiplicative equivalent, the ***quotiential***:

$$^{0'}\!q_a f(x) :\equiv \frac{f(x+0')}{f(x)} \equiv e^{d\ln f(x)}.$$

The inverse of the quotiential is the ***prodegrant***:

$$^{0'}\!\!\int^{a} f(x) :\equiv \prod_{k=1}^{\frac{a}{0'}} f(0'k) \equiv e^{\int \ln x}.$$

From the quotiential and differential we derive two ***quotient derivatives***, the ***geometric derivative*** and the ***bilogarithmic derivative***:

$$\sqrt[dx]{qf(x)} \equiv qf(x)^{\frac{1}{dx}} \equiv e^{\frac{d\ln f(x)}{dx}} \equiv e^{\frac{df(x)}{f(x)dx}},$$

$$\log_{qx} qf(x) \equiv e^{\frac{df(x)}{d\ln x}} \equiv e^{\frac{xdf(x)}{dx}}.$$

We also derive two ***product integrals***: the ***geometric integral*** or ***type 1 product integral***, and the ***bilogarithmic integral***:

$$\int_a^b f(x)^{dx} :\equiv \frac{\displaystyle\int^b f(x)^{dx}}{\displaystyle\int^a f(x)^{dx}}$$

$$\equiv e^{\int_a^b \ln f(x)\,dx}$$

$$\int_a^b qx^{f(x)} :\equiv \frac{\displaystyle\int^b qx^{f(x)}}{\displaystyle\int^a qx^{f(x)}}$$

$$\equiv e^{\int_a^b f(x)\,d\ln x}$$

Somewhat ambiguously, the symbol $\prod$ is usually used elsewhere instead of $\int$.

Volterra, who first investigated product integrals [V], orignially defined what is now called the ***type 2 product integral***:

$$\prod_a^b [1 + f(x)\,dx] :\equiv e^{\int_a^b f(x)\,dx} \equiv \int_a^b e^{f(x)\,dx}$$

The inverse of the type 2 product integral is the ***logarithmic derivative***:

$$\frac{f'(x)}{f(x)} \equiv \frac{d\ln f(x)}{dx} \equiv \frac{df(x)}{x\,dx}.$$

The ***partial quotiential*** is given by:

$${}^{0'}\varphi_x f(x,y) :\equiv \frac{f(x+0',y)}{f(x,y)} \equiv e^{{}^{0'}\partial_x \ln f(x,y)}.$$

Higher order derivatives and integrals

The simple form of the equipoint derivative lends itself to direct calculaton of higher order derivatives. These derivatives are are also simple quotients, with $dx^n :\equiv (dx)^n$ in the denominator.

$$\boldsymbol{f^{(n)}(x) = \frac{\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}f(x+0'k)}{0'^n}}$$

PROOF. Computing higher order derivatives is mainly a matter of computing the numerator $d^n f(x)$, which is an iterated application of the differential operator:

$$\begin{aligned} f''(x) &= \frac{d^2}{dx^2}f(x) \\ &= \frac{d[d[f(x)]]}{0'^2} \\ &= \frac{d[f(x+0')-f(x)]}{0'^2} \\ &= \frac{[f(x+2\cdot 0')-f(x+0')]-[f(x+0')-f(x)]}{0'^2} \\ &= \frac{f(x+2\cdot 0')-2f(x+0')+f(x)}{0'^2}. \end{aligned}$$

The expansion of these operators is similar to expansion of the binomial power

$$\begin{aligned} (a-b)^n &= \sum_{k=0}^{n}(-1)^k\binom{n}{k}a^{n-k}b^k \\ &= \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}a^k b^{n-k}. \end{aligned}$$

In derivatives, $f(x+0'k)$ corresponds to $a^k b^{n-k}$: the n-th derivative is

$$\begin{aligned}
f^{(n)}(x) &= \frac{d^n}{dx^n} f(x) \\
&= \frac{\sum_{k=0}^{n} (-1)^k \binom{n}{k} f\left(x + [n-k]0'\right)}{0'^n} \\
&= \frac{\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(x + 0'k)}{0'^n},
\end{aligned}$$

which can be proved by induction:

$$f^{(0)}(x) = f(x) = \binom{0}{0} (-1)^0 f(x - 00'),$$

and

$$\begin{aligned}
\frac{d^{n+1}}{dx^{n+1}} f(x) &= \frac{d}{dx}\frac{d^n}{dx^n} f(x) = \frac{1}{0'}\left[\frac{d^n}{dx^n} f(x-0') - \frac{d^n}{dx^n} f(x)\right] \\
&= \frac{1}{0'^{n+1}}\left[\sum_{k=0}^{n} (-1)^k \binom{n}{k} f\left(x + [n+1-k]0'\right) - \sum_{k=0}^{n} (-1)^k \binom{n}{k} f\left(x + [n-k]0'\right)\right] \\
&= \frac{1}{0'^{n+1}}\left[\binom{n}{0} f\left(x + [n+1]0'\right)\right. \\
&\qquad + \sum_{k=1}^{n} (-1)^k \left[\binom{n}{k} + \binom{n}{k-1}\right] f\left(x + [n+1-k]0'\right) \\
&\qquad \left. + \binom{n}{n} f(x)\right] \\
&= \frac{1}{0'^{n+1}}\left[\binom{n+1}{0} f\left(x + [n+1]0'\right)\right. \\
&\qquad + \sum_{k=1}^{n} (-1)^k \binom{n+1}{k} f\left(x + [n+1-k]0'\right) \\
&\qquad \left. + \binom{n+1}{n+1} f(x)\right] \\
&= \frac{1}{0'^{n+1}} \sum_{k=0}^{n+1} (-1)^k \binom{n+1}{k} f\left(x + [n+1-k]0'\right). \square
\end{aligned}$$

$$\boldsymbol{\underbrace{\int \cdots \int}_{n} f(x)\, dx^n = f^{(-n)}(x) = \sum_{k=n}^{\infty'} \binom{k-1}{k-n} f(x - 0'k)\; 0'^n.}$$

PROOF. Since the binomial theorem extends to negative exponents, we can extend this result to integrals. In this case, the upper limit on the summation is infinite:

$$\begin{aligned}
(a-b)^{-n} &= \sum_{k=0}^{\infty'} (-1)^k \binom{-n}{k} a^{-n-k} b^k \\
&= \sum_{k=0}^{\infty'} (-1)^{2k} \binom{n+k-1}{k} a^{-n-k} b^k \\
&= \sum_{k=0}^{\infty'} \binom{n+k-1}{k} a^{-n-k} b^k.
\end{aligned}$$

For $n = 1$, this becomes

$$(a-b)^{-1} = \sum_{k=0}^{\infty'} a^{-1-k} b^k$$

and

$$\begin{aligned}
f^{(-1)}(x) &= \frac{d^{-1}}{dx^{-1}} f(x) \\
&= \sum_{k=0}^{\infty'} f(x - (k+1)0')\ 0' \\
&= \sum_{k=1}^{\infty'} f(x - 0'k)\ 0'.
\end{aligned}$$

Taking $0' = \frac{a-x}{\infty'}$, the above summation matches the definition of the definite integral:

$$f^{(-1)}(x) = \sum_{k=1}^{\infty'} f(x - 0'k)\ 0' = \int_a^x f(t)\, dt.$$

Since ∞' is independent of $0'$, a is arbitrary, and this expression is actually a class of functions of x, each expressed as a definite integral with a fixed lower limit and a variable upper limit. This matches the second definition of the indefinite integral $\int f(x)\, dx$ given in Differentials and integrants above.

Higher order integrals are obtained through other negative powers of binomials:

$$\underbrace{\int \cdots \int}_{n} f(x)\,dx^n = f^{(-n)}(x) = \frac{d^{-n}}{dx^{-n}} f(x)$$

$$= \sum_{k=0}^{\infty'} (-1)^k \binom{-n}{k} f(x - (k+n)0')\, 0'^n$$

$$= \sum_{k=0}^{\infty'} \binom{k+n-1}{k} f(x - (k+n)0')\, 0'^n$$

$$= \sum_{k=n}^{\infty'} \binom{k-1}{k-n} f(x - 0'k)\, 0'^n. \;\square$$

L'Hôpital's rule

L'HÔPITAL'S RULE FOR 0/0: If functions f and g are continuous at c and $f(c) = g(c) = 0$, then $\dfrac{f(c)}{g(c)} = \dfrac{f'(c)}{g'(c)}$.

The equipoint version of L'Hôpital's rule evaluates the function $q(x)$ at c in unfolded arithmetic, since $f(c) = g(c) = 0$ in folded arithmetic is insufficient to compute $\dfrac{f(c)}{g(c)}$ as a single value.

PROOF. Since f is continuous at c, we have $f(c + d) = f(c + 0') = f(c) = 0 = f(d) - f(c)$, and similarly for g. Then

$$\frac{f(c)}{g(c)} = \frac{f(c+0')}{g(c+0')}$$

$$= \frac{f(c+0') - f(c)}{g(c+0') - g(c)}$$

$$\equiv \frac{\frac{f(c+0')-f(c)}{0'}}{\frac{g(c+0')-g(c)}{0'}}$$

$$\equiv \frac{f'(c)}{g'(c)}.$$

Since f and g are continuous and $f(c) = g(c) = 0$, $f'(c)$ and $g'(c)$ are finite. If both $f'(c)$ and $g'(c)$ are zero, then we can iterate the rule until we find some n for which either $f^{(n)}(c)$ or $g^{(n)}(c)$ or both are nonzero. If both derivatives are zero for all n, then the rule does not give a single value for $q(c)$. $\square$

L'Hôpital's rule for ∞/∞: If functions f and g are continuous at c, $f(c) = g(c) = \infty$, and f and g are finite in some punctured neighborhood around c, then $\dfrac{f(c)}{g(c)} = \dfrac{f'(c)}{g'(c)}$.

Proof. Let $c' :\equiv c+0'$. Since f and g are continuous and infinite at an isolated point c, the unfolded f and g must take on every unfolded infinite value within the unfolded space around c.

Let $\infty' :\equiv \frac{1}{0'}$ and let $c + 0''$ be an arbitrary point within the unfolded space around c. Within this space, the magnitudes of f and g strictly decrease monotonically as the magnitude of $0''$ increases. It is therefore possible to choose $0''$ so that both

$$\log_{\infty'} |f(c' + 0'')| < \log_{\infty'} |g(c')|$$
$$\log_{\infty'} |g(c' + 0'')| < \log_{\infty'} |g(c')|$$

This means that $f(c' + 0'')$ and $g(c')$ are distinguishable from finite multiples of themselves at different sensitivity levels, and similarly for $g(c' + 0'')$ and $g(c')$. Since the magnitudes of $f(c' + 0'')$ and $g(c' + 0'')$ are less than that of $g(c')$, we have

$$\frac{f(c' + 0'')}{g(c')} =' 0$$
$$\frac{g(c' + 0'')}{g(c')} =' 0$$

We then compute

$$\begin{aligned}
\frac{f(c)}{g(c)} = \frac{f(c')}{g(c')} &\equiv \frac{\frac{f(c')}{g(c')}}{1} \\
&= \frac{\frac{f(c')}{g(c')} - \frac{f(c'+0'')}{g(c')}}{1 - \frac{g(c'+0'')}{g(c')}} \\
&\equiv \frac{f(c' + 0'') - f(c')}{g(c' + 0'') - g(c')} \\
&\equiv \frac{f'(c')}{g'(c')} = \frac{f'(c)}{g'(c)}. \;\square
\end{aligned}$$

This proof only requires that $g(c)$ be infinite. If $f(c)$ is finite, then the rule still applies but is not needed, since $\dfrac{f(c)}{g(c)} = 0$ by ordinary extended arithmetic.

Since numeristic division and logarithms are unrestricted, it is easy to extend the rule to other indeterminate forms.

- If $f(c)g(c)$ is of the form $0 \cdot \infty$, then use the rule on $\dfrac{f(x)}{\frac{1}{g(x)}}$ or $\dfrac{\frac{1}{f(x)}}{g(x)}$.

- If $f(c) - g(c)$ is of the form $\infty - \infty$, then use the rule on $e^{f(c)-g(c)} = \dfrac{e^{f(c)}}{e^{g(c)}}$.

- If $f(c)^{g(c)}$ is of the form 0^0, 1^∞, or ∞^0, then use the rule on $\ln f(c)^{g(c)} = g(c) \ln f(c)$.

Power series

In the following, we define σ as an integration operator with a fixed lower bound and a variable upper bound:

$$\sigma_a f(t) := \int_a^t f(t)\, dt$$

and

$$\sigma f(t) := \sigma_0 f(t) = \int_0^t f(t)\, dt$$

Powers of σ denote repeated integration or differentiation:

$$\begin{aligned}
\sigma^n f(t) &= \int_0^t \int_0^{u_n} \int_0^{u_{n-1}} \cdots \int_0^{u_3} \int_0^{u_2} f(u)\, du_1\, du_2\, \ldots\, du_{n-2}\, du_{n-1}\, du_n \\
\sigma^{-n} f(t) &= \frac{d^n}{du^n} f(u) \Big|_{u=t} \\
\sigma^0 f(t) &= f(t).
\end{aligned}$$

We are now ready to derive a compact formula for power series of an analytic function.

POWER SERIES: For an analytic function f,

$$\boldsymbol{f(t) = e^{(t-a)\sigma^{-1}} f(a).}$$

PROOF. We start by integrating and differentiating f repeatedly.

$$
\begin{aligned}
\sigma_a \sigma^{-1} f(t) &= f(t) - f(a) \\
\sigma_a \sigma^{-2} f(t) &= \sigma^{-1} f(t) - \sigma^{-1} f(a) \\
\sigma_a^2 \sigma^{-2} f(t) &= f(t) - f(a) - (t-a)\sigma^{-1} f(a) \\
\sigma_a^2 \sigma^{-3} f(t) &= \sigma^{-1} f(t) - \sigma^{-1} f(a) - (t-a)\sigma^{-2} f(a) \\
\sigma_a^3 \sigma^{-3} f(t) &= f(t) - f(a) - (t-a)\sigma^{-1} f(a) - \frac{1}{2}(t-a)^2 \sigma^{-2} f(a) \\
&\dots \\
\sigma_a^n \sigma^{-n} f(t) &= f(t) - f(a) - (t-a)\sigma^{-1} f(a) - \frac{1}{2}(t-a)^2 \sigma^{-2} f(a) - \dots \\
&\quad - \frac{1}{n!}(t-a)^n \sigma^{-n} f(a).
\end{aligned}
$$

We then take the infinite case of this series and regard it as an operator ψ on f. We do similar operations on this series and find that it leaves the series unchanged.

$$
\begin{aligned}
\psi f(t) &:= \sigma_a^\infty \sigma^{-\infty} f(t) \\
&= f(t) - f(a) - (t-a)\sigma^{-1} f(a) - \frac{1}{2}(t-a)^2 \sigma^{-2} f(a) - \dots \\
\psi \sigma f(t) &= \sigma f(t) - \sigma f(a) - (t-a)\sigma^{-2} f(a) - \frac{1}{2}(t-a)^2 \sigma^{-3} f(a) - \dots \\
\sigma_a \psi \sigma f(t) &= f(t) - f(a) - (t-a)\sigma^{-1} f(a) - \frac{1}{2}(t-a)^2 \sigma^{-2} f(a) - \dots \\
&= \psi f(t).
\end{aligned}
$$

For all infinitely differentiable f and all a, we now have $\psi f(t) = \sigma_a \psi \sigma f(t)$, or $\psi \sigma f(t) = \sigma_a \psi f(t)$. Since $\sigma_a \sigma f(t) = \sigma \sigma_a f(t)$, by the definition of ψ we have $\psi \sigma f(t) = \sigma \psi f(t) = \sigma_a \psi f(t) = \sigma \psi f(t) - \sigma \psi f(a)$. Subtracting, $\sigma \psi f(a) = 0$ for all a, i.e. $\sigma \psi f$ is the zero function. Hence $\psi f(t)$ must also be the zero function for all f, i.e. ψ is the zero operator. So

$$
\begin{aligned}
f(t) &= f(a) - (t-a)\sigma^{-1} f(a) - \frac{1}{2}(t-a)^2 \sigma^{-2} f(a) - \dots \\
&= \sum_{n=0}^{\infty} \frac{(t-a)^n}{n!} \sigma^{-n} f(a) \\
&= e^{(t-a)\sigma^{-1}} f(a). \ \square
\end{aligned}
$$

SINGULARITIES

Offset derivatives

In Differentials and integrants, we defined the differential of a function f at a finite point x with respect to a zero $0'$, denoted ${}^{0'}df(x)$, is the difference $f(x+0') - f(x)$. The equipoint handling of singularities sometimes requires a variant of this differential.

If f is continuous at x, the differential is zero: By continuity, $f(x+0') = f(x)$, so ${}^{0'}df(x) :\equiv f(x+0') - f(x) = 0$. Since dx is the differential of the identity function $f(x) = x$, it too is always zero.

If $f(x)$ is infinite or discontinuous, $df(x)$ may be nonzero. Previous chapters have assumed that differentials of dependent variables are zero, but most results continue to hold if they are nonzero. Exceptions include the two fundamental theorems of calculus, which do not hold at poles, as described below.

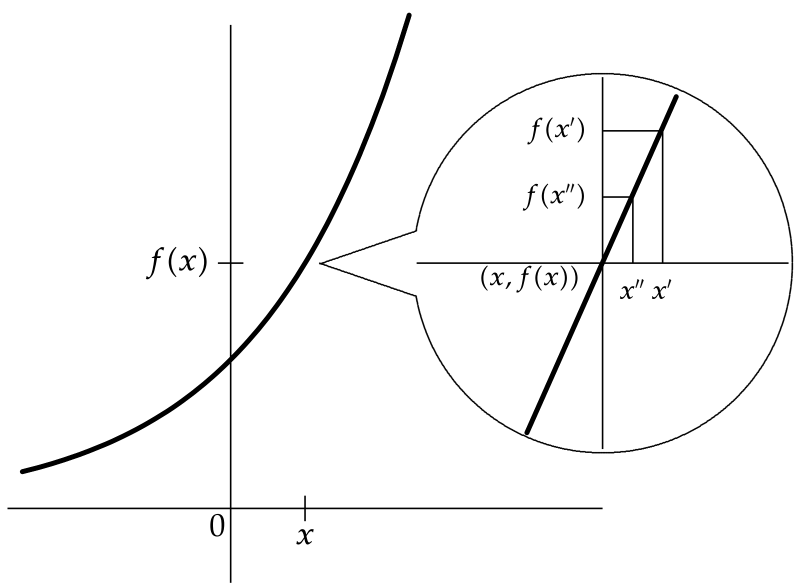

FIG. 11:
Calculation of derivative
with offset differentials

Occasionally the differential ${}^{0'}df(x) :\equiv f(x+0') - f(x)$, or derivatives that use it, do not yield a determinate result. In such cases, we may use the fact that the slope of an analytic curve at a finite point x can be computed with any two points within the microscope. This is shown in Figure 11, where we use the two points

$$\begin{aligned}(x', f(x')) &:\equiv (x+0', f(x+0')) \\ (x'', f(x'')) &:\equiv (x+0'', f(x+0''))\end{aligned}$$

The differentials along the two axes in the microscope are called ***offset differentials***, and the derivative using them is called an ***offset derivative***, with the following notations:

$$\begin{aligned}{}^{0'}_{0''}df(x) &:\equiv f(x+0') - f(x+0'') \\ {}^{0'}_{0''}f'(x) :\equiv \frac{{}^{0'}_{0''}df(x)}{{}^{0'}_{0''}dx} &\equiv \frac{f(x+0') - f(x+0'')}{0' - 0''}\end{aligned}$$

The quantities $0'$ is called the ***upper displacement*** and $0''$ the ***lower displacement***. The first type of differential, with only an upper displacement, is called a ***original differential***, since the lower displacement is the origin of the microscope. As shown in Figure 11, for a finite analytic function, the curve becomes a straight line in the microscope, so a original derivative and an offset derivative yield the same result.

Letting $0''' :\equiv 0' - 0''$, we have

$$\frac{{}^{0'}_{0''}df(x)}{{}^{0'}_{0''}dx} \equiv \frac{f(x+0') - f(x+0'')}{0' - 0''} \equiv \frac{f(x+0''+0''') - f(x+0'')}{0'''} \equiv {}^{0'''}f'(x+0'').$$

This form of an offset derivative shows that it can be considered as the derivative of an offset. As with original derivatives, if an offset derivative in this form is independent of its upper displacement, we omit it and write $f'(x+0'')$.

The above definitions apply only to finite x. For infinte x, we use the fact that for $x = 0$, ${}^{0'}_{0''}df(x) :\equiv f(0') - f(0'')$. For infinite x, then, we define

$$\begin{aligned}
\infty''' &:\equiv \frac{1}{0'''} :\equiv \frac{1}{0'} - \frac{1}{0''} \equiv: \infty' - \infty'' \\
{}^{0'}_{0''}df(x) &:\equiv f\left(\frac{1}{0'}\right) - f\left(\frac{1}{0''}\right) \equiv f(\infty') - f(\infty'') \\
&\equiv f\left(\frac{1}{0''} + \frac{1}{0'''}\right) - f\left(\frac{1}{0''}\right) \equiv f(\infty'' + \infty''') - f(\infty'') \\
{}^{0'}_{0''}f'(x) &:\equiv \frac{{}^{0'}_{0''}df(x)}{{}^{0'}_{0''}dx} \equiv \frac{f\left(\frac{1}{0'}\right) - f\left(\frac{1}{0''}\right)}{\frac{1}{0'} - \frac{1}{0''}} \equiv \frac{f(\infty') - f(\infty'')}{\infty' - \infty''} \\
&\equiv \frac{f\left(\frac{1}{0''} + \frac{1}{0'''}\right) - f\left(\frac{1}{0''}\right)}{\frac{1}{0'''}} \equiv \frac{f(\infty'' + \infty''') - f(\infty'')}{\infty'''} \equiv {}^{0'''}f'\left(\frac{1}{0''}\right) \equiv {}^{0'''}f'(\infty'') \\
&\equiv f'\left(\frac{1}{0''}\right) \equiv f'(\infty'') \text{ if independent of } 0'''
\end{aligned}$$

Offset derivatives are not always inverse with integrals and should only be used when original derivatives do not yield a determinate result. This is clarified further in following sections, especially Poles.

Definition of singularity

A class x is ***integrous*** if there is a bijection between the elements of x and some subset of the integers. Examples are 5, ± 1 and $2\pi\mathbb{N}$. This concept is further discussed in Class count comparisons.

A class x is ***determinate*** if it is nonempty and integrous.

A class is ***semideterminate*** if it is not empty, not determinate, and not full. An example is the interval $[-1, +1]$.

A class is ***indeterminate*** if it is full.

A function f is ***regular*** or ***analytic*** on a region A if:

- $f(x)$ and its original derivatives $f^{(n)}(x)$ are determinate and continuous for every $x \in A$;
- $f(x)$ is equal to some value of the power series $e^{(x-a)\sigma^{-1}} f(a)$ for every $x, a \in A$.

The numeristic theory of infinite series shows how most infinite series, even convergent ones, are multivalued. See [CD].

An ***ordinary point*** of a function f is any point x in a region where where f is regular. A ***singularity*** of f is any other point, i.e. where any of the above conditions fails.

A function f is ***semiregular*** on a region A if:

- $f(x)$ and $f^{(n)}(x)$ are determinate and continuous for every nonsingular x in A;
- The offset $f(x+0')$ and offset derivatives $f^{(n)}(x+0')$ are semideterminate and semiuniform for every singular x in A;
- $f(x)$ is equal to some value of the power series $e^{(x-a)\sigma^{-1}} f(a)$ for every $x, a \in A$, where the power series is calculated with original values and derivatives for nonsingular a and offset values and derivatives for singular a.

A ***semiordinary*** point of a function f is any point x in a region where f is semiregular. An ***irregularity*** of f is any other point.

Types of singularity

A singularity is ***isolated*** if there is a punctured perfinite-size neighborhood that contains no singularies. This means that the unfolding of the singularity contains only one singularity. In this chapter we discuss the following four types of isolated singularity:

- ***Removable discontinuity***: f has a removable discontinuity at p if the offset values $f(p+0')$ is uniform, but the function is discontinuous, i.e. $f(p+0') \neq f(p)$. Examples discussed below are the punctured constant function, the Kronecker delta function, and the Dirac delta function.

- ***Jump discontinuity***: f has a jump discontinuity at p if the offset value $f(p+0')$ is semiuniform but not uniform, and the function is discontinuous. Examples discussed below are the absolute value function and its derivative, the step function.

- ***Pole***: f has a pole at p if $f(x) = \dfrac{g(x)}{h(x)}$, g and h are regular, h has a root (zero) at p, and the multiplicity of the root p of h is finite. An example is the reciprocal function, discussed below in Poles.

- ***Essential singularity***: f has an essential singularity at p if it has a singularity that is not any of the above three types. An example is the function $\sin \frac{1}{x}$, discussed below in Function $\sin \frac{1}{x}$.

There are many types of nonisolated singularities. Some examples are given in Other singularities, but they are not analyzed in detail.

This chapter also gives an example of a function which is singular everywhere in conventional analysis but is regular in equipoint analysis. See Weierstrass function.

Absolute value function

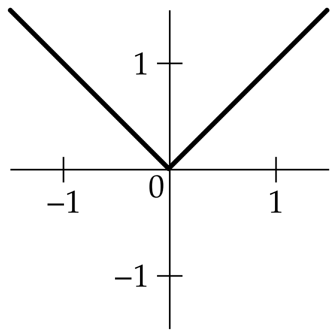

FIG. 12: Absolute value function $a(x) := |x|$

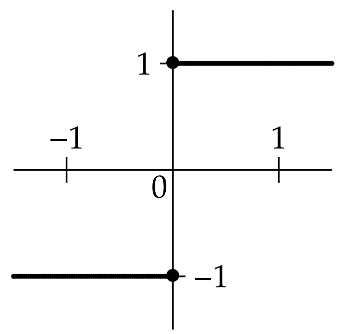

FIG. 13: Derivative of absolute value function $a(x) := |x|$, $a'(x) = \operatorname{sgn}_3 x$

The absolute value function $a(x) := |x|$ is shown in Figure 12. Its derivative is the step function shown in Figure 13. The derivative has a jump discontinuity at 0.

In the region $x > 0$, we have $a(x) = x$, $a'(x) = 1$, and the power series about any p in this region is $e^{(x-p)\sigma^{-1}} a(p) = p + (x - p) = x$. The function is therefore regular in this region. Similarly, it is regular in the region $x < 0$.

For any region that includes $x = 0$, the derivative is not uniform, since it has two values at 0:

$$\begin{aligned}
a'_{0'}(0) &\equiv \frac{a(0') - a(0)}{0'} \equiv \frac{0'}{0'} = 1 \\
a'_{-0'}(0) &\equiv \frac{a(-0') - a(0)}{-0'} \equiv \frac{0'}{-0'} = -1 \\
a_{\mathbb{R}0'} &= \pm 1.
\end{aligned}$$

$a(x)$ is therefore not regular for any region which includes $x = 0$. However, $a'(x)$ is semiuniform, and $a(x)$ is therefore semiregular everywhere.

The derivative of a similar step function is discussed below in Dirac delta function. As discussed in that section, a step function can be made analytic at the unfolded level. In the same way, the absolute value function, as the integral of a step function, can also be made unfolded analytic.

As a complex function, the derivative of a is the unit circle: for any $0' \in' 0$,

$$
\begin{aligned}
a'_{0'}(0) &\equiv \frac{a(0') - a(0)}{0'} \equiv \frac{|0'|}{0'} = \operatorname{sgn} 0' \\
a_{\mathbb{C}0'} &= e^{i\mathbb{R}}.
\end{aligned}
$$

Multivalued complex derivatives are discussed further in Complex derivative.

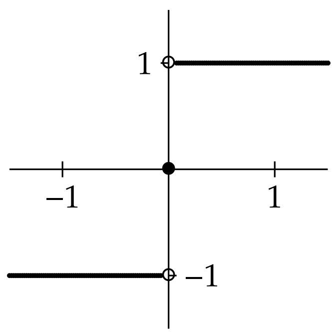

FIG. 14:
Conventional signum function $f(x) = \operatorname{sgn}_1 x$

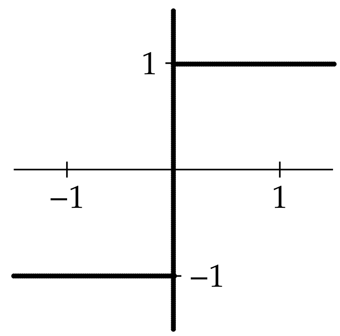

FIG. 15:
Alternate signum function $f(x) = \operatorname{sgn}_2 x$

The derivative f' is an alternate form of the signum function. The standard form, shown in Figure 14, is

$$
\operatorname{sgn}_1 x := \begin{cases} -1 & \text{for } x < 0 \\ 0 & \text{for } x = 0 \\ +1 & \text{for } x > 0. \end{cases}
$$

In [CN] we developed an alternate form, shown in Figure 15:

$$
\operatorname{sgn}_2 x := \frac{|x|}{x}
$$

There we computed $\operatorname{sgn}_2 0 = \not\infty$ because we had not yet developed unfoldings. In terms of this monograph then, the proper definition would be

$$
\operatorname{sgn}_2 x := \frac{|x|}{x}{\sim}1
$$

If we used unfolded division, which is the default for $x = 0$, then for the above derivative we have

$$
\begin{aligned}
\operatorname{sgn}_3 x &:= f'(x) = \frac{|x|}{x} \\
\operatorname{sgn}_3 0 &= \pm 1.
\end{aligned}
$$

This third form allows us to calculate signum for infinite numbers:

$$\begin{aligned}
\text{Projectively extended real numbers } (\widehat{\mathbb{R}}) &: \quad \operatorname{sgn}_3 \infty = \pm 1 \\
\text{Affinely extended real numbers } (\overline{\mathbb{R}}) &: \quad \operatorname{sgn}_3(+\infty) = +1 \\
& \quad \operatorname{sgn}_3(-\infty) = -1 \\
\text{single projectively extended complex numbers } (\widetilde{\mathbb{C}}) &: \quad \operatorname{sgn}_3 \infty = e^{i\mathbb{R}} \\
\text{double projectively extended complex numbers } (\widehat{\mathbb{C}}) &: \quad \operatorname{sgn}_3 \left(\infty e^{ir}\right) = \pm e^{ir} \\
\text{Affinely extended complex numbers } (\overline{\mathbb{C}}) &: \quad \operatorname{sgn}_3 \left(\infty e^{ir}\right) = e^{ir}
\end{aligned}$$

Punctured constant function

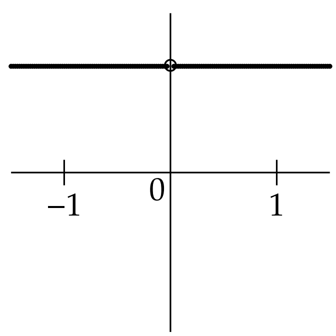

FIG. 16: Punctured constant function $p(x)$

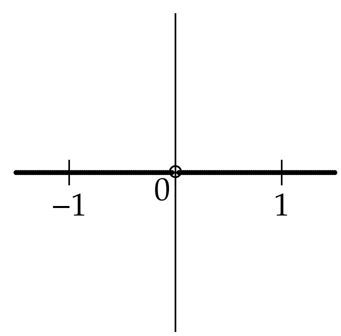

FIG. 17: Derivative of punctured constant function $p'(x)$

A function with a missing point, a point where the function has no value, is shown in Figure 16. This is a punctured constant function:

$$p(x) := \begin{cases} 1 & \text{for } x \neq 0 \\ \emptyset & \text{for } x = 0. \end{cases}$$

The function p has a removable discontinuity at 0, since $p(0') = p(0'') = 1$ for all unfolded elements $0'$ and $0''$, but $1 = p(0') \neq p(0) = \emptyset$.

The derivative p', shown in Figure 17, also has a missing point:

$$p'(0) \equiv \frac{p(0') - p(0)}{0'} \equiv \frac{1 - \emptyset}{0'} = \emptyset.$$

The offset derivative, as defined in Offset derivatives, yields a value everywhere:

$${}_{0''}p'(0) \equiv \frac{p(0') - p(0'')}{0'} \equiv \frac{1 - 1}{0' - 0''} = 0.$$

$p(x)$ is irregular for any any region that includes the singularity at $x = 0$. Since $p(0)$ is empty, it is not determinate, and $p(x)$ cannot be regular or semiregular.

Singularities at infinity

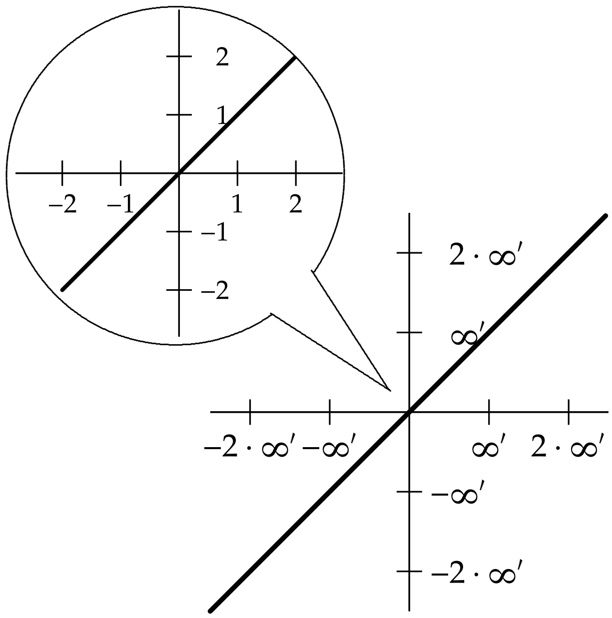

FIG. 18: Identity function $I(x) := x$
with microscope view of finite plane
within origin of unfolded infinite plane

Even a very simple function such as $I(x) := x$ has a singularity at infinite values. This function is shown in Figure 18, which shows the finite plane in a microscope and the unfolded infinite plane in the macroscope. For clarity, we use the affinely extended real numbers, and set $\infty' :\equiv \frac{1}{0'}$. The origin of the unfolded infinite line is $\frac{1}{0 \cdot 0'} = \infty \cdot (\pm\infty')$. This is a pair of points infinitely removed from the origin of the macroscope.

The original derivative at $x = +\infty$ therefore uses $\infty \cdot \infty'$ as a lower displacement, but this yields an indeterminate result:

$$^{\infty'}I'(x) \equiv \frac{(\infty + \infty') - \infty}{\infty'} \equiv \frac{(\infty - \infty) + \infty'}{\infty'} = \not\propto.$$

An offset derivative yields

$$^{\infty'}_{\infty''}I'(x) \equiv \frac{\infty'' - \infty'}{\infty'' - \infty'} = 1.$$

Since the original derivative is indeterminate but the offset derivative is determinate, $I(x)$ is only semiregular in any region that includes an infinite value.

For another example, we take the exponential function $\exp(x) := e^x$. At $x = -\infty$, an original derivative is sufficient:

$$^{\infty'}\exp'(x) \equiv \frac{e^{-\infty+\infty'} - e^{-\infty}}{\infty'} =' \frac{e^{-\infty} - e^{-\infty}}{\infty'} =' \frac{0-0}{\infty'} = 0.$$

But at $x = +\infty$, an offset derivative is required:

$$^{\infty'}\exp'(x) \equiv \frac{e^{\infty+\infty'} - e^{\infty}}{\infty'} =' \frac{e^{\infty} - e^{\infty}}{\infty'} =' \frac{\infty - \infty}{\infty'} = \not\phi,$$

$$^{\infty'}_{\infty''}\exp'(x) \equiv \frac{e^{\infty''} - e^{\infty'}}{\infty'' - \infty'} =' \frac{\infty'''}{\infty'' - \infty'} = \infty.$$

Therefore, in the affinely extended real numbers, $exp(x)$ has a singularity at $+\infty$ but not at $-\infty$, and is regular in any region that includes $-\infty$ but only semiregular in a region that includes $+\infty$.

Every singularity at an infinite value is nonisolated, since $\infty + r = \infty$ for all perfinite r, and any punctured perfinite size neighborhood of the infinite value is still within the same infinite value.

Kronecker delta function

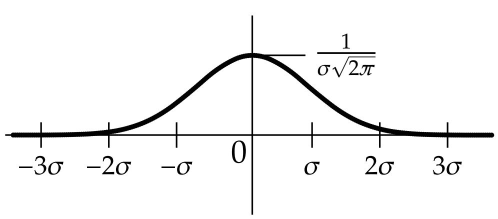

FIG. 19: Normal distribution function $\phi(x)$ with standard deviation σ

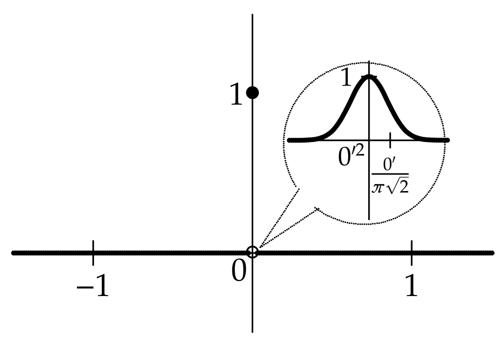

FIG. 20: Kronecker delta function $\delta_{0,x}$ as proper unfolded normal distribution

The ***Kronecker delta function*** has a very simple definition:

$$\delta_{a,b} := \begin{cases} 1 & \text{for } a = b \\ 0 & \text{for } a \neq b. \end{cases}$$

The function $\delta_{x,0}$ has a removable discontinuity at 0, since $f(0') = 0$ for all unfolded elements $0'$, but $f(0) = 1$. Put another way, $\lim_{x\to 0} \delta_{x,0}$ exists and is 0. In equipoint terms, $\lim_{x\to 0}$ means $f(x + 0')$, and to say it exists means that $f(x + 0')$ is single valued and independent of $0'$. See Limits above.

The Kronecker delta function is not regular in any region that includes the singularity at $x = 0$, since the original and offset derivatives there do not agree: the original derivative is infinite while the offset derivatives are zero. The function is not semiregular in these regions, since the power series using offset derivatives at the singularity do not equal the function. Hence the function is irregular in these regions.

The Kronecker delta function, and any function with this type of discontinuity, can be made regular at the unfolded level, by constructing a proper unfolded regular function which folds into this function. Figure 20 shows one way of doing this, by constructing a normal distribution with an infinitesimal standard deviation.

Figure 19 shows the standard normal distribution $\phi(x)$ with standard deviation σ:

$$\begin{aligned}\phi_\sigma(x) &:= \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}} \\ &= \frac{1}{\sigma\sqrt{2\pi}} \sum_{n=0}^{\infty} \frac{x^{2n}}{n!2^n\sigma^{2n}} \\ &= \frac{1}{\sigma\sqrt{2\pi}} \left(1 - \frac{x^2}{2\sigma^2} + \frac{x^4}{8\sigma^4} - \frac{x^6}{48\sigma^6} + \ldots\right).\end{aligned}$$

We can then define the Kronecker delta in terms of $\phi(x)$, as graphed in Figure 20:

$$\begin{aligned}\delta_{x,0} &:\equiv 0'\sqrt{\pi}\,\phi_{\frac{0'}{\pi\sqrt{2}}}(x) \\ &\equiv e^{-\frac{x^2}{0'^2}} \\ &\equiv \sum_{n=0}^{\infty} \frac{x^{2n}}{n!0'^{2n}} \\ &\equiv 1 - \frac{x^2}{0'^2} + \frac{x^4}{2\cdot 0'^4} - \frac{x^6}{6\cdot 0'^6} + \ldots.\end{aligned}$$

Dirac delta function

The ***Dirac delta function*** or ***unit impulse function*** $\delta(\cdot)$ has many definitions. Two qualities of δ which should follow from any definition are:

$$\delta(x) = 0 \text{ for } x \neq 0,$$

$$\int_{-\infty}^{+\infty} \delta(x)\, dx = 1.$$

These two conditions imply an infinite value for $\delta(0)$. In conventional analysis, this does not allow δ to be a function. δ is instead defined as a distribution or generalized function. Here we consider the Dirac delta to be a function with an infinite value at 0, a removable discontinuity.

We can define the delta function as the derivative of the ***Heaviside step function***, also called the ***unit step function***. This also has several definitions, but for the moment, we will use the left-continuous form:

$$H(x) := \begin{cases} 0 & \text{for } x \le 0 \\ 1 & \text{for } x > 0. \end{cases}$$

The derivative is

$$\delta_{0'}(x) :\equiv H'_{0'}(x) \equiv \frac{{}^{0'}dH(x)}{{}^{0'}dx}.$$

In this expression, ${}^{0'}dH(x) \equiv 1$ for any $0' >_{\sim 0'} 0'^2$, i.e. for any $0'$ on the right side of unfolded 0. When it is divided by ${}^{0'}dx \equiv 0'$, the result is infinite.

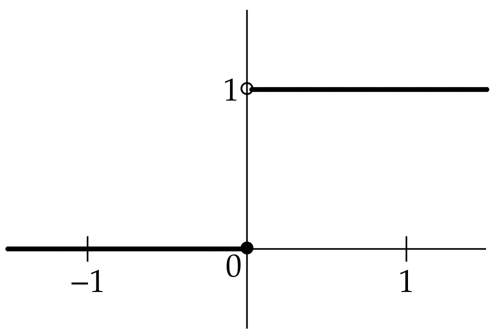

FIG. 21:
Heaviside step function $H(x)$

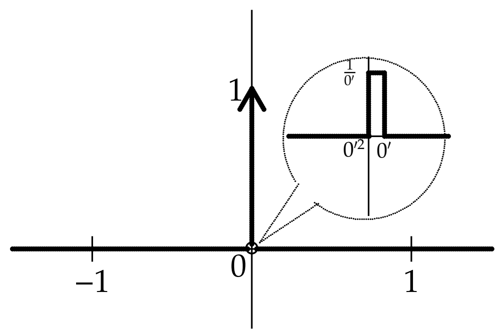

FIG. 22:
Dirac delta function $\delta(x)$

Figure 22 shows the infinite value at $\delta(0)$. The microscope in this figure expands infinitely in the x direction and *contracts* infinitely in the y direction. The rectangle in the microscope is infinitely tall and infinitely narrow, and its total area is 1.

The Dirac delta function is a proper unfolded function, i.e. is not an unfolding of any folded function. If we fold the function, we lose some property in folded arithmetic. In this case, we would have $\delta(x) = \infty$ at $x = 0$ and zero elsewhere, which would give an indeterminate value for the integral $\int_{-\infty}^{+\infty} \delta(x)\, dx$.

In the unfolded form, the properties of $\delta_0(x)$ are independent of $0'$. We can regard δ as a class of proper unfolded functions, and we can drop the subscripts and write

$$\delta(x) :\equiv H'(x) \equiv \frac{dH(x)}{dx}.$$

Figures 21 and 22 show the left-continuous form of $H(x)$ and the corresponding $\delta(x)$. There are several alternatives, a few of which are:

1. In the right-continuous form of $H(x)$, $H(0) = 1$, and the rectangle in the microscope of $\delta(0)$ is to the left of $0'^2$ instead of the right. The difference is only in the unfolded arithmetic; the folded properties of $\delta(x)$ remain the same.

2. If we define $H(0) = \frac{1}{2}$, then $H(x) = \frac{1+\operatorname{sgn} x}{2}$, and the microscope rectangle of $\delta(0)$ is half on the left and half on the right of $0'^2$. Again, this makes no difference to the folded properties of $\delta(x)$.

3. If we allow $H(x)$ to be multivalued and set $H(0) = [0,1]$, the unit interval, then the graph of $H(x)$ is a continuous path and can be parameterized with a single valued function. Since $H(0)$ is a multivalued class, then $\delta(0)$ is multivalued also, the class $\{[0,1]\delta_1(0)\}$, where $\delta_1(x)$ is the single valued $\delta(x)$ defined above. One of the members of this class, $0\delta_1(0)$, is itself multivalued, since $0\delta_1(0) \equiv 0'\mathbb{R}\frac{1}{0'} \equiv \mathbb{R}$. The other values, $\{(0,1]\delta_1(0)\}$, yield all the infinite multiples of $\delta_1(0)$ up to $\delta_1(0)$ itself. Therefore the graph of $\delta(x)$ is also a continuous path and can be parameterized with a single valued function.

4. Define $\delta(x)$ as a proper unfolded normal distribution, and $H(x)$ as its integral, as discussed below.

Under the first definition, the derivative of $\delta(x)$ can be computed in superunfolded arithmetic. We must compute the derivative at the two sides of the rectangle, first at the infinitely increasing step function at $0'^2$, and secondly at the infinitely decreasing step function at $\frac{1}{0'}$. The result, $\dfrac{d\delta(x)}{dx}$, is a second order proper unfolded function, and $\dfrac{d^{(n)}\delta(x)}{dx^{(n)}} = \dfrac{d^{(n+1)}H(x)}{dx^{(n+1)}}$, is $(n+1)$-th order proper unfolded.

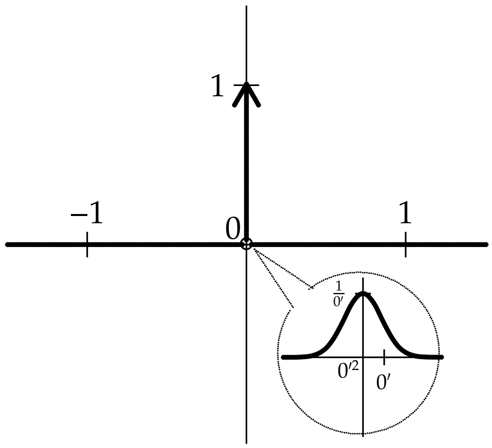

FIG. 23: Dirac delta function $\delta(x)$ as proper unfolded normal distribution

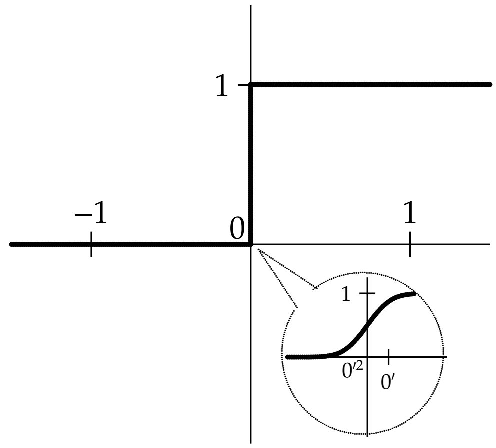

FIG. 24: Heaviside step function $H(x)$ as proper unfolded cumulative normal distribution

Like the Kronecker delta function, the Dirac delta function is not regular in any region that includes the singularity at $x = 0$, since the original and offset derivatives there do not agree, and it is not semiregular, since power series using offset derivatives at the singularity do not equal the function. Hence the function is irregular in these regions.

We made the Kronecker delta function regular at the unfolded level by constructing it as a normal distribution with an infinitely small standard deviation. A similar technique can be used with the Dirac delta function, as shown in Figure 23. In this case, we want the integral under the function to remain unity, so again we use $\sigma = \frac{0'}{\pi\sqrt{2}}$, but without any additional scaling:

$$\begin{aligned}
\delta(x) &:\equiv \phi_{\frac{0'}{\pi\sqrt{2}}}(x) \\
&= \frac{1}{0'\sqrt{\pi}} e^{-\frac{x^2}{0'^2}} \\
&= \frac{1}{0'\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{x^{2n}}{n!0'^{2n}} \\
&= \frac{1}{0'\sqrt{\pi}} \left(1 - \frac{x^2}{0'^2} + \frac{x^4}{2 \cdot 0'^4} - \frac{x^6}{6 \cdot 0'^6} + \dots\right).
\end{aligned}$$

To do the same for the Heaviside step function, we naturally choose the integral of the normal distribution, the cumulative normal distribution:

$$\begin{aligned}\Phi_\sigma(x) &:\equiv \int_{-\infty}^{x} \phi_\sigma(u)\,du \\ &= \int_{-\infty}^{x} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{u^2}{2\sigma^2}}\,du \\ &= \frac{1}{2} + \frac{1}{\sigma\sqrt{2\pi}} \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)n!2^n\sigma^{2n}} \\ &= \frac{1}{2} + \frac{1}{\sigma\sqrt{2\pi}} \left(x - \frac{x^3}{6\sigma^2} + \frac{x^5}{40\sigma^4} - \frac{x^7}{336\sigma^6} + \ldots \right).\end{aligned}$$

We can then redefine the Heaviside step function in terms of $\Phi(x)$, as graphed in Figure 24:

$$\begin{aligned}H(x) &:\equiv 0'\sqrt{\pi}\,\Phi_{\frac{0'}{\pi\sqrt{2}}}(x) \\ &\equiv \frac{1}{0'\sqrt{\pi}} \int_{-\infty}^{x} e^{-\frac{u^2}{0'^2}}\,du \\ &\equiv \frac{1}{2} + \frac{1}{0'\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)n!0'^{2n}} \\ &\equiv \frac{1}{2} + \frac{1}{0'\sqrt{\pi}} \left(x - \frac{x^3}{3\cdot 0'^2} + \frac{x^5}{10\cdot 0'^4} - \frac{x^7}{42\cdot 0'^6} + \ldots \right).\end{aligned}$$

Using the normal distribution in this way to define $\delta(x)$ has a significant advantage over previously discussed methods, in which $\delta(x) :\equiv 0 \cdot 0'$ for $x \neq 0$. With those methods, $\int_{-\infty}^{+\infty} \delta(x)\,dx$ could be more than 1. For example, $\delta(x) :\equiv 0'^2$ for $x \neq 0$, and if ∞ is unfolded as $\frac{1}{(dx)^3}$, then $\int_{-\infty}^{+\infty} \delta(x)\,dx \equiv 1 + 2 \cdot \frac{1}{0'^3} \cdot 0'^2 \cdot 0' = 3$. But when $\delta :\equiv \phi_{\frac{0'}{\pi\sqrt{2}}}(x)$, at $x = \infty$ it has the form $ke^{-\infty}$, which is smaller than any power of $0'$. Thus this form cannot affect the value of the integral, i.e. we will always have $\int_{-\infty}^{+\infty} \delta(x)\,dx = 1$.

The objection may be raised that $\int_{-\infty}^{+\infty} \delta(x)\,dx$ could mean $\int_{-e^\infty}^{+e^\infty} \delta(x)\,dx$ or some such expression. However, this integral should be rewritten as $\int_{-\infty}^{+\infty} \delta(e^u)e^u\,du$, as we generally assume that ∞ in integration limits is proportional to an integer power of the differential.

Poles

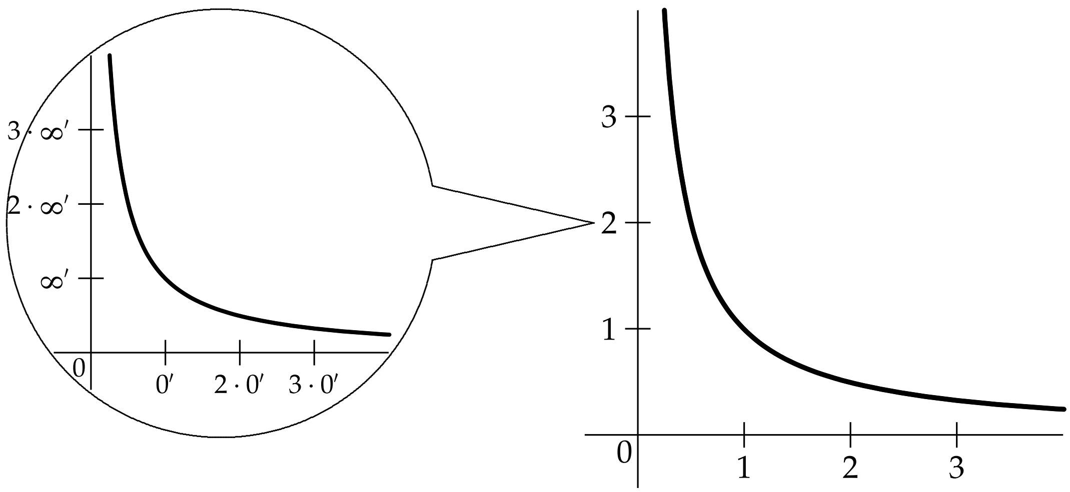

FIG. 25: Reciprocal function $r(x) := \frac{1}{x}$
with microscope view of y axis infinitely expanded
in x direction and infinitely compressed in y direction

In Types of singularity, we defined a pole of a function f as a point p such that $f(x) = \dfrac{g(x)}{h(x)}$, g and h are analytic, $h(x)$ has a root (zero) at p, and the multiplicity of the root is finite.

Here we discuss the simplest pole, the function $r(x) := \frac{1}{x}$ at the point $x = 0$. Figure 25 shows a graph of $r(x)$ and a microscope of the y-axis, which is infinitely expanded in the x direction and infinitely compressed in the y direction. In curves like the one in Figure 7, the curve becomes straight in the microscope, but in Figure 25, the curve keeps its asymptote along the vertical axis. This remains the case no matter how many times the curve is superunfolded.

This leads to an indeterminacy in the derivative:

$$
\begin{aligned}
r'(0) :\equiv \frac{r(0+0') - r(0)}{0'} &\equiv \frac{\frac{1}{0+0'} - \frac{1}{0}}{0'} \\
=' \frac{\frac{1}{0} - \frac{1}{0}}{0'} &\equiv \frac{\infty - \infty}{0'} = \not\infty.
\end{aligned}
$$

The offset derivative however is determinate:

$$r'(x+0'') :\equiv \frac{r(0''+0')-r(0'')}{0'} \equiv \frac{\frac{1}{0''+0'}-\frac{1}{0''}}{0'}$$
$$\equiv \frac{0''-0''-0'}{0'0''(0''+0'')} \equiv \frac{-1}{0''(0''+0')} =' \frac{-1}{0''^2} = \begin{cases} \infty & \text{in } \widehat{\mathbb{R}} \\ -\infty & \text{in } \overline{\mathbb{R}} \end{cases}.$$

$r(x)$ is therefore not regular in any region that includes the pole, but it is semiregular, since the offset derivatives are determinate and semiuniform, and the power series using them yields the function. For perfinite a and a zero $0'$ we have:

$$f(x) = e^{(x-a)\sigma^{-1}} f(a) = \frac{1}{a} - \frac{x-a}{a^2} + \frac{(x-a)^2}{a^3} - \ldots = \sum_{k=0}^{\infty} \frac{(-1)^k (x-a)^k}{a^{k+1}}$$
$$f(x) = e^{(x-0')\sigma^{-1}} f(0') = \frac{1}{a} - \frac{x-0'}{0'^2} + \frac{(x-0')^2}{0'^3} - \ldots = \sum_{k=0}^{\infty} \frac{(-1)^k (x-0')^k}{0'^{k+1}}$$

An indeterminacy problem also occurs in the integral. A curve like the one in Figure 8 becomes flat in the microscope, but $r(x)$ keeps its asymptote in all unfoldings. If we integrate $r(x)$ from $0'$ to a positive point p, we obtain the infinite result $\ln p - \ln 0' = +\infty$ without a problem, but if we try to integrate through the pole, we run into an indeterminacy. An attempt to integrate from $-p$ to $+p$, for example, would lead to the following:

$$\int_{-p}^{+p} r(x)dx \equiv \int_{-p}^{-0'} r(x)dx + \int_{-0'}^{0\cdot 0'} r(x)dx + \int_{0\cdot 0'}^{+0'} r(x)dx + \int_{+0}^{+p} r(x)dx$$

The indeterminacy occurs with either of the middle two pieces, $\displaystyle\int_{-0'}^{0\cdot 0'} r(x)dx$ and $\displaystyle\int_{0\cdot 0'}^{+0'} r(x)dx$. The second of these two we can see in the microscope of Figure 25 as the area under the curve from the origin $0\cdot 0'$ to $0'$. If the curve were flat, we could use a rectangle with the right side as the height, $0'\cdot\infty' \equiv 1$, but this value is clearly too small in this case. Using the left side as the height gives $0'\cdot\infty\cdot\infty' \equiv 0'\cdot\infty \equiv \not\infty$ in the projectively extended real numbers and $0'\cdot\infty\cdot\infty' \equiv 0'\cdot\infty \equiv |\not\infty|$ in the affinely extended real numbers. If we use the trapezoidal estimate, we still obtain an indeterminacy: $0'\left(\dfrac{\infty\cdot\infty'+\infty'}{2}\right) \equiv 0'\left(\dfrac{(\infty+1)\infty'}{2}\right) \equiv 0'\cdot\infty = \not\infty$ or $|\not\infty|$. Further unfoldings yield the same indeterminate result, since $0'^n \dfrac{1}{0\cdot 0'^n} = \dfrac{1}{0} = \infty$ for any n.

Any approximation to this area that involves the left endpoint gives in an indeterminacy, and any approximation that does not is inaccurate. Therefore we cannot integrate directly through this pole. The same problem occurs with any other pole.

This leaves us with two alternatives:

1. In real space, integrate piecewise, once to the left of the pole, and once to the right.

2. In complex space, integrate around the pole.

The antiderivative of $\frac{1}{x}$ is $\ln x$, but since this is imaginary for negative x, it cannot be used directly in real analysis. Instead we use the fact that $\ln x = \ln(|x| \operatorname{sgn} x) = \ln|x| + \ln \operatorname{sgn} x$ and integrate either completely on the positive side of the real axis or completely on the negative side. In this case, the $\ln \operatorname{sgn} x$ terms cancel, and the effective antiderivative is $\ln|x|$.

This also shows algebraically why integration cannot be carried out through a pole, which we saw above geometrically. Since $\operatorname{sgn} x = \not{0}$, any sum of $\ln x$ through the origin is indeterminate.

In complex analysis, the antiderivative is $\ln x$, and the path of integration is connected. This is discussed in detail in Complex poles.

Axial function

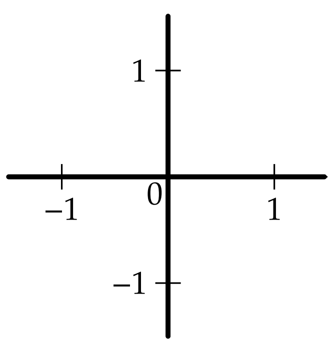

FIG. 26:
Axial function $A(x)$

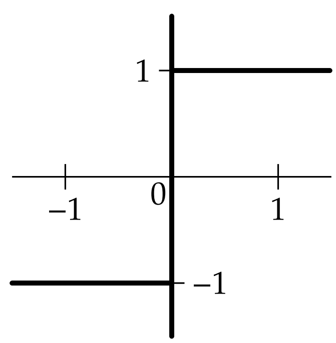

FIG. 27:
Sample of integral of axial function $A^{(-1)}(x)$

Figure 26 shows the ***axial function*** $A(x) := \frac{0}{x}$, a multivalued function whose graph coincides with both the horizontal and vertical axes:

$$A(x) = \begin{cases} 0 & \text{for } x \neq 0 \\ \not{0} & \text{for } x = 0 \end{cases}$$

As a multivalued function, the continuity of $A(x)$ is of three types, as defined in Continuity:

- ***Classwise continuity***: The axial function is classwise discontinuous at 0 because $A(0') = \{0\}$ and $A(0) = \not\infty$.

- ***Conjunctive continuity***: The axial function is conjunctively discontinuous at 0 because $\{0\}$ cannot be mapped bijectively to $\not\infty$.

- ***Disjunctive continuity***: The axial function is disjunctively continuous at 0 because $A(0') = 0 \in \not\infty = A(0)$.

The singularity of $A(x)$ at 0 is a removable singularity. $A(x)$ is not regular in any region that includes the singularity at $x = 0$, since $A(0)$ is indeterminate. The function is not semiregular in these regions, since the power series using offset derivatives at the singularity equal zero. Hence the function is irregular in these regions.

The derivative of $A(x)$ is $A(x)$, since $A'(x) = \dfrac{dA(x)}{dx} = \dfrac{d}{dx}\dfrac{0}{x} = \dfrac{0}{-x^2} = \dfrac{0}{x^2} = \dfrac{0}{x}$. Since the original derivative is indeterminate at 0, as it is at a pole, the Fundamental Theorems of Calculus do not hold here. See Poles for a detailed discussion of this point.

To compute the integral of $A(x)$:

- In real space, integrate piecewise on the left and right sides. This allows us to choose independent constants of integration for the right and left integrals. One possible integral of $A(x)$ is shown in Figure 27.

- In complex space, integrate around the singularity. See Complex axial function.

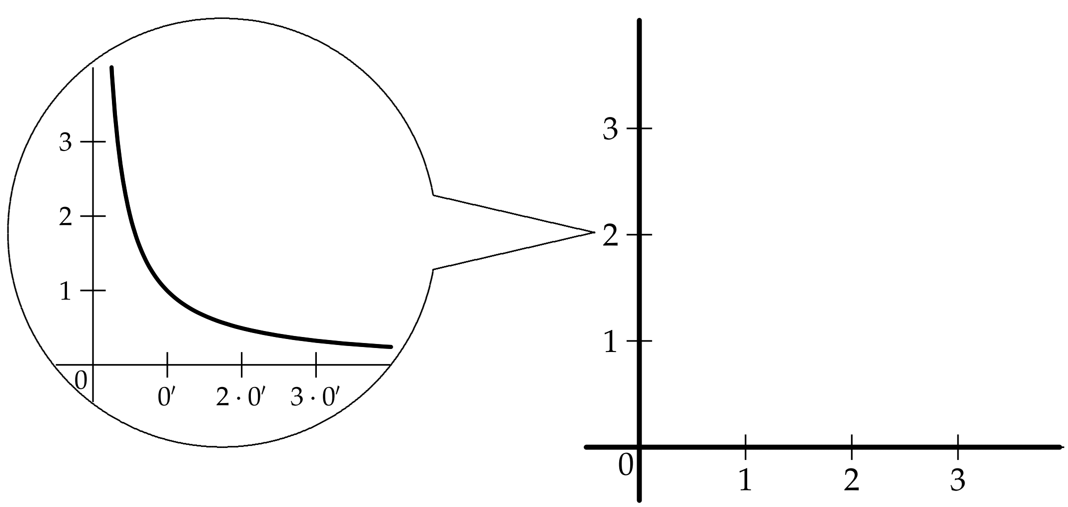

FIG. 28: Axial function $A(x)$
as proper unfolded reciprocal function

The axial function can be made semiregular at the unfolded level by choosing a proper unfolding. In this case, we use the proper unfolding $A(x) :\equiv \dfrac{0'}{x}$, shown in Figure 28.

Function $\sin \frac{1}{x}$

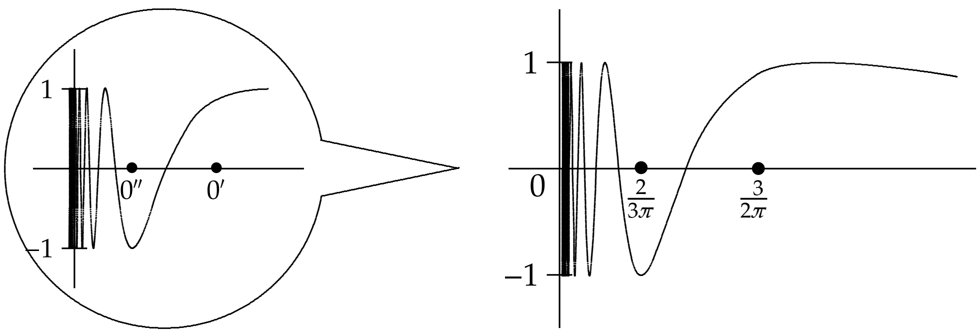

FIG. 29: Essential singularity of $S(x) := \sin \frac{1}{x}$ with microscope view of y axis infinitely expanded in x direction and unchanged scale in y direction

The function $S(x) := \sin \dfrac{1}{x}$ is graphed in Figure 29. Like the pole in Figure 25, $S(x)$ at 0 maintains its shape from macroscope to microscope. Within the microscope, $S(0)$ takes on every value within the interval $[-1, +1]$. Algebraically we can see this by observing that $\infty + r = \infty$ for every real perfinite r, so $\sin \infty = S(0) = [-1, +1]$.

Since $S(0)$ is not determinate, it is a singularity. An offset value $S(0')$ can be any point within $[-1, +1]$, so the offset values are not uniform or semiuniform, and the singularity is not a removable discontinuity or jump discontinuity.

The following calculation shows that the singularity is also not a pole. As defined in Types of singularity, a pole of a function f is a point p such that $f(x) = \dfrac{g(x)}{h(x)}$, g and h are analytic, $h(x)$ has a root (zero) at p, and the multiplicity of the root is finite. The following converts the power series for $S(x)$ to a fraction, using the ***Pochhammer symbol*** $(n)_r$ to denote the ***falling factorial*** function $\dfrac{n!}{(n-r)!}$, for which $(n)_n = (n)_{n-1} = n!$, $(n)_1 = n$,

$(n)_0 = 1$.

$$\begin{aligned}
\sin\frac{1}{x} &= \frac{1}{x} - \frac{1}{3!x^3} + \frac{1}{5!x^5} - \dots \\
&= \frac{3!x^2 - 1}{3!x^3} + \frac{1}{5!x^5} - \dots \\
&= \frac{5!x^4 - (5)_2 x^2 + 1}{5!x^5} - \frac{1}{7!x^7} + \dots \\
&= \frac{7!x^6 - (7)_4 x^4 + (7)_2 x^2 - 1}{7!x^7} + \frac{1}{9!x^9} - \dots \\
&= \frac{\sum_{k=0}^{\infty'} (-1)^k (2\infty' + 1)_{2(\infty'-k)} x^{2(\infty'-k)}}{(2\infty' + 1)! x^{2\infty'+1}}
\end{aligned}$$

The denominator of the final fraction has a root at $x = 0$ of infinite multiplicity. Since the singularity is not a removable discontinuity, jump discontinuity, or pole, it is an essential singularity.

The derivative $S'(x) = -\frac{\cos\frac{1}{x}}{x^2}$ is indeterminate at the singularity, but the antiderivative $\int S(x)dx = x\sin x + \int_{\frac{1}{x}}^{\infty} \frac{\cos t}{t} dt + k$ is determinate.

Weierstrass function

Weierstrass gave an example of a class of functions that are continuous everywhere but differentiable nowhere in conventional analysis. We now examine a function which is simpler but still shows the essential features of the original Weierstrass functions:

$$W(x) := \sum_{n=0}^{\infty} \frac{\sin(2^n x)}{2^n} = \sin x + \frac{\sin 2x}{2} + \frac{\sin 4x}{4} + \dots$$

Conventional analysis cannot differentiate this function because

$$\liminf_{\delta\to 0} \frac{W(x+\delta) - W(x)}{\delta} > \limsup_{\delta\to 0} \frac{W(x+\delta) - W(x)}{\delta},$$

at every point, and thus

$$W'(x) = \lim_{\delta\to 0} \frac{W(x+\delta) - W(x)}{\delta}$$

does not exist.

Equipoint analysis is not have any such requirement. It requires only that a function be defined on an interval. Then, using an unfolded $0'$-level arithmetic, it computes

$$W'(x) = \frac{W(x+0') - W(x)}{0'}.$$

The numeristic approach to infinite series, developed in [CD], combined with the algebraic definition of differentials developed here, allows distribution over infinite sums, so

$$W'(x) = \sum_{n=0}^{\infty} \cos(2^n x).$$

Figures 30 and 31 show $W(x)$ and $W'(x)$.

While the derivative $W'(x)$ is can be calculated at every point within an unfolding of x and is single valued everywhere, the offset values $W(x + 0')$ vary with each $0'$, so $W(x)$ is not continuous anywhere. Thus every point is a singularity, each of which is nonisolated.

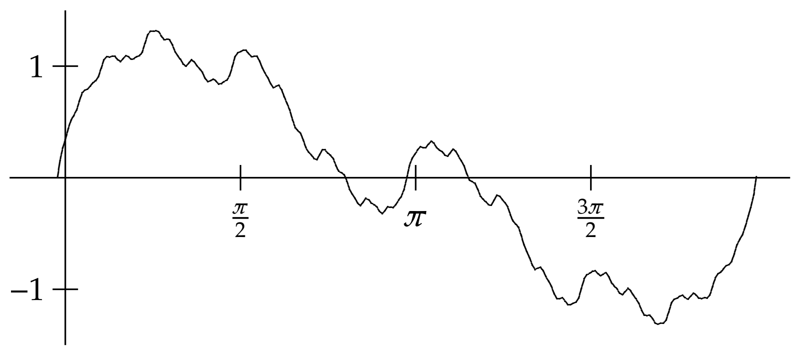

FIG. 30:
Weierstrass-like function $W(x)$

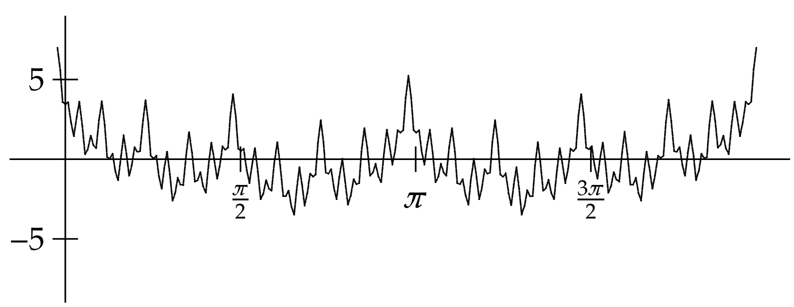

FIG. 31:
Derivative of Weierstrass-like function

Fourier transform

The Fourier transform, in the unitary asymmetric form, maps a function $f(x)$ to the transform $\widehat{f}(k)$ by

$$\widehat{f}(k) = \int_{-\infty}^{+\infty} f(x) e^{-2\pi ikx}\, dx.$$

We will not redevelop Fourier theory here but only note the Fourier transform of some proper unfolded functions. These are variations of the two elementary transforms

$\mathbf{f}(\mathbf{x})$	$\widehat{\mathbf{f}}(\mathbf{k})$
1	$\delta(k)$
$e^{2\pi iax}$	$\delta(k-a)$

The proper unfolded variations are

$\mathbf{f}(\mathbf{x})$	$\widehat{\mathbf{f}}(\mathbf{k})$
0	$0'\delta(k) = \delta_{k,0}$
$0e^{2\pi iax}$	$0'\delta(k-a) = \delta_{k,a}$

These two results assume that the width and height of $\delta(0)$ are $0'$ and $\frac{1}{0'}$. These connect the Dirac delta $\delta(x-a)$ with the Kronecker delta $\delta_{x,a}$.

Other singularities

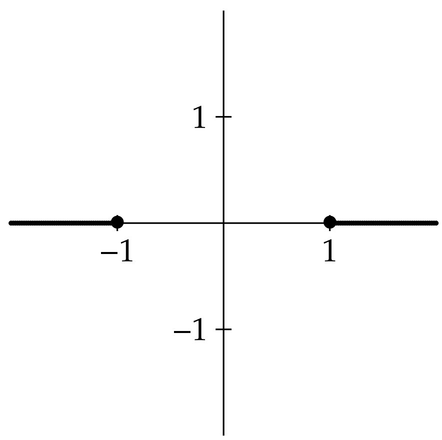

FIG. 32:
Gapped interval in $G(x)$

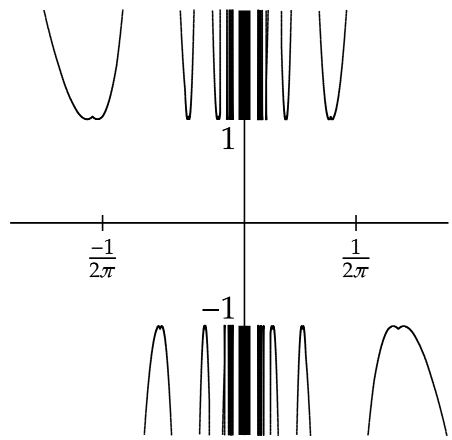

FIG. 33:
Accumulation point
of poles of $C(x) = \csc \frac{1}{x}$

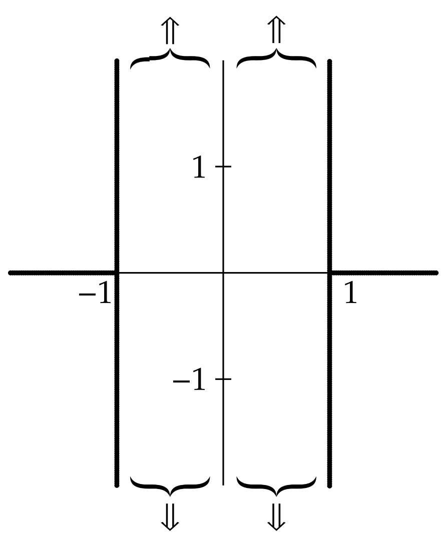

FIG. 34:
Interval of poles
in $J(x) = \frac{1}{x^\infty}$

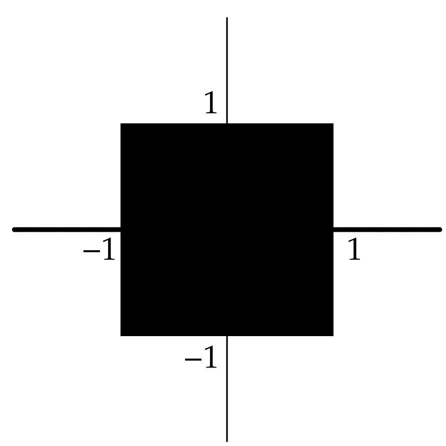

FIG. 35:
Interval of intervals
in $T(x) = \sin \frac{1}{x^\infty}$

Nonisolated singularities. Examples of nonisolated singularities are shown in Figures 32 through 35.

- Figure 32: A gapped interval in the function
$$G(x) := \begin{cases} 0 & \text{for } |x| \geq 1 \\ \emptyset & \text{for } |x| < 1 \end{cases}$$

- Figure 33: An accumulation point of poles at $x = 0$ in the function $C(x) := \csc \dfrac{1}{x}$. Every neighborhood around $x = 0$, and the unfolded point itself, has an infinite number of poles.

- Figures 34: An interval of poles in the function $J(x) := \dfrac{1}{x^\infty}$. For every $x \in [-1, +1]$, $J(x)$ is a pole. For $x = \pm 1$, $J(x) = \not\infty$, and elsewhere $J(x) = 0$.

- Figure 35: An interval of intervals in the function $T(X) = \sin \dfrac{1}{x^\infty}$. For every point $x \in [-1, +1]$, $T(x) = [-1, +1]$. Elsewhere, $T(x) = 0$.

- Rational test function. See Using class counts in derivatives and integrals.

Complex singularities. The following are analyzed in the Complex functions chapter.

- Complex poles, the complex ananlogs of real poles described above.

- The complex axial function, the complex analog of the real axial function described above.

- The complex function $e^{\frac{1}{x}}$, which includes the real function $\sin \frac{1}{x}$ described above.

COMPLEX FUNCTIONS

Complex derivative

The complex derivative is similar to the real derivative but allows folded and unfolded complex numbers and extended complex numbers and functions.

It is single valued and finite if the real and imaginary partial derivatives are single valued, finite, and analytic. It may be multivalued otherwise. For example, at $x = 0$, for real x,

$$\frac{d|x|}{dx} = \pm 1,$$

while for complex z,

$$\frac{d|z|}{dz} = e^{i\mathbb{R}}.$$

The general complex derivative (possibly multivalued, infinite, and/or non-analytic) is as follows. For a complex function $f(z)$ we first define

$$\begin{aligned} f(z) &\equiv \operatorname{Re} f(\operatorname{Re} z + i \operatorname{Im} z) + i \operatorname{Im} f(\operatorname{Re} z + i \operatorname{Im} z) \\ &\equiv g(x, y) + i h(x, y). \end{aligned}$$

We then have

$$\begin{aligned} \frac{{}^{0'}df(z)}{{}^{0'}dz} &\equiv \frac{g(\operatorname{Re}(z + 0'), \operatorname{Im}(z + 0')) - g(\operatorname{Re} z, \operatorname{Im} z)}{0'} \\ &\quad + i \frac{h(\operatorname{Re}(z + 0'), \operatorname{Im}(z + 0')) - h(\operatorname{Re} z, \operatorname{Im} z)}{0'} \\ &\equiv \frac{g(x + \operatorname{Re} 0', y + \operatorname{Im} 0') - g(x, y)}{0'} \\ &\quad + i \frac{h(x + \operatorname{Re} 0', y + \operatorname{Im} 0') - h(x, y)}{0'} \end{aligned}$$

$$
\begin{aligned}
&\equiv \frac{g(x+\operatorname{Re}0', y+\operatorname{Im}0') - g(x, y+\operatorname{Im}0')}{\operatorname{Re}0'} \cdot \frac{\operatorname{Re}0'}{0'} \\
&+ \frac{g(x, y+\operatorname{Im}0') - g(x,y)}{\operatorname{Re}0'} \cdot \frac{\operatorname{Re}0'}{0'} \\
&+ i\frac{h(x+\operatorname{Re}0', y+\operatorname{Im}0') - h(x, y+\operatorname{Im}0')}{\operatorname{Im}0'} \cdot \frac{\operatorname{Im}0'}{0'} \\
&+ i\frac{h(x, y+\operatorname{Im}0') - h(x,y)}{\operatorname{Im}0'} \cdot \frac{\operatorname{Im}0'}{0'} \\
&\equiv \frac{\partial g(x,y)}{\partial x} \cdot \frac{\operatorname{Re}0'}{0'} + \frac{\partial g(x,y)}{\partial y} \cdot \frac{\operatorname{Im}0'}{0'} \\
&+ i\frac{\partial h(x,y)}{\partial x} \cdot \frac{\operatorname{Re}0'}{0'} + i\frac{\partial h(x,y)}{\partial y} \cdot \frac{\operatorname{Im}0'}{0'} \\
&\equiv \frac{\partial g(x,y) + i\partial h(x,y)}{\partial x} \cdot \frac{\operatorname{Re}0'}{0'} + \frac{\partial g(x,y) + i\partial h(x,y)}{\partial y} \cdot \frac{\operatorname{Im}0'}{0'} \\
&\equiv \frac{\partial f(z)}{\partial \operatorname{Re} z} \cdot \frac{\cos\arg 0'}{\operatorname{sgn} 0'} + \frac{\partial f(z)}{\partial \operatorname{Im} z} \cdot \frac{\sin\arg 0'}{\operatorname{sgn} 0'}.
\end{aligned}
$$

If f is analytic, then this becomes

$$
\begin{aligned}
\frac{df(z)}{dz} &\equiv \frac{\partial f(z)}{\partial \operatorname{Re} z} \cdot \frac{\cos\arg 0'}{\operatorname{sgn} 0'} + i\frac{\partial f(z)}{\partial \operatorname{Re} z} \cdot \frac{\sin\arg 0'}{\operatorname{sgn} 0'} \\
&\equiv \frac{\partial f(z)}{\partial \operatorname{Re} z} \cdot \frac{\operatorname{Re}0' + i\operatorname{Im}0'}{0'} \\
&\equiv \frac{\partial f(z)}{\partial \operatorname{Re} z} \cdot \frac{0'}{0'} \\
&\equiv \frac{\partial f(z)}{\partial \operatorname{Re} z} \\
&\equiv -i\frac{\partial f(z)}{\partial \operatorname{Im} z} \cdot \frac{\cos\arg 0'}{\operatorname{sgn} 0'} + \frac{\partial f(z)}{\partial \operatorname{Im} z} \cdot \frac{\sin\arg 0'}{\operatorname{sgn} 0'} \\
&\equiv -i\frac{\partial f(z)}{\partial \operatorname{Im} z}.
\end{aligned}
$$

As an example of the general complex derivative, let $f(z) = 3\operatorname{Re} z + 2i\operatorname{Im} z$, which is not analytic. We then have

$$
\frac{{}^{0'}df(z)}{{}^{0'}dz} = \frac{3\cos\arg 0' + 2i\sin\arg 0'}{\operatorname{sgn} 0'}.
$$

Letting $\theta := \arg 0'$, this becomes

$$\begin{aligned}\frac{{}^{0'}df(z)}{{}^{0'}dz} &\equiv \frac{3\cos\theta + 2i\sin\theta}{e^{i\theta}} \\ &\equiv [\cos^2\theta + 2] - i[\cos\theta\sin\theta]\end{aligned}$$

Letting $x := \operatorname{Re}\frac{df}{dz}$ and $y := \operatorname{Im}\frac{df}{dz}$ gives

$$\begin{aligned}\left(x - \frac{5}{2}\right)^2 &= -\cos^2\theta\sin^2\theta + \frac{1}{4} \\ \left(x - \frac{5}{2}\right)^2 + y^2 &= \frac{1}{4}.\end{aligned}$$

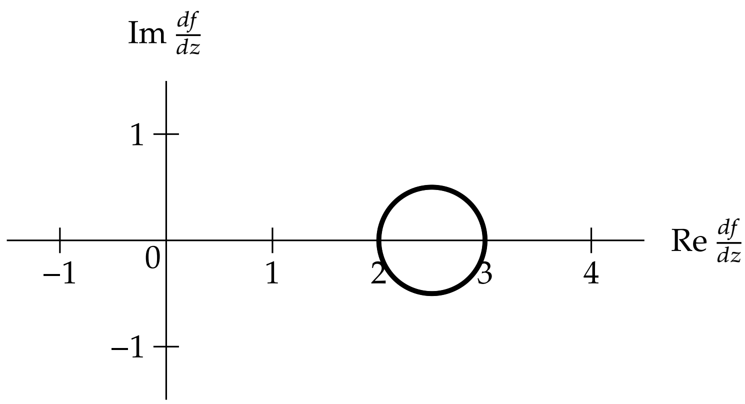

FIG. 36:
Derivative of $f(z) = 3\operatorname{Re} z + 2i\operatorname{Im} z$

The derivative is the class of all points on a circle with radius $\frac{1}{2}$ and centered on $\frac{5}{2}$, shown in Figure 36. Since the partial derivatives are constant with respect to z, so is the total derivative. While the derivative does not depend on z, it does depend on $0'$. When $\arg 0' = 0$, the derivative is 3, but when $\arg 0' = \frac{\pi}{2}$, the derivative is 2.

The Cauchy integral formula

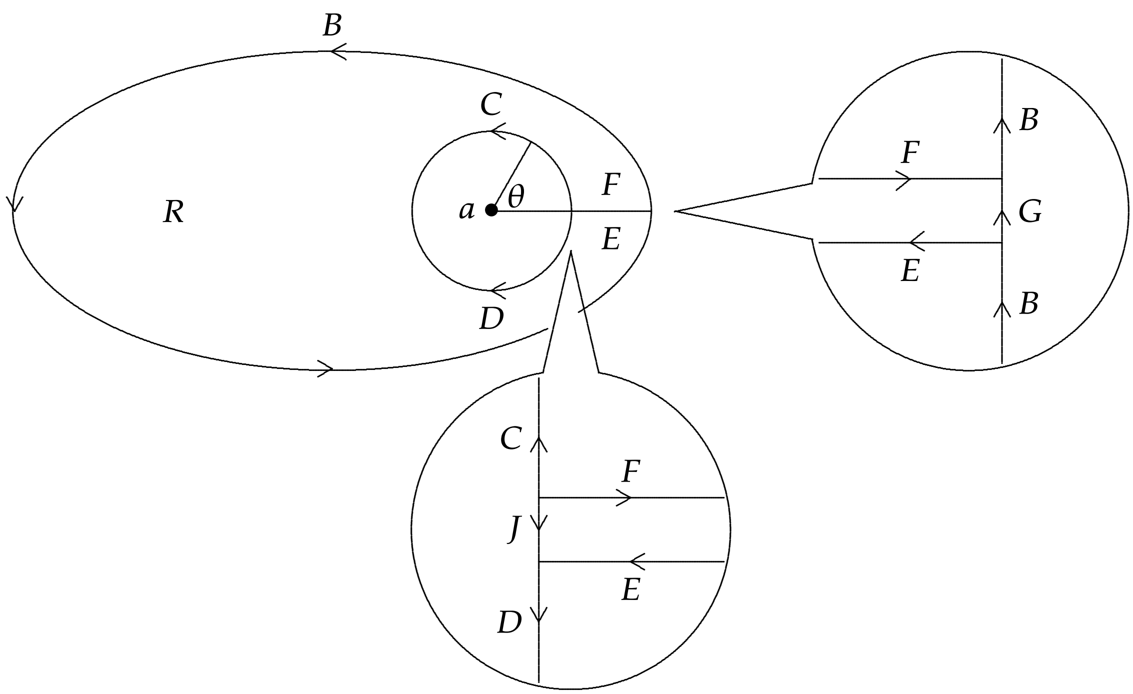

FIG. 37:
Contours for path independence
around a singularity

The Cauchy integral formula is a theorem of complex analysis that is conventionally proved with limits. Below is an equipoint proof. First we derive a preliminary theorem which is also used in the derivation of Laurent series coefficients.

Given a function f that is analytic within a region R with boundary B, with the possible exception that f is not analytic at some point $a \in R$, and any closed path C within R that goes once around a, then $\int_B f(z)\,dz = \int_C f(z)\,dz$, assuming that we integrate along B and C in the same direction.

PROOF. We start by drawing the contours shown in Figure 37:

- Without loss of generality, assume B is directed counterclockwise.
- Draw a path D coincident with C but directed clockwise.
- Draw a directed line E from any point on B to any point on C, and line F, separated from line E by a distance of $0'$, directed out from C.

- Let G be the infinitesimally short portion of B between E and F, and let H be the portion of B with G removed. Then $H :\equiv B \setminus G =' B$.

- Let J be the infinitesimally short portion of D between E and F, and let K be the portion of D with J removed. Then $K :\equiv D \setminus J =' D$.

- Let L be the concatenation of, in order, H, E, K, and F. That is, start from the point where F meets B, go almost all the way around B to E, go in on E to D, go almost all the way around D to F, and go out on F to the starting point at B.

L is a closed contour which does not include a. By the Cauchy-Goursat integral theorem, we then have

$$\begin{aligned}
0 &= \int_L f(z)\,dz \\
&\equiv \int_H f(z)\,dz + \int_E f(z)\,dz + \int_K f(z)\,dz + \int_F f(z)\,dz \\
&\equiv \int_H f(z)\,dz + \int_K f(z)\,dz \\
&= \int_B f(z)\,dz + \int_D f(z)\,dz \\
&\equiv \int_B f(z)\,dz - \int_C f(z)\,dz,
\end{aligned}$$

or

$$\int_B f(z)\,dz = \int_C f(z)\,dz. \ \Box$$

We note that the curves B and C can be finite, infinite, or infinitesimal, with lengths of E and F to match, and the separators G and J infinitesimal compared to B and C.

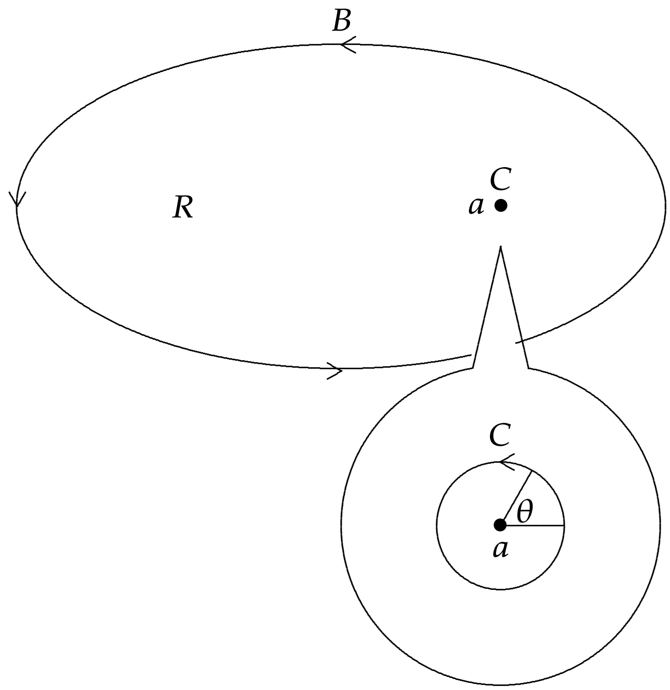

FIG. 38:
Contours for proof of the
Cauchy integral formula

CAUCHY INTEGRAL FORMULA: Given a function $f(z)$ that is analytic in a simply connected region R and on its boundary B, and given a point $a \in R$, then

$$\boldsymbol{f(a) = \frac{1}{2\pi i}\int_B \frac{f(z)\,dz}{z-a}.}$$

PROOF. We draw contours as in Figure 38. Without loss of generality, we again assume B is directed counterclockwise. We draw an infinitesimal circle C, of diameter $0'$, also directed counterclockwise, around a. By the previous theorem, the integral around the boundary B equals the integral around the circle C. To integrate around C, since it is infinitesimal, we make the substitution

$$z \equiv a + 0'e^{i\theta}$$
$$dz \equiv 0'ie^{i\theta}$$

and compute

$$
\begin{aligned}
\int_C \frac{f(z)\,dz}{z-a} &\equiv \int_C \frac{f\left(a+0'ie^{i\theta}\right)0'ie^{i\theta}d\theta}{0'e^{i\theta}} \\
&\equiv i\int_C f\left(a+0'ie^{i\theta}\right)d\theta \\
&= if(a)\int_C d\theta \\
&= 2\pi i f(a)
\end{aligned}
$$

or

$$
f(a) = \frac{1}{2\pi i}\int_B \frac{f(z)\,dz}{z-a}. \ \square
$$

We now make the point a variable and rewrite this theorem as

$$
f(z) = \frac{1}{2\pi i}\int_B \frac{f(w)\,dw}{w-z}.
$$

CAUCHY INTEGRAL FORMULA FOR DERIVATIVES: **Given the same conditions as in the previous theorem,**

$$
\boldsymbol{f^{(n)}(z) = \frac{n!}{2\pi i}\int_B \frac{f(w)\,dw}{(w-z)^{n+1}}.}
$$

PROOF. Since an integral is an infinite series and a derivative is a quotient difference, and since we establish in the numeristic theory of infinite series [CD] that we can handle them much as we do finite series, e.g. they commute, associate, and distribute as finite series do, we can calculate simply:

$$
\begin{aligned}
\frac{df(z)}{dz} &= \frac{d}{dz}\frac{1}{2\pi i}\int_B \frac{f(w)\,dw}{w-z} \\
&\equiv \frac{1}{2\pi i0'}\int_B \frac{f(w)\,dw}{w-(z-0')} - \frac{1}{2\pi i0'}\int_B \frac{f(w)\,dw}{w-z} \\
&\equiv \frac{1}{2\pi i0'}\int_B \left[\frac{f(w)}{w-(z-0')} - \frac{f(w)}{w-z}\right] dw \\
&\equiv \frac{1}{2\pi i}\int_B \frac{d}{dz}\frac{f(w)}{w-z}\,dw \\
&= \frac{1}{2\pi i}\int_B \frac{f(w)\,dw}{(w-z)^2}.
\end{aligned}
$$

Further differentiations yield the theorem. $\square$

Taylor and Laurent series

The numeristic theory of infinite series [CD] also establishes that certain series, when summed through extended numersitic arithmetic, are valid not only where they converge, but also where they diverge. The following is one such series, which is valid everywhere in the complex plane, even though it converges only within the unit circle:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots = \sum_{k=0}^{\infty} x^k$$

We now derive two alternate forms of this series, which we will use in the proofs of the following two theorems:

$$\begin{aligned}
\frac{1}{w-z} &\equiv \frac{1}{(w-a)-(z-a)} \\
&\equiv \frac{\frac{1}{w-a}}{1-\frac{z-a}{w-a}} \\
&\equiv \frac{1}{w-a}\left[1 + \frac{z-a}{w-a} + \left(\frac{z-a}{w-a}\right)^2 + \left(\frac{z-a}{w-a}\right)^3 + \ldots\right] \\
&\equiv \frac{1}{w-a} + \frac{z-a}{(w-a)^2} + \frac{(z-a)^2}{(w-a)^3} + \frac{(z-a)^3}{(w-a)^4} + \ldots, \\
\frac{-1}{w-z} &\equiv \frac{1}{(z-a)-(w-a)} \\
&\equiv \frac{\frac{1}{z-a}}{1-\frac{w-a}{z-a}} \\
&\equiv \frac{1}{z-a}\left[1 + \frac{w-a}{z-a} + \left(\frac{w-a}{z-a}\right)^2 + \left(\frac{w-a}{z-a}\right)^3 + \ldots\right] \\
&\equiv \frac{1}{z-a} + \frac{w-a}{(z-a)^2} + \frac{(w-a)^2}{(z-a)^3} + \frac{(w-a)^3}{(z-a)^4} + \ldots.
\end{aligned}$$

TAYLOR SERIES COEFFICIENTS: Given a function $f(z)$ that is analytic within a simply connected region R and a point $a \in R$, then, for any $z \in R$:

$$\begin{aligned}
f(z) &= f(a) + (z-a)f'(a) + (z-a)^2\frac{f''(a)}{2!} + (z-a)^3\frac{f^{(3)}(a)}{3!} + \ldots \\
&= \sum_{k=0}^{\infty}(z-a)^k\frac{f^{(k)}(a)}{k!}.
\end{aligned}$$

PROOF. Let C be any closed path around a. C could be infinitesimal. Using the first of the above identities and the n-th derivative Cauchy integral formula, we compute

$$\begin{aligned}
f(z) &= \frac{1}{2\pi i}\int_C \frac{f(w)\,dw}{w-z} \\
&= \frac{1}{2\pi i}\int_C \frac{f(w)\,dw}{w-a} + \frac{z-a}{2\pi i}\int_C \frac{f(w)\,dw}{(w-a)^2} + \frac{(z-a)^2}{2\pi i}\int_C \frac{f(w)\,dw}{(w-a)^3} + \ldots \\
&= f(a) + (z-a)f'(a) + (z-a)^2\frac{f''(a)}{2!} + (z-a)^3\frac{f^{(3)}(a)}{3!} + \ldots \\
&= \sum_{k=0}^{\infty}(z-a)^k\frac{f^{(k)}(a)}{k!}. \;\square
\end{aligned}$$

LAURENT SERIES COEFFICIENTS: Given a function $f(z)$ that is analytic within a region R between an outer boundary B (which may be infinite) and an inner boundary C (which may be infinitesimal), and given a point a inside C (so that $a \notin R$), then, for any $z \in R$:

$$\boldsymbol{f(z) = \sum_{k=-\infty}^{\infty}\frac{(z-a)^k}{2\pi i}\int_B \frac{f(w)\,dw}{(w-a)^{k+1}}.}$$

PROOF. Following Figure 37, let L be the concenation of, in order, H, E, K, and F. Let z be any point in R and let $g(w) :\equiv \frac{f(w)}{w-z}$. Then

$$\begin{aligned}
\int_L g(w)\,dw &\equiv \int_H g(w)\,dw + \int_E g(w)\,dw + \int_K g(w)\,dw + \int_F g(w)\,dw \\
&\equiv \int_H g(w)\,dw + \int_K g(w)\,dw \\
&= \int_B g(w)\,dw + \int_D g(w)\,dw \\
&\equiv \int_B g(w)\,dw - \int_C g(w)\,dw.
\end{aligned}$$

Since L encloses z, and using both of the above identies,

$$
\begin{aligned}
f(z) &= \frac{1}{2\pi i}\int_L g(w)\,dw \\
&= \frac{1}{2\pi i}\int_B g(w)\,dw - \frac{1}{2\pi i}\int_C g(w)\,dw \\
&= \frac{1}{2\pi i}\int_B \frac{f(w)\,dw}{w-z} - \frac{1}{2\pi i}\int_C \frac{f(w)\,dw}{w-z} \\
&= \frac{1}{2\pi i}\int_B \frac{f(w)\,dw}{w-a} + \frac{z-a}{2\pi i}\int_B \frac{f(w)\,dw}{(w-a)^2} + \frac{(z-a)^2}{2\pi i}\int_B \frac{f(w)\,dw}{(w-a)^3} + \dots \\
&\quad + \frac{1}{2\pi i(z-a)}\int_C f(w)\,dw + \frac{1}{2\pi i(z-a)^2}\int_C f(w)(w-a)\,dw \\
&\quad + \frac{1}{2\pi i(z-a)^3}\int_C f(w)(w-a)^2\,dw + \dots \\
&= \sum_{k=0}^{\infty} \frac{(z-a)^k}{2\pi i}\int_B \frac{f(w)\,dw}{(w-a)^{k+1}} + \sum_{k=1}^{\infty} \frac{1}{2\pi i(z-a)^k}\int_C f(w)(w-a)^{k-1}\,dw.
\end{aligned}
$$

The integrals in the second sum do not enclose any singuarities, so the path B can be used instead of C. Hence

$$
f(z) = \sum_{k=-\infty}^{\infty} \frac{(z-a)^k}{2\pi i}\int_B \frac{f(w)\,dw}{(w-a)^{k+1}}. \square
$$

Complex poles

In Types of singularity, we defined a pole of a function f as a point x such that $f(x) = \dfrac{g(x)}{h(x)}$, g and h are analytic, $h(x)$ has a root (zero) at x, and the multiplicity of the root is finite.

In numeristics, every elementary function is defined over the whole complex plane, even at its singularities. Since a function may be defined at a singularity, the domain of such a function may still be simply connected.

As in conventional analysis, we transform a contour by parameterizing it into a directed real interval.

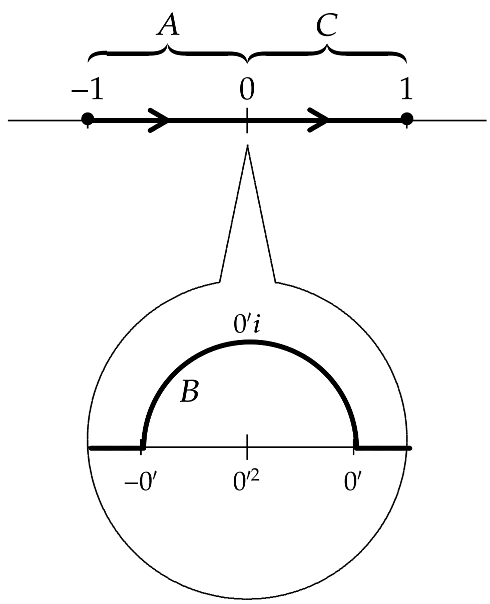

FIG. 39:
Contour G consisting of
three portions A, B, C

In real analysis, as described in real poles, the effective antiderivative of $\dfrac{1}{x}$ is $\ln|x|$, which assumes that we integrate either completely on the negative side of the real axis or completely on the positive side.

In complex space, the antiderivative is $\ln x$, and the path of integration is connected. A path which includes an infinitesimal region is shown in Figure 39. This path, contour G, has three portions that link two points, -1 and $+1$:

A: A path along the real axis from -1 to $-0'$,

B: An infinitesimal semicircle around the origin from $-0'$ to $+0'$, and

C: A path along the real axis from $+0'$ to $+1$.

In real space, we must omit portion B and use the effective antiderivative $\ln|x|$. Although this path includes all but one point of the real interval $[-1,+1]$, the resulting integral of $\frac{1}{x}$ differs from the complex version:

$$\int_{-1}^{+1}\frac{dx}{x} \equiv \int_{A+C}\frac{dx}{x} \equiv \ln|x|\Big|_{-1}^{-0'} + \ln|x|\Big|_{+0'}^{+1} \equiv [-\infty' - 0] + [0 + \infty'] = 0.$$

In complex space, we can include portion B and use the actual antiderivative $\ln x$:

$$\begin{aligned}\int_G \frac{dz}{z} &\equiv \int_{A+B+C}\frac{dz}{z} \equiv \int_{-1}^{-0'}\frac{dz}{z} + \int_{\pi}^{0} d\theta + \int_{1}^{0'}\frac{dz}{z} \equiv \ln z\Big|_{-1}^{-0'} + \theta\Big|_{\pi}^{0} + \ln z\Big|_{+0'}^{+1}\\ &\equiv \big[\ln(-0') - \ln(-1)\big] + [0-\pi] + \big[\ln 1 - \ln 0'\big]\\ &\equiv \ln 0' - \pi - \ln 0' = -\pi.\end{aligned}$$

Additional windings around the pole, inside this same infinitesimal complex space hidden within the real line, give the class of values $(2\mathbb{Z}+1)\pi i$. This agrees with the Fundamental Theorems of Calculus:

$$\int_G \frac{dz}{z} = \ln z\Big|_{-1}^{+1} = \ln 1 - \ln(-1) = \frac{2\mathbb{Z}+1}{2}$$

With other poles, the discrepancy between real and complex integrals happens whenever the integral of portion B is nonzero. Some examples of this integral:

$$\int_B \frac{dx}{x} = \frac{2\mathbb{Z}+1}{2}$$

$$\int_B \frac{dx}{x^2} = \infty$$

$$\int_B \frac{dx}{x^3} = 0$$

Complex axial function

The axial function is defined:

$$A(z) = \begin{cases} 0 & \text{for } z \neq 0 \\ \not\infty & \text{for } z = 0 \end{cases}$$

Its name derives from the real version of this function, discussed in Axial function. The graph of the real function coincides the coordinate axes. The complex version of this function coincides with the plane that contains the two axes of the domain, and the plane containing the two axes of the range.

In real space, an integral through the origin yields an indeterminacy, as it does with a pole. To integrate from one side of the origin to the other, we have to integrate piecewise on each side, which yields two independent constants of integration.

In complex space, we can integrate on a path around the origin, as with did with a complex pole. This yields a single constant of integration.

Function $e^{\frac{1}{x}}$

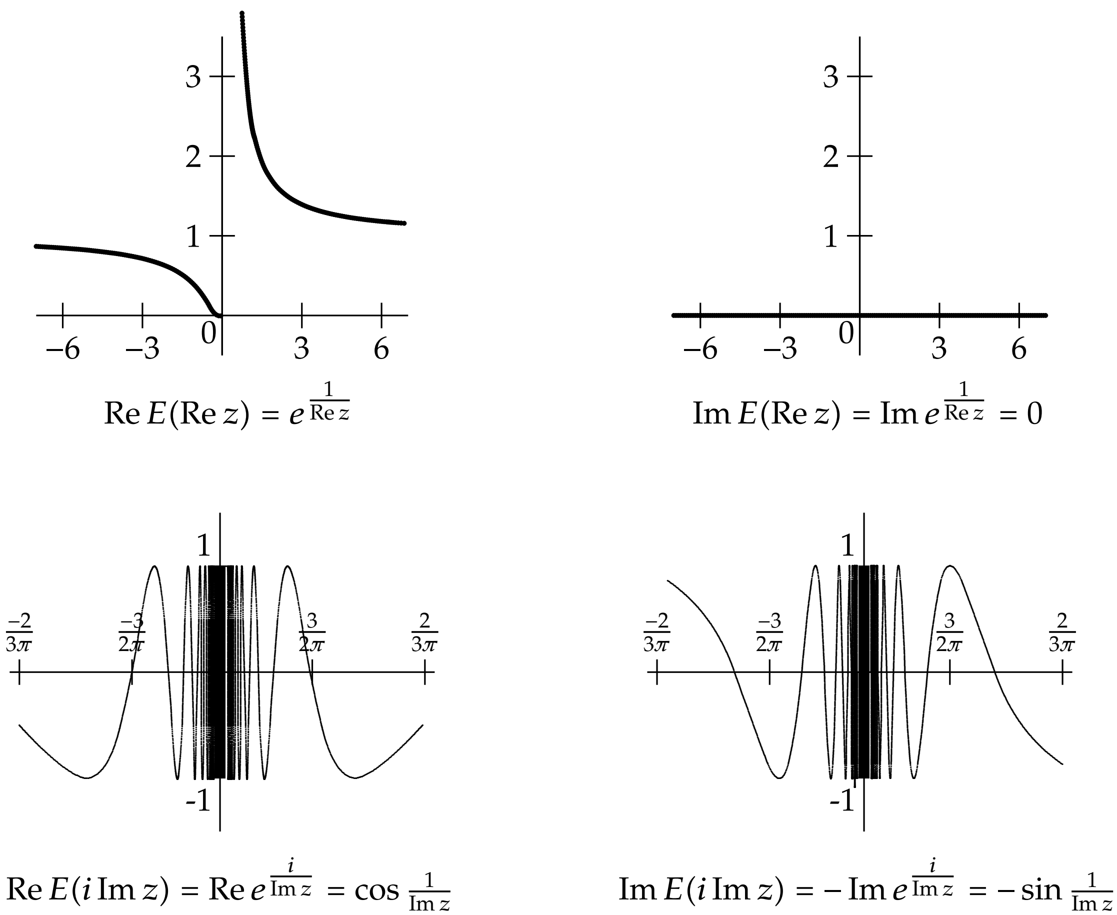

FIG. 40: Essential singularity of $E(z) := e^{\frac{1}{z}}$

The function $E(z) := e^{\frac{1}{z}}$ is graphed via real and imaginary parts in Figure 40. $E(x)$ is a complex version of the function $S(x)$ investigated in Function $\sin \frac{1}{x}$.

As a real function, $E(z)$ has a jump discontinuity at $z = 0$, but as a complex function, this is not the case. The offset values $E(0')$ for imaginary $0'$ can be any value within $[-1, +1]$, so the offset values are not semiuniform, and the singularity is not a removable discontinuity or jump discontinuity. The following shows that it is not a pole.

$$\begin{aligned}
e^{\frac{1}{z}} &= 1 + \frac{1}{z} + \frac{1}{2!x^2} + \dots \\
&= \frac{x+1}{x} + \frac{1}{2!x^2} + \dots \\
&= \frac{2!x^2 + (2)_1 x + 1}{2!x^2} + \frac{1}{x^3} + \dots \\
&= \frac{3!x^3 + (3)_2 x^2 + (3)_1 x + 1}{3!x^3} + \frac{1}{x^4} + \dots \\
&= \sum_{k=0}^{\infty'} (\infty')_{\infty'-k} x^{\infty'-k} \infty'! x^{\infty'}
\end{aligned}$$

The denominator of the final fraction has a root at $z = 0$ of infinite multiplicity. Thus the singularity is an essential singularity.

Picard's theorem: If a complex function f has an essential singularity at x, then $f(x) = \not\phi$.

We will not prove the general case of this theorem, but we will show that it holds for $E(z)$, i.e. that $E(0) = e^{\frac{1}{0}} = \not\phi$. To do this, we will show that for given any z, we can find $0'$ such that $z = e^{\frac{1}{0'}}$. If we write $z = re^{i\theta} = e^{\frac{1}{0'}}$, then we are looking for r and θ such that $\ln r + i\theta$ is infinite.

For infinite or zero r, $\ln r$ is infinite, and θ can be any value. For perfinite r, θ must be infinite. Since, as we saw in Function $\sin \frac{1}{x}$, $\sin \infty = [-1, +1]$, then for an unfolded infinite θ, $e^{i\theta}$ is on the unit circle as it is for finite θ. Thus, for any z, $\ln r + i\theta = \infty'$ for some complex infinite value ∞', and $0' =' \dfrac{1}{\infty'}$. □

CLASS COUNTS

Class count comparisons

In set theory, the notation $\#C$ means the cardinality of the set C. Since numeristics does not use the concept of cardinality, we use this notation to mean simply the number of elements in the class C, which we call a ***class count***.

Of course, the counts of all infinite classes are the same value:

$$\#\mathbb{N} = \#\mathbb{Q} = \#\mathbb{R} = \#\mathbb{C} = \infty.$$

On the other hand, as with other infinite quantities, ratios and other operations between two infinite values may be finite. Much as the value of $\dfrac{dy}{dx}$ depends on the relation between x and y in unfolded space, so $\#C - \#D$, $\dfrac{\#C}{\#D}$, and other such expressions may depend on the relation between the counts of C and D in unfolded space, which in turn depends on the relation between C and D.

Comparing $\mathbb{N}$ and $\mathbb{N}^+$

As an example, consider $\mathbb{N}$ and $\mathbb{N}^+$. If we map $\mathbb{N}$ to $\mathbb{N}^+$ with a function f that takes $0, 1, 2, 3, 4, \ldots$ to $1, 2, 3, 4, \ldots$, i.e. $f : n \mapsto n + 1$, then f is bijective, and we write $f \vdash (\#\mathbb{N} \equiv \#\mathbb{N}^+)$ ("through f"), or

$$f \vdash \#\mathbb{N} - \#\mathbb{N}^+ = 0.$$

But if we use $g : 0 \mapsto 1$ and $n \mapsto n$ for $n \geq 1$, then g is bijective with the one exception $g^{-1}(1) = \{0, 1\}$, so we have $g \vdash (\#\mathbb{N} \equiv \#\mathbb{N}^+ + 1)$, or

$$g \vdash \#\mathbb{N} - \#\mathbb{N}^+ = 1.$$

Comparing $\mathbb{N}$ and $\mathbb{Z}$

If we use $f : \mathbb{N} \to \mathbb{Z}$ which takes $0, 1, 2, 3, 4, \ldots$ to $0, 1, -1, 2, -2, \ldots$, i.e.

$$n \mapsto \begin{cases} 0 & \text{for } n = 0 \\ m & \text{for } n = 2m \\ -m & \text{for } n = 2m + 1, \end{cases}$$

then this is bijective and we write

$$f \vdash \frac{\#\mathbb{N}}{\#\mathbb{Z}} = 1.$$

But if $\mathbb{N}$ is mapped to $\mathbb{Z}$ by $g : 0, 1, 2, \ldots \mapsto 0, \pm 1, \pm 2, \ldots$ i.e. $n \mapsto \pm n$, then this is a one-to-two map, except for $n = 0$. Letting $\infty' :\equiv \#\mathbb{N}$, we have

$$g \vdash \frac{\#\mathbb{N}}{\#\mathbb{Z}} \equiv \frac{\infty'}{2\infty' - 1} = \frac{1}{2}.$$

In both these cases, g seems to be a more natural map, since it embeds the domain into the range and thus preserves structure better than f. We will henceforward look for such functions.

In general, for a class count comparison map m:

- To be a natural measure of class counts, m should be as close to an embedding as possible and should preserve order and other properties.

- If $m : A \to B$ is bijective except that m elements in A are mapped to a single element in B, then $m \vdash (\#A + n \equiv \#B)$, or $m \vdash \#A - \#B = n$.

- If $m : A \to B$ is injective and takes every element in A to n elements in B, then $m \vdash (\#B \equiv n\#A)$, or $m \vdash \frac{\#B}{\#A} = n$.

- If $m : A^n \to B$ is bijective, then $m \vdash (\#A^n \equiv \#B)$, or $m \vdash \log_{\#B} \#A = n$.

- If $m : A \to D$ is bijective, and $D = \{d | d : B \to n\}$, then $m \vdash (\#A \equiv n^{\#B})$, or $m \vdash \sqrt[\#B]{\#A} = n$.

Comparing $\mathbb{Z}$ and $\mathbb{Q}$

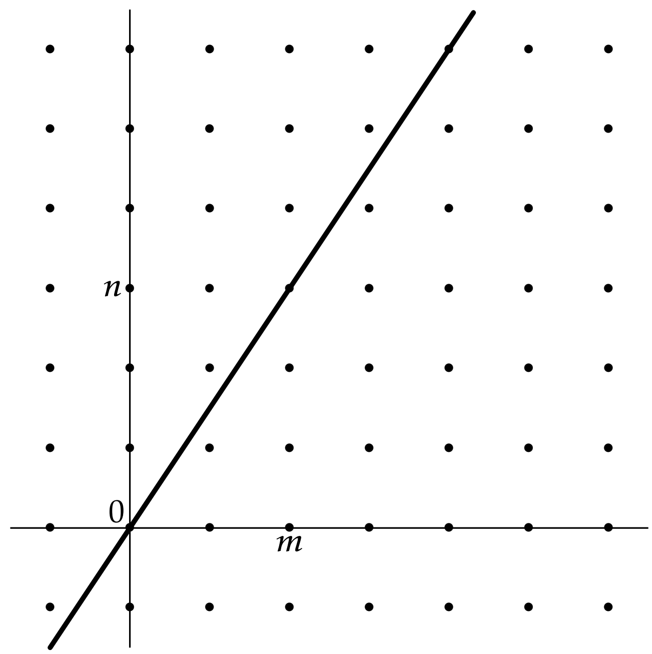

FIG. 41:
Map s from (m, n) to line of slope $\frac{m}{n}$

A class x is ***integrous*** if there is a bijection between the elements of x and some subset of the integers. Classes with a finite number of elements, $\mathbb{N}$, and $\mathbb{Z}$, are all integrous. In set theory, such classes are called ***countable*** or ***denumerable***, but from a numeristic point of view, these terms are misleading, since we can count and compare the number of elements of any class, including the nonintegrous classes $\mathbb{R}$ and $\mathbb{C}$.

Through a well known diagonal technique, it is possible to construct a bijection d^+ between $\mathbb{N}$ and $\mathbb{Q}^+$, and a similar bijection d between $\mathbb{Z}$ and $\mathbb{Q}$. This means that

$$d^+ \vdash (\#\mathbb{N} \equiv \mathbb{Q}^+)$$

$$d \vdash (\#\mathbb{Z} \equiv \mathbb{Q})$$

and that $\mathbb{Q}^+$ and $\mathbb{Q}$ are integrous. But d is not a natural map, since among other things, it does not preserve order.

For a more natural map, we define $s : \mathbb{N}^2 \to \mathbb{Q}^+$. Figure 41 shows this map geometrically. s takes $(m, n) \in \mathbb{N}^2$ to the line through (m, n) and the origin, which has slope $\dfrac{n}{m} \in \mathbb{Q}^+$. This is a many-to-one map with duplicates whenever m and n are not coprime (relatively prime).

Cesàro and others (see [H75, thm. 332, p. 269]) have shown that the probability of two random integers being coprime is $\frac{1}{\zeta(2)} = \frac{6}{\pi^2}$, where ζ is the Riemann zeta function. Therefore

$$\begin{aligned} s &\vdash \frac{\#\mathbb{Q}^+}{\#\mathbb{N}^2} = \frac{6}{\pi^2} \\ s_2 &\vdash \frac{\#\mathbb{Q}}{\#\mathbb{N}^2} = \frac{3}{\pi^2} \\ &\text{where } s_2 : \mathbb{N}^2 \longrightarrow \mathbb{Q} \\ s_3 &\vdash \frac{\#\mathbb{Q}}{\#\mathbb{Z}^2} = \frac{3}{2\pi^2} \\ &\text{where } s_3 : \mathbb{Z}^2 \longrightarrow \mathbb{Q}. \end{aligned}$$

Comparing $\mathbb{N}$ and $\mathbb{R}$

We first map $\mathbb{N}$ to the half-open unit interval $I = [0, 1)$ through base two radix representations (base two decimals). The expansion of $r \in I$ consists of a radix (decimal) point followed by an infinite string of binary digits. Such a string can be considered a map $k : \mathbb{N} \longrightarrow 2$. We define a class K of all possible such k, and then we define a map $j : K \longrightarrow I, k \mapsto r$.

K has $2^{\#\mathbb{N}}$ elements, each of which maps to a unique r, except for duplicates of the form $0.(digits)0111\ldots = 0.(digits)1000\ldots$, which appear at $\#\mathbb{N}$ unique positions. Hence

$$j \vdash \left(\#I \equiv 2^{\#\mathbb{N}} - \#\mathbb{N}\right).$$

In order to cover the real line, we make $\#\mathbb{Z}$ copies of I, using the map $y : \mathbb{R} \longrightarrow I, r \mapsto r - \lfloor r \rfloor$. Hence

$$\begin{aligned} k, y &\vdash (\#\mathbb{R} \equiv \#\mathbb{Z}\#I) \\ k, j, y &\vdash \left(\#\mathbb{R} \equiv \#\mathbb{Z}\left[2^{\#\mathbb{N}} - \#\mathbb{N}\right]\right). \end{aligned}$$

Letting $\infty' :\equiv \#\mathbb{N}$, we have

$$\begin{aligned} k, j, y \vdash \sqrt[\#\mathbb{N}]{\#\mathbb{R}} &\equiv \sqrt[\#\mathbb{N}]{\#\mathbb{Z}}\ \sqrt[\#\mathbb{N}]{2^{\#\mathbb{N}} - \#\mathbb{N}} \\ &\equiv (2\infty')^{\frac{1}{\infty'}} \left(2^{\infty'} - \infty'\right)^{\frac{1}{\infty'}} \\ &\equiv e^{\frac{\ln 2\infty'}{\infty'}} e^{\frac{\ln 2^{\infty'} - \infty'}{\infty'}} \equiv e^{\frac{2}{2\infty'}} e^{\frac{(\ln 2)2^{\infty'} - 1}{2^{\infty'} - \infty'}} \\ &=' e^{0'} e^{\frac{(\ln 2)^2 2^{\infty'}}{(\ln 2)2^{\infty'} - 1}} =' e^{\frac{(\ln 2)^3 2^{\infty'}}{(\ln 2)^2 2^{\infty'}}} = 2. \end{aligned}$$

A similar result holds for any radix. For a general j_b which uses a radix b instead of 2, we obtain

$$k, j_b, y \vdash \sqrt[\#\mathbb{N}]{\#\mathbb{R}} = b.$$

Comparing $\mathbb{R}$ and $\mathbb{C}$

If we define $p : \mathbb{C} \longrightarrow \mathbb{R}^2, a + bi \mapsto (a, b)$, then

$$p \vdash \frac{\#\mathbb{C}}{\#\mathbb{R}^2} = 1.$$

Using class counts in derivatives and integrals

We now use class counts and other equipoint arguments to calculate the derivative and integral of the ***rational test function***, defined as:

$$q(x) := \begin{cases} 1 & \text{for } x \in \mathbb{Q} \\ 0 & \text{for } x \notin \mathbb{Q}. \end{cases}$$

First we calculate $\frac{\#\mathbb{Q}}{\#\mathbb{R}}$, using the maps s, k, j, and y from the previous section:

$$\begin{aligned} s, k, j, y \vdash \frac{\#\mathbb{Q}}{\#\mathbb{R}} &\equiv \frac{\frac{3}{\pi^2}\#\mathbb{N}^2}{2^{\#\mathbb{N}}} \\ &\equiv \frac{\frac{3}{\pi^2}\#\mathbb{N}^2}{\left(1 + \#\mathbb{N}\ln 2 + \frac{(\#\mathbb{N}\ln 2)^2}{2!} + \frac{(\#\mathbb{N}\ln 2)^3}{3!} + \frac{(\#\mathbb{N}\ln 2)^4}{4!} + \ldots\right)} \\ &\equiv \frac{\frac{3}{\pi^2}}{\frac{1}{\#\mathbb{N}^2} + \frac{\ln 2}{\#\mathbb{N}} + \frac{(\ln 2)^2}{2!} + \frac{\#\mathbb{N}(\ln 2)^3}{3!} + \frac{\#\mathbb{N}^2(\ln 2)^4}{4!} + \ldots} \\ &= \frac{\frac{3}{\pi^2}}{0'^2 + 0'\ln 2 + \frac{(\ln 2)^2}{2!} + \frac{\infty'(\ln 2)^3}{3!} + \frac{\infty'^2(\ln 2)^4}{4!} + \ldots} = 0 \end{aligned}$$

Next we investigate the continuity of $q(x)$ using the definition above, namely that $q(x)$ is continuous at x when $q(x + 0') = q(x)$.

To compute $q(x + 0')$, pick an unfolded integer ∞' and let $0' \equiv 10^{-\infty'}$. The ∞'-th digit of the decimal representation of x is the origin of the space unfolded with $0'$. Call this digit d. The unit in this place has the value $0'$.

If x is rational, the decimal preserves the repetend of x, even in the ∞'-th place. Adding the unit $0'$ to x changes d to $d + 1$ for $d < 9$, and 9 to 0 with a finite number of carries, with one exception noted later. With this change of at least one digit, the repetend is broken, and the number is no longer rational. Hence $q(x + 0') = 0 \neq q(x)$, and $q(x)$ is discontinuous at x.

The exception to the above process occurs when the repetend is 9, in which case there are an infinite number of carries. The 9s to the left of the ∞'-th place change to 0, but the digits to the right of this place remain 9. In this case also, the repetend is broken, and $q(x)$ is discontinuous at x.

If x is irrational, the same thing occurs, except that there is never a repetend of 9, because there is never any repetend. So there are at most a finite number of carries, the folded digits are never affected, and $q(x + 0')$ is also irrational at the folded level. Hence $q(x + 0') = 0 = q(x)$, and $q(x)$ is continuous at x.

This differs from the conclusion of conventional analysis, which says that the function is discontinuous everywhere because the limit $\lim_{x \to a} q(x)$ does not exist at any point. Here, while there are an infinite number of discontinuities in any finite interval, the function is continuous at most points, since

$$\begin{aligned} s,k,j,y &\vdash \frac{\mathbb{Q}}{\mathbb{R}} = 0 \\ s,k,j,y &\vdash \frac{\mathbb{R} \setminus \mathbb{Q}}{\mathbb{R}} = 1. \end{aligned}$$

We can now compute the derivative of $q(x)$:

$$\begin{aligned} s,k,j,y \vdash {}_{0'}\frac{dq(x)}{dx} &\equiv q'_{0'}(x) \\ &\equiv \begin{cases} {}^{0'^2}\delta'_{x+0',x} & \text{for } x \in \mathbb{Q} \\ 0 & \text{for } x \notin \mathbb{Q}. \end{cases} \\ &\equiv \begin{cases} 0'^2\delta'_{0'^2}(0') & \text{for } x \in \mathbb{Q} \\ 0 & \text{for } x \notin \mathbb{Q}. \end{cases} \\ &\equiv \begin{cases} \delta'_{0'^2}\left(0' + \frac{0'^2}{2}\right) - \left(0' - \frac{0'^2}{2}\right) & \text{for } x \in \mathbb{Q} \\ 0 & \text{for } x \notin \mathbb{Q}. \end{cases} \end{aligned}$$

$\displaystyle\int_0^a q(x)\,dx$ is simply the ratio $\frac{\mathbb{Q}}{\mathbb{R}}$, which we have already seen is 0.

INFINITE INTEGERS AND RATIONAL NUMBERS

Given any sensitivity unit $0'$, the unfolded infinite numbers $\dfrac{\mathbb{R}}{0'}$ contain unfolded integers, rationals, and irrationals. Similarly, the unfolded infinitesimals numbers $\mathbb{R}0'$ contain unfolded rationals and irrationals.

If M is an unfolded infinite integer and n is a folded finite integer, then $\dfrac{M}{n}$ is an ***infinite rational***, and $\dfrac{n}{M}$ is an ***infinitesimal rational*** or ***zero rational***.

If M and N are both unfolded infinite integers, then $\dfrac{M}{N}$ may be either rational or irrational, and either infinite, perfinite, or zero.

A perfinite irrational can be unfolded into a class which includes such ratios. For example, $\sqrt{2} = 1.414\ldots$, when unfolded, includes elements of the form $1.414\ldots d(\infty')$, where $d(n)$ is the n-th digit:

$$1.414\ldots d(\infty') =' \frac{1414...d(\infty')}{10^{\infty'}}$$

$$1.414\ldots d(\infty'+1) =' \frac{1414...d(\infty'+1)}{10^{\infty'+1}}$$

$$1.414\ldots d(2\infty') =' \frac{1414...d(2\infty')}{10^{2\infty'}}$$

CONVERGENT SERIES PARADOX

In this analysis, infinite series and definite integrals are both simple sums with an infinite number of terms. In a definite integral, all the terms are zero, while in an infinte series, an infinite number of terms are nonzero. When the terms can be directly compared, this may lead to a paradoxical condition wherein both a series and an integral yield a finite result. For example, consider that

$$\sum_{n=1}^{\infty} 2^{-n} = 1 < \int_0^1 2\,dx = \sum_{n=1}^{\infty'} \frac{2}{\infty'} = 2,$$

even though, if we look at individual terms,

$$2^{-n} \geq \frac{2}{\infty}$$

for all n, with equality holding only for infinite n.

In the numerisitic theory of infinite series, we find that convergent series such as the one above actually have *two* values, one finite and one infinite. The infinite value arises when we consider the series to be an infinite sum of strictly positive values, and the finite value comes from the identification of $+\infty$ and $-\infty$ in the projectively extended real numbers. This is explained in detail in [CD] and [CR].

QUANTUM RENORMALIZATION

Renormalization is a procedure used in quantum physics to "tame" infinities that occur in quantum formulas. The correctness of values derived through renormalization is well verified experimentally, but the mathematics of this procedure is poorly understood, and therefore its theoretical validity is controversial.

Equipoint analysis should improve the understanding of quantum renormalization. The following example may show this. Although realistic examples of quantum renormalization usually involve very difficult formulas, we use here a very simplified example given by Klauber [Kl, p. 305]. The problem is to evaluate

$$\int_{-\infty}^{\infty} x^2\,dx.$$

In conventional analysis, this evaluation is done in two main steps, regularization and renormalization. In this example, the regularized form is

$$\lim_{\Lambda\to\infty} \int_{-\Lambda}^{\Lambda} x^2\,dx.$$

This still diverges, so we renormalize by multiplying by the factor $\frac{1}{\Lambda^3}$. Then we have

$$\lim_{\Lambda\to\infty} \frac{1}{\Lambda^3} \int_{-\Lambda}^{\Lambda} x^2\,dx = \frac{2}{3}.$$

In equipoint analysis, we avoid limits and directly write

$$\frac{1}{\infty'^3} \int_{-\infty'}^{\infty'} x^2\,dx = \frac{2}{3}.$$

We can also separate the factor $\frac{1}{\infty'^3}$ from the integral and evaluate these expressions separately as infinite quantities.

See also [T, ch. 18] and [D] for elementary introductions to quantum renormalization.

APPENDIX: COMPARISON OF EQUIPOINT WITH OTHER THEORIES OF ANALYSIS

Comparison of equipoint and conventional analysis

The differences between equipoint analysis and conventional limit-based analysis have been observed throughout this monograph. Here we summarize these differences. In equipoint analysis:

- There are one or more infinite numeric values at the folded level.
- An expression can represent a single value or a multivalued numeristic class.
- As a result of the previous two points, all operations are definable. An expression which is syntactically correct has a value and is never regarded as meaningless.
- Infinite and infinitesimal quantities are directly handled through an extended multiple-level number system rather than indirectly through limits.
- Infinite and infinitesimal operations are simple algebraic expressions.
- Proofs are shorter and more general, sometimes considerably so.
- The Leibnitz derivative and Riemann integral suffice for any real or complex function. There is no need for constructs such as the Lebesgue integral.

For comparison, below are definitions and examples of the derivative and definite integral in coventional limit-based analysis. For equipoint equivalents, see Defintion of derivative and Defintion of definite integral above.

Conventional definition of derivative:

$$\frac{df(x)}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Sample application of this definition:

$$
\begin{aligned}
\frac{dx^2}{dx} &= \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} \\
&= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} \\
&= \lim_{h\to 0} \frac{2xh + h^2}{h} \\
&= \lim_{h\to 0} (2x + h) \\
&= 2x.
\end{aligned}
$$

Conventional definition of definite integral:

$$
\int_a^b f(x)\, dx = \lim_{N\to\infty} \sum_{k=1}^{N} f\left(a + \frac{k(b-a)}{N}\right) \frac{b-a}{N}.
$$

Sample application of this definition:

$$
\begin{aligned}
\int_0^u 2x\, dx &= \lim_{N\to\infty} \sum_{k=1}^{N} 2\frac{ku}{N}\frac{u}{N} \\
&= \lim_{N\to\infty} \frac{2u^2}{N^2} \sum_{k=1}^{N} k \\
&= \lim_{N\to\infty} \frac{2u^2}{N^2} N \frac{N+1}{2} \\
&= \lim_{N\to\infty} u^2 \left(1 + \frac{1}{N}\right) \\
&= u^2.
\end{aligned}
$$

Comparison of equipoint and nonstandard analysis

Equipoint analysis has much in common with nonstandard analysis and has borrowed many of its concepts, including the microscope method of diagramming. In equipoint analysis:

- There are one or more infinite numeric values at the folded (standard) level.
- An expression can represent a single value or a multivalued numeristic class.
- As a result of the previous two points, all operations are definable. An expression which is syntactically correct has a value and is never regarded as meaningless.

- There are an infinite number of levels of sensitivity, rather than the two levels of standard and nonstandard.

- Multiple levels of relations allow us to dispense with the standard part function.

- An unfolded space is the expansion of a point, rather than of the infinitesimal neighborhood around the point.

For source material on nonstandard analysis, see [R74], [KE] and [KF].

For comparison, below are definitions and examples of the derivative and definite integral in nonstandard analysis. For equipoint equivalents, see Defintion of derivative and Defintion of definite integral above.

Nonstandard definition of derivative:

$$\frac{df(x)}{dx} = \mathrm{st}\left(\frac{f(x+\varepsilon) - f(x)}{\varepsilon}\right),$$

where ε is an infinitesimal, which in nonstandard analysis is nonzero but smaller than all nonzero reals, and $st()$ is the standard part function, which maps a number of the form $a + \varepsilon$ to a, where a is real.

Sample application of this definition:

$$\begin{aligned}
\frac{dx^2}{dx} &= \mathrm{st}\left(\frac{(x+\varepsilon)^2 - x^2}{\varepsilon}\right) \\
&= \mathrm{st}\left(\frac{x^2 + 2x\varepsilon + \varepsilon^2 - x^2}{\varepsilon}\right) \\
&= \mathrm{st}\left(\frac{2x\varepsilon + \varepsilon^2}{\varepsilon}\right) \\
&= \mathrm{st}(2x + \varepsilon) \\
&= 2x.
\end{aligned}$$

Nonstandard definition of definite integral:

$$\int_a^b f(x)\,dx = \mathrm{st}\left(\sum_{k=1}^{H} f\left(a + \frac{k(b-a)}{H}\right)\frac{b-a}{H}\right).$$

Sample application of this definition:

$$\begin{aligned}
\int_0^u 2x\,dx &= \mathrm{st}\left(\sum_{k=1}^{H} 2\frac{ku}{H}\frac{u}{H}\right)\\
&= \mathrm{st}\left(\frac{2u^2}{H^2}\sum_{k=1}^{H} k\right)\\
&= \mathrm{st}\left(\frac{2u^2}{H^2}H\frac{H+1}{2}\right)\\
&= \mathrm{st}\left(u^2\left(1+\frac{1}{H}\right)\right)\\
&= \mathrm{st}\left(u^2(1+\varepsilon)\right)\\
&= u^2.
\end{aligned}$$

Comparison of equipoint and ultrasmall/relative analysis

Relative analysis rests on a version of set theory to which is added a new set theoretic relation. Relative analysis uses this relation to build set theoretic proper classes of numbers, which are levels of numbers much like sensitivity levels in equipoint analysis.

What are called infinites and infinitesimals at unfolded levels in equipoint analysis are called ultralarge and ultrasmall in relative analysis.

In equipoint analysis, equality at each level is relative to a unit, which at unfolded levels may be an infinitesimal $0'$, and this equality is denoted $='$. Relative analysis has a very similar notion of equality, which is relative to a given level, a proper class usually denoted as V, and this equality is denoted $\simeq_V$.

Unlike equipoint analysis, relative analysis has no infinite elements at the real level (corresponding to the folded arithmetic of equipoint analysis) and no numeristic classes, so some operations must be left undefined, as in conventional analysis.

Relative analysis says that variables “appear” at certain levels, where equipoint analysis would say that they become distinguishable.

Equipoint analysis, and numeristics generally, does not say that numbers appear or disappear, because it regards numbers as eternal and omnipresent. Numbers exist in both subjective and objective realms and are not simply defined into existence. The definition of a number creates the *concept* of a number, but not the number itself. Were this not so, mathematics would be only a game, not a science.

Relative analysis definition of derivative:

$$f'(x) := n\left(\frac{f(x+h) - f(x)}{h}\right),$$

where $n(x)$ or $n_V(x)$ denotes the ***neighbor*** of x, the unique real number ultraclose to x at level V.

Relative analysis definition of definite integral:

$$\int_a^b f(x)\,dx := n\left(\sum_{i=0}^{N-1} f(x_i)h\right),$$

for h ultrasmall and N ultralarge, $h := \frac{b-a}{N}$ and $x_i := a + ih$.

For equipoint equivalents, see Equipoint defintion of derivative and Equipoint defintion of definite integral.

For source material on relative analysis, see [H10], [OD09], and [OD11].

Comparison of equipoint and smooth infinitesimal analysis

Like equipoint analysis, smooth infinitesimal analysis uses only simple algebra for its infinitesimal operations. The most significant differences are that smooth infinitesimal analysis uses intuitionistic logic, a single level of equality, no infinite values, and thus no division by infinitesimals.

Smooth infinitesimal analysis has two postulates for infinitesimals:

$$(\exists! D)(\forall \varepsilon) f(\varepsilon) = f(0) + D$$

$$[(\forall \varepsilon)\varepsilon a = \varepsilon b] \Rightarrow a = b$$

Using intuitionistic logic, these postulates imply $\varepsilon \neq 0$ and $\neg(\varepsilon \neq 0)$, but the second is not equivalent to $\varepsilon = 0$, i.e. ε equality does not obey the law of excluded middle. These postulates also imply $\varepsilon^2 = 0$, i.e. infinitesimals are nilpotent.

This approach leads to formulae such as

$$f(x+\varepsilon) - f(x) = \varepsilon f'(x)$$

as the definition of derivative, which is only implicit and has to be proved to exist. Since there is no division by infinitesimals, we cannot use notation such as $\frac{dy}{dx}$ or $f'(x) = \frac{f(x+\varepsilon)-f(x)}{\varepsilon}$.

Integration is even less direct, being given not by a formula but by an ***Integration Principle***: Given a smooth function $f : [0,1] \to \mathbb{R}$, there exists a unique smooth function $g : [0,1] \to \mathbb{R}$, such that $g' = f$ and $g(0) = 0$.

The equipoint equivalents of these definitions are explicit formulas given in Defintion of derivative and Defintion of definite integral.

For source material on smooth infinitesimal analysis, see [BI], [BP], and [L].

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