Equinfinitesimal Analysis: A Numeristic Approach to Calculus

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ABSTRACT

This paper extends the concepts of numeristics to analysis. It develops a theory of analysis based on infinitesimals which are all exactly equal to zero, and infinite values that are their reciprocals. Concepts derive from Maharishi Mahesh Yogi's Vedic Mathematics, Charles Musès's analysis of zero and infinity, and Abraham Robinson's non-standard analysis. This theory uses multiple levels of sensitivity to evaluate equality and defines derivatives and integrals solely in terms of elementary arithmetic operations.

> Non-standard analysis frequently simplifies substantially the proofs, not only of elementary theorems, but also of deep results. This is true, e.g., also for the proof of the existence of invariant subspaces for compact operators, disregarding the improvement of the result; and it is true in an even higher degree in other cases. This state of affairs should prevent a rather common misinterpretation of non-standard analysis, namely the idea that it is some kind of extravagance or fad of mathematical logicians. Nothing could be farther from the truth. Rather, there are good reasons to believe that non-standard analysis, in some version or other, will be the analysis of the future.—Kurt Gödel [Gö]

> [Srinivasa Ramanujan] sometimes spoke of 'zero' as the symbol of the Absolute (Nirguna Brahman) of the extreme monistic school of Hindu Philosophy, that is, the reality to which no qualities can be attributed, which cannot be defined or described by words and is completely beyond the reach of the human mind; according to Ramanujan, the appropriate symbol was the number 'zero', which is the absolute negation of all attributes. He looked on the number 'infinity' as the totality of all possibilities which was capable of becoming manifest in reality and which was inexhaustible. According to Ramanujan, the product of infinity and zero would supply the whole set of finite numbers. Each act of creation, as far as I could understand, could be symbolized as a particular product of infinity and zero, and from each such product would emerge a particular individual of which the appropriate symbol was a particular finite number. . . . He spoke with such enthusiasm about the philosophical questions that sometimes I felt he would have been better pleased to have succeeded in establishing his philosophical theories than in supplying rigorous proofs of his mathematical conjectures.—P. C. Mahalanobis about Srinivasa Ramanujan [Ml]

DEFINITION AND SCOPE

Equinfinitesimal analysis is application of numeristic principles to analysis. Numeristics is introduced in [CaN]; the present paper is a sequel.

Broad conceptualizations for numeristics come from Maharishi Mahesh Yogi in his Vedic Mathematics. These sources are summarized in [CaS] and include especially [My]. Equinfinitesimal analysis extends these concepts into mathematical methods, using ideas from non-standard analysis and Charles A. Musès. Non-standard analysis is a modern theory of infinitesmals and is described in more details below. Musès supplied some key insignts for refining the ideas of non-standard analysis in [Mu72]. The present paper aims to be self-sufficient and not require familiarity with any of these sources.

The term *equinfinitesimal analysis* means that the infinitesimals it uses are exactly equal to zero. This concept is briefly introduced by Musès in [Mu65, p. 178–179] and [Mu72]. The nearly identical term *equi-infinitesimal analysis* was used by Musès in [Mu59] and [Mu60].

NON-STANDARD ANALYSIS

Non-standard analysis has its roots in the original development of calculus in terms of infinitesimals by Leibnitz in the 17th century. In the intervening centuries, calculus was found to be very useful, but the explanation of it in terms of infinitesimals did not satisfy very many mathematicians. With the increasing demand for rigor in the 19th century, the theory of infinitesimals was replaced by the now standard theory based on set theory and limits.

In 1960, Abraham Robinson resurrected the theory of infinitesimals by developing it as a modern set theoretic system he called non-standard analysis. Jerome Keisler used the principles of non-standard analysis in his elementary calculus textbook [KeE] and undergraduate analysis text [KeF]. Yet non-standard analysis has not achieved widespread use.

ORIGIN OF EQUINFINITESIMAL ARITHMETIC

As explained in [CaN], numeristics is based on the infinite and the experience of the silent, unmanifest point of infinity, *samādhi* or zero. In numeristics this is conceptualized to give an arithmetic of 0 and ∞, including unrestricted multiplication and division by these quanities, such as $\frac{1}{0}$, $\frac{0}{0}$, and $0 \cdot \infty$.

Beyond the experience of the point of infinity is the experience of this point opening up into a vast inner space distinct from and much richer than ordinary space, and the contraction of this space back into ordinary space. Equinfinitesimal analysis conceptualizes some aspects of the expanded space by considering it as a space of points all having the value of zero from the perspective of ordinary space but distinct from its own perspective. This allows us to do calculus with infinitesimals that are exactly zero, and numeric infinites that are reciprocals of these zeros. We formalize the multiple perspectives as levels of sensitivity, as explained next.

SENSITIVITY

Levels of sensitivity

In equinfinitesimal analysis, equality is evaluated relative to a given level of sensitivity. At the default level of sensitivity, we distinguish real numbers, integers, rational numbers, etc. as elements. When, as described in the previous section, zero opens up into a space of zeros, these zeros are all exactly equal to zero at the default level of sensitivity, but at a greater level of sensitivity, zero is multivalued, and the individual zeros are distinct elements. This is shown pictorially below.

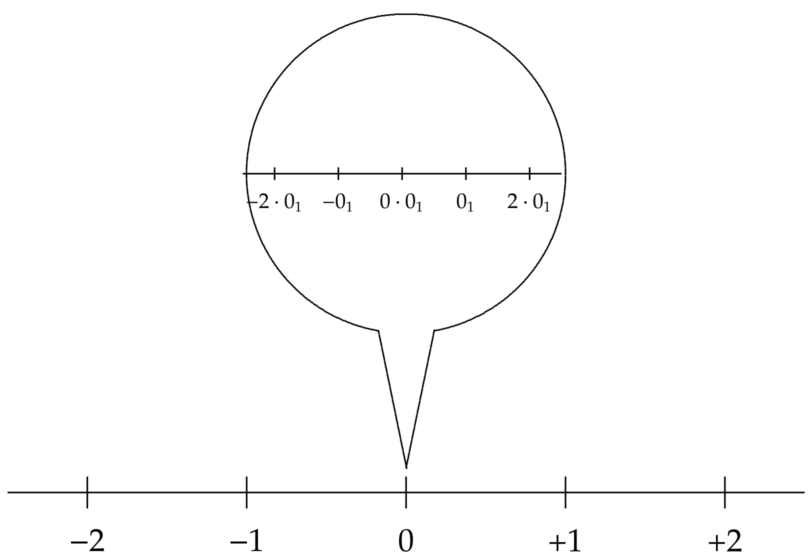

FIG. 1:
Real number line with microscope
view of line of zeros within 0

In Figure 1, we have the ordinary real number line with two of the points, the real numbers 0 and 1, expanded into a space. When the real number 0 is expanded into a space, we find multiple zeros in that space. One of those zeros we denote as 0_1, and we show also some multiples of 0_1. We could denote one of these other points, say $3 \cdot 0_1$, as 0_2. From the point of view of the real numbers, $0_1 = 0_2 = 0$, but from the point of view of the expanded space, $0_1 \neq 0_2$ and $0_1, 0_2 \in 0$. We call these different points of view *levels of sensitivity*.

An important aspect of the expanded space around 0 is that each of the individual values of 0 in that space has a well defined ratio with every other point in that space. For instance, if $0_2 = 3 \cdot 0_1$, then $\dfrac{0_2}{0_1} = 3$. This also means that the expanded space is ordered in analogously to real space, e.g. $0_2 > 0_1$ at the expanded level of sensitivity.

The origin of the expanded space around 0, the value $0 \cdot 0_1$, can open up into a more expanded space where the units are 0_1^2. This process can continue indefinitely, leading to an infinite number of levels of sensitivity.

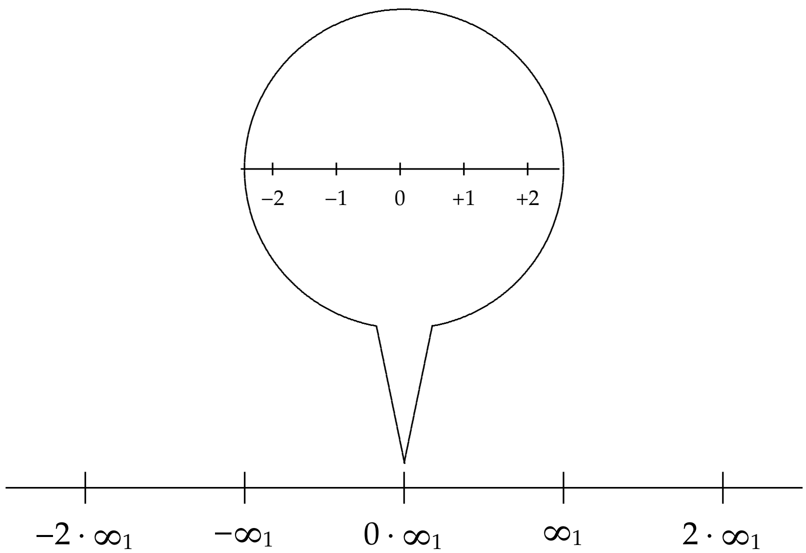

FIG. 2:
Line of infinities with microscope view of
real number line within $0 \cdot \infty_1$

Inversely to the expansion of a point of real space, the real number line can collapse into a point within a space of infinities. This is shown in Figure 2. Within this space, we may have e.g. $\infty_2 = 3 \cdot \infty_1$, $\dfrac{\infty_2}{\infty_1} = 3$, $\infty_2 > \infty_1$ at the higher level of sensitivity.

Just as we define $\infty := \dfrac{1}{0}$ in numeristics, in equinfinitesimal analysis we define $\infty_n := \dfrac{1}{0_n}$.

Measure of sensitivity

We measure a level of sensitivity by identifying a distinguishable unit, a number that does not lie at the origin. In the examples above, such units are $0_1, 0_2, \infty_1, \infty_2$. We write relations in which such units are distinguished from the origin with the modifier sen, e.g. $a + 0_1 \neq a \operatorname{sen} 0_1$. The default unit of sensitivity is 1; $a = b$ means $a = b \operatorname{sen} 1$, unless otherwise specified. In the above examples, $0_1 = 0_2 \operatorname{sen} 1$, while $0_1 \neq 0_2 \operatorname{sen} 0_1$.

At the sensitivity level 0_1, if we were to expand the origin $0 \cdot 0_1$ to a greater level of sensitivity, the sensitivity unit would be 0_1^2, and we would have $0_1^2 \neq 0_2^2 \operatorname{sen} 0_1^2$, whereas $0_1^2 = 0_2^2 \operatorname{sen} 0_1$.

The sensitivity notation is defined as follows. if s is perfinite or a zero unit, $a = b \operatorname{sen} s$ is equivalent to $\dfrac{a}{s} = \dfrac{b}{s}$; if s is perfinite or an infinite unit, $a = b \operatorname{sen} s$ is equivalent to $as = bs$. Another notation for $a = b \operatorname{sen} s$ is $a \overset{s}{=} b$.

Besides =, sensitivity notation can also be used with other relations, including $<, >, \leq, \geq, \neq, \subset, \supset$, and $\in$. Interestingly, $\subseteq$ and $\supseteq$ are not subject to sensitivity, since $a \subseteq b$, if it holds at any sensitivity level, holds across all of them.

Sensitivity is a relation modifier. It is not an operator, except in the sense that it changes one relation to another. It does not operate on numbers, and it does not modify operations on numbers. An increased level of sensitivity allows us to detect differences that were already there but below the threshold of detection. A decreased level of sensitivity allows us to ignore such differences.

Here are a few more examples of sensitivity notation.

$$0 + 0 = 2 \cdot 0 = 0 \operatorname{sen} 1,$$
$$\infty + \infty = 2 \cdot \infty = \infty \operatorname{sen} 1,$$
$$0_1 + 0_1 = 2 \cdot 0_1 \neq 0_1 \operatorname{sen} 0_1,$$
$$\infty_1 + \infty_1 = 2 \cdot \infty_1 \neq \infty_1 \operatorname{sen} 0_1,$$
$$0_1 + 0_1^2 = 0_1 \operatorname{sen} 0_1,$$
$$0_1 + 0_1^2 \neq 0_1 \operatorname{sen} 0_1^2.$$

The definition of sensitivity means that $ab = c \operatorname{sen} s$ is equivalent to $\dfrac{c}{a} = b \operatorname{sen} s$ only for $s = 1$. For example, if $0_2 = 2 \cdot 0_1 \operatorname{sen} 0_1$, then $0_2/0_1 = 2 \neq 2 + 0_1 \operatorname{sen} 0_1$, even though $(2 + 0_1)0_1 = 0_2 + 0_1^2 = 0_2 \operatorname{sen} 0_1$.

Translation of sensitivity worlds

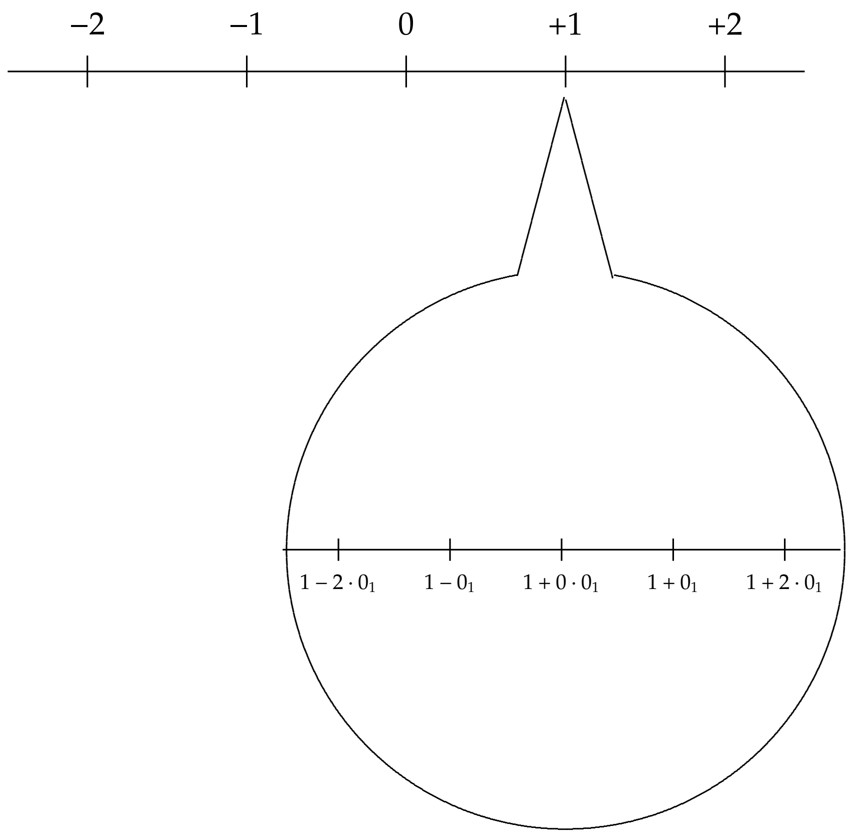

FIG. 3:
Real number line with microscope
view of line of +1's within +1

Since $a+0=a$ for any real a, every real number a can open up into a space consisting of a plus zeros. Figure 3 shows this for $a=1$. Just as we locate 0_1 within 0 and use it to perform a sensitive arithmetic within 0, we can also locate a_1 within a and use it as the base of a sensitive arithmetic within a.

Following are examples of arithmetic within a finite real a.

$$\begin{aligned} a - a &= 0 \operatorname{sen} 1, \\ a + 0_1 &= a \operatorname{sen} 1, \\ a_1 &= a \operatorname{sen} 1, \\ a + 0_1 &\neq a \operatorname{sen} 0_1, \\ a_1 - a_1 &= 0_1 \operatorname{sen} 0_1, \\ a_1 &\subset a \operatorname{sen} 0_1, \\ a_1 &\in a \operatorname{sen} 0_1. \end{aligned}$$

Subscriptless notation

The distinction between a and a_1 as used in the above discussion is always important. At sensitivity level 0_1, 0 is the class of all finite real multiples of 0_1, and 0_1 is an element of this class. Similar distinctions hold between and a and a_1, including between ∞ and ∞_1.

However, in many cases our discussion will reference only a_1 and not a. As long as the distinction between a and a_1 is implicitly clear, we can drop the subscript on a_1 and denote it a instead. In such a discussion, if we need to refer to the original class a, we can use the notation ${}_0a$.

Since this notation reduces clutter, usually without ambiguity, we will use it frequently from now on. This means that whenever an expression explicitly or implicitly is performed at sensitivity 0_1, the notation 0 means 0_1, and a means a_1.

In subscriptless notation, some of the above formulas are notated:

$$
\begin{aligned}
0 + 0 = 2 \cdot 0 &= 0 \operatorname{sen} 1, \\
\infty + \infty = 2 \cdot \infty &= \infty \operatorname{sen} 1, \\
0 + 0 = 2 \cdot 0 &\neq 0 \operatorname{sen} 0, \\
\infty_1 + \infty_1 = 2 \cdot \infty_1 &\neq \infty_1 \operatorname{sen} 0, \\
0 + 0^2 &= 0 \operatorname{sen} 0, \\
0 + 0^2 &\neq 0 \operatorname{sen} 0^2, \\
a + 0 &\neq a \operatorname{sen} 0, \\
a - a &= 0 \operatorname{sen} 0, \\
a &\subset a \operatorname{sen} 0, \\
a &\in a \operatorname{sen} 0.
\end{aligned}
$$

Elements and levels of sensitivity

At a given level of sensitivity, an element is a single value, a value within which there are no smaller values; $a \in b \operatorname{sen} c$ is equivalent to $a \subseteq b$ and $\sim \exists d \subset a \operatorname{sen} c$. $a \subset b \operatorname{sen} s$ is equivalent to $a \subseteq b$ and $a \neq b \operatorname{sen} s$. Examples: $0, 2 \cdot 0, 3 \cdot 0 \in {}_0 0 \operatorname{sen} 0$; $0^2, 2 \cdot 0^2, 3 \cdot 0^2 \in 0 \operatorname{sen} 0^2$; $0^2 = 0 \operatorname{sen} 0$; $0^2 = 0$. In the last example, a default level equality, 0 refers to ${}_0 0$.

Transfer principle and transfer identities

Some arithmetic identities are not affected by sensitivity levels. For instance, $(a+b)^2 = a^2 + 2ab + b^2$ is valid for any a and b at any sensitivity level, while $a + 0 = a$ is valid only at some levels.

When an identity holds at all sensitivity levels, we say that it *transfers* from one level to another. Generally, an identity that holds at the default level of sensitivity transfers to all levels, unless there is an explicit reason for it not to transfer. We refer to this as the *transfer principle*.

An identity that transfers we refer to as a *transfer identity* and we denote it with one of several notations: $(a+b)^2 = a^2 + 2ab + b^2 \operatorname{tran}$, or $(a+b)^2 \stackrel{\text{tran}}{=} a^2 + 2ab + b^2$, or $(a+b)^2 \equiv a^2 + 2ab + b^2$. In this paper, we use the $\equiv$ notation for transfer equality and use := for definition.

Examples:

$$(x+0)^2 = x^2 + 2x0 + 0^2 \operatorname{tran} = x^2 + 2x0 = x^2 \operatorname{sen} 1,$$
$$(x+0)^2 = x^2 + 2x0 + 0^2 \operatorname{tran} = x^2 + 2x0 \neq x^2 \operatorname{sen} 0,$$
$$(x+0)^2 = x^2 + 2x0 + 0^2 \operatorname{tran} \neq x^2 + 2x0 \neq x^2 \operatorname{sen} 0^2.$$

It is easy to see that $a = b \operatorname{tran}$ is equivalent to $(\forall c)\, a = b \operatorname{sen} c$, and that $a \neq b \operatorname{tran}$ is equivalent to $(\exists c)\, a \neq b \operatorname{sen} c$.

Transfer of power series

Power series expansions of functions generally transfer. This enables us to properly evaluate sensitivity levels in non-polynomial functions. Example:

$$\begin{aligned} e^{0x} &= 1 \operatorname{sen} 1 \\ &= 1 + 0x \operatorname{sen} 0 \\ &= 1 + 0x + \frac{0^2 x^2}{2} \operatorname{sen} 0^2 \\ &= \sum_{k=1}^{\infty} \frac{0^k x^k}{k!} \operatorname{sen} 0^\infty. \end{aligned}$$

The power series $\dfrac{1}{1-x} = 1 + x + x^2 + \ldots = \sum_{k=0}^{\infty} x^k$ has an important implication for levels of sensitivity at infinite numbers, e.g. $\operatorname{sen} \infty$. Since $1 + 0$ is evaluated at the same sensitivity level as 0, $\dfrac{1}{1-0}$ is evaluated at the same sensitivity level as $\dfrac{1}{0}$. By virtue of the power series for $\dfrac{1}{1-x}$, all of these are evaluated at the same sensitivity level, i.e. $a = b \operatorname{sen} s$ is equivalent to $\dfrac{1}{a} = \dfrac{1}{b} \operatorname{sen} \dfrac{1}{s}$.

Index of sensitivity

We have expressed sensitivity levels as powers of an arbitrary infinitesimal 0 or 0_1, and we have found that sensitivity level 0_n is equivalent to sensitivity level ∞_n. We now define the *index of sensitivity* as the logarithm of these powers, relative to 0, as $|\log_0 s|$, where s is the sensitivity level. A higher index indicates a finer sensitivity level. Sensitivity level 1 has an index of 0, the coarsest possible index.

It is easy to see that $a = b \operatorname{sen} s$ implies $a = b \operatorname{sen} t$ only if s is equal to or finer than t, i.e. the index of s is greater than the index of t, or $|\log_0 s| \geq |\log_0 t|$. If s is coarser than t, then we could have $a \neq b \operatorname{sen} t$. Inversely, $a \neq b \operatorname{sen} s$ implies $a \neq b \operatorname{sen} t$ only if s is coarser than t; if s is finer than t, then we could have $a = b \operatorname{sen} d$.

Continuity

We define continuity as follows. A function f is continuous at x iff $f(x+0) = f(x) \operatorname{sen} 1$ for every 0, i.e. $f(x+0_1) = f(x) \operatorname{sen} 1$ for every $0_1 \in 0$.

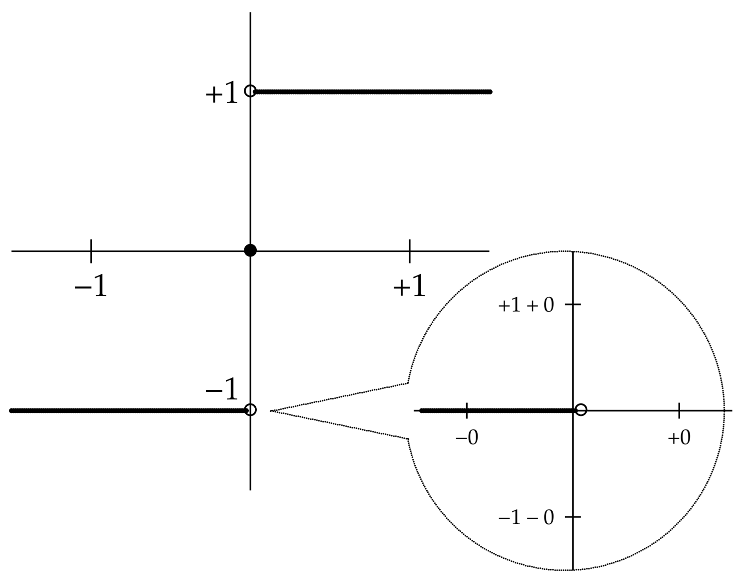

FIG. 4:
Discontinuity of signum function
$\operatorname{sgn} x$ at $x = 0$

Figure 4 shows an example of the signum function

$$\operatorname{sgn} x \equiv \begin{cases} -1 & \text{for } x < 0 \\ 0 & \text{for } x = 0 \\ +1 & \text{for } x > 0. \end{cases}$$

This function is discontinuous at $x = 0$ because $f(x - 0) = -1$, $f(x) = 0$, and $f(x + 0) = +1$ for $0 > 0^2 \operatorname{sen} 0$.

If f is continuous at x, then f is locally linear: $f(x + 0) - f(x) = f(x + 0(k + 1)) - f(x + 0k)$ for real k.

DEFINITIONS OF DERIVATIVE AND INTEGRAL

Equinfinitesimal defintion of derivative

The equinfinitesimal derivative directly calculates the rate of change at a point using sensitivity levels and the transfer principle.

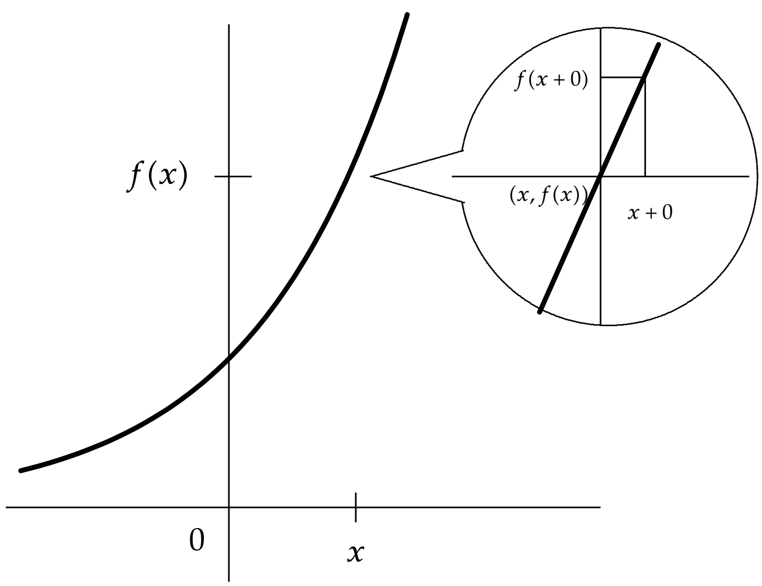

FIG. 5:
Calculation of derivative
as slope within a pont

Figure 5 shows a curve $y = f(x)$ and a microscope view of the point $(x, f(x))$. Within the point, the curve is infinitely magnified and becomes a straight line. The Δx of this line is an infinitesimal 0, the sensitivity level of the microscope, and the Δy of this line is $f(x+0) - f(x)$. The slope of the line, and the derivative of $f(x)$ at x, is

$$f'(x) = \frac{df(x)}{dx} = \frac{f(x+0) - f(x)}{0}.$$

As an example of this calculation:

$$\begin{aligned} f(x) &= x^2 \\ \frac{df(x)}{dx} &= \frac{(x+0)^2 - x^2}{0} \\ &= \frac{x^2 + 2 \cdot 0x + 0^2 - x^2}{0} \\ &= \frac{2 \cdot 0x}{0} \\ &= 2x. \end{aligned}$$

Since $\frac{0^2}{0} = 0$ and $\frac{x^2 - x^2}{0} = \frac{{}_0 00}{0} = 0$, these terms vanish from the final result.

Other defintions of derivative

For comparison, below are definitions and examples of the derivative in nonstandard and coventional limit based analysis.

Nonstandard definition of derivative:

$$\frac{df(x)}{dx} = \mathrm{st}\left(\frac{f(x+\varepsilon) - f(x)}{\varepsilon}\right),$$

where ε is an infinitesimal, which in nonstandard analysis is nonzero but smaller than all nonzero reals, and $st()$ is the standard part function, which maps a number of the form $a + \varepsilon$ to a, where a is real.

Conventional definition of derivative:

$$\frac{df(x)}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Nonstandard derivative of $f(x) = x^2$:

$$\begin{aligned}
\frac{dx^2}{dx} &= \mathrm{st}\left(\frac{(x+\varepsilon)^2 - x^2}{\varepsilon}\right) \\
&= \mathrm{st}\left(\frac{x^2 + 2x\varepsilon + \varepsilon^2 - x^2}{\varepsilon}\right) \\
&= \mathrm{st}\left(\frac{2x\varepsilon + \varepsilon^2}{\varepsilon}\right) \\
&= \mathrm{st}(2x + \varepsilon) \\
&= 2x.
\end{aligned}$$

Conventional derivative of $f(x) = x^2$:

$$\begin{aligned}
\frac{dx^2}{dx} &= \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} \\
&= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} \\
&= \lim_{h\to 0} \frac{2xh + h^2}{h} \\
&= \lim_{h\to 0} (2x + h) \\
&= 2x.
\end{aligned}$$

Equinfinitesimal defintion of definite integral

The equinfinitesimal integral directly calculates an area as an infinite sum of zero width rectangles.

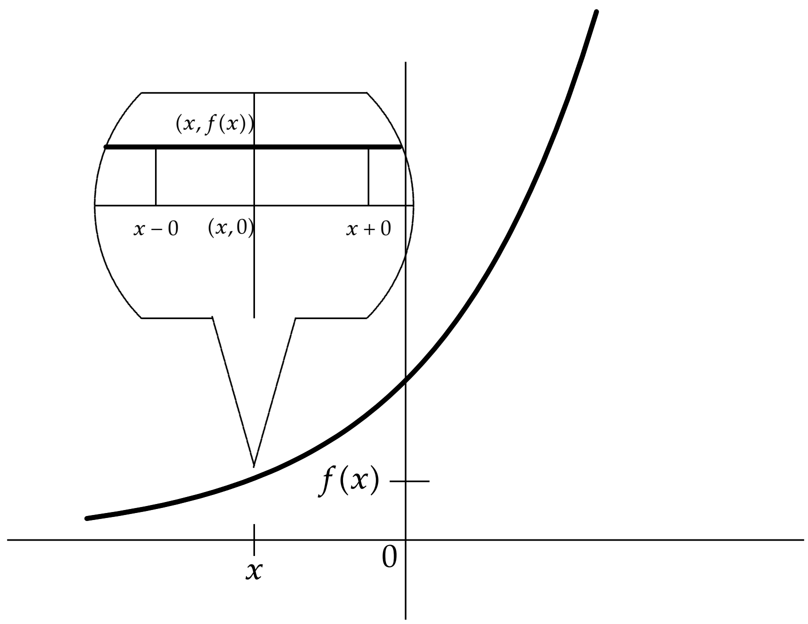

FIG. 6:
Calculation of integral
as sum of zero width rectangles

Figure 6 shows a curve $y = f(x)$ and a microscope view of the sliver at x an infinitely thin area under the curve $f()$ at the point x. The microscope view expands the sliver in the x direction but not in the y direction. Within the sliver, the curve becomes a flat line, and sliver is a rectangle with height $f(x)$ and width $x + 0 - x = 0$. The total area under the curve from $x = a$ to $x = b$ is the sum of the areas of these slivers, and the number of these slivers is $\dfrac{b-a}{0} = (b-a)\infty$. The total area from a to b, and the definite integral of $f(x)$ from a to b, is

$$\int_a^b f(x)dx = \sum_{k=1}^{\infty} f\left(a + \frac{k(b-a)}{\infty}\right)\frac{b-a}{\infty}.$$

As an example of this calculation:

$$\begin{aligned}\int_0^u 2x\,dx &= \sum_{k=1}^{\infty} 2\frac{ku}{\infty}\frac{u}{\infty}\\ &= \frac{2u^2}{\infty^2}\sum_{k=1}^{\infty} k\\ &= \frac{2u^2}{\infty^2}\infty\frac{\infty+1}{2}\\ &= u^2\left(1+\frac{1}{\infty}\right)\\ &= u^2(1+0)\\ &= u^2.\end{aligned}$$

Other defintions of definite integral

Nonstandard definition of definite integral:

$$\int_a^b f(x)dx = \operatorname{st}\left(\sum_{k=1}^{H} f\left(a+\frac{k(b-a)}{H}\right)\frac{b-a}{H}\right).$$

Conventional definition of definite integral:

$$\int_a^b f(x)dx = \lim_{N\to\infty}\sum_{k=1}^{N} f\left(a+\frac{k(b-a)}{N}\right)\frac{b-a}{N}.$$

Nonstandard definite integral of $f(x) = 2x$:

$$\begin{aligned}\int_0^u 2x dx &= \operatorname{st}\left(\sum_{k=1}^{H} 2\frac{ku}{H}\frac{u}{H}\right)\\ &= \operatorname{st}\left(\frac{2u^2}{H^2}\sum_{k=1}^{H} k\right)\\ &= \operatorname{st}\left(\frac{2u^2}{H^2}H\frac{H+1}{2}\right)\\ &= \operatorname{st}\left(u^2\left(1+\frac{1}{H}\right)\right)\\ &= \operatorname{st}\left(u^2(1+\varepsilon)\right)\\ &= u^2.\end{aligned}$$

Conventional definite integral of $f(x) = 2x$:

$$\begin{aligned}\int_0^u 2x dx &= \lim_{N\to\infty}\sum_{k=1}^{N} 2\frac{ku}{N}\frac{u}{N}\\ &= \lim_{N\to\infty}\frac{2u^2}{N^2}\sum_{k=1}^{N} k\\ &= \lim_{N\to\infty}\frac{2u^2}{N^2}N\frac{N+1}{2}\\ &= \lim_{N\to\infty} u^2\left(1+\frac{1}{N}\right)\\ &= u^2.\end{aligned}$$

Differentiability and integrability

At a point of discontinuity, a function is still differentiable; at such a point, the derivative is infinite. The reverse is also possible; a function can be integrated across an interval that contains singularities.

The only condition in which a derivative or integral does not exist is if the original function does not exist. The notation $\{\}$ can be used to denote the value of any function, derivative, or integral that is not defined at a point.

Infinite bounds on integrals and path integral

In the equinfinitesimal definition of integral, $b - a$ may be infinite if either of the limits a or b is infinite. In this case, we simply choose a 0 of a high enough sensitivity that $\dfrac{b-a}{0}$ is infinitesimal, e.g. $\dfrac{1}{(b-a)^2}$.

A key feature of numeristics is the equality of $+\infty$ and $-\infty$ at sensitivity level 1, but of course there is a big difference between these two as bounds of integration. From the numeristic point of view, bounds of integration implicitly establish a *path* of integration: integrating from $-\infty$ to $+\infty$ integrates through 0 and all the finite values, integrating from 0 to $+\infty$ integrates through all the positive finite values, etc. The equinfinitesimal integral along a path $x = P(t)$, where t runs from a to b, is given by

$$\int_P f(x)dx = \int_{t=a}^{b} f(P(t))dP(t) = \int_a^b f(P(t))\frac{dP(t)}{dt}dt.$$

EQUINFINITESIMAL PROOF OF FUNDAMENTAL THEOREM OF CALCULUS

For the following equinfinitesimal proof of the fundamental theorem of calculus, we assume the following: (1) the splitting property $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$ for all c, which is easily proved from the definition of the definite integral; (2) a corollary, the zero property $\int_a^a f(x)dx = 0$ for all a; (3) another corollary, the reversal property $\int_a^b f(x)dx = -\int_b^a f(x)dx$. We do *not* assume the mean value theorem.

We recall that the definite integral is defined as

$$\int_a^b f(x)dx = \sum_{k=1}^{\infty} f\left(a + \frac{k(b-a)}{\infty}\right)\frac{b-a}{\infty},$$

and the derivative as

$$\frac{df(x)}{dx} = \frac{f(x+0) - f(x)}{0}.$$

We now compute the derivative of the definite integral. For any constant c,

$$\begin{aligned}
\frac{d}{dx}\int_c^x f(u)du &= \frac{\int_c^{x+0} f(u)du - \int_c^x f(u)du}{0} \\
&= \frac{\int_c^x f(u)du + \int_{x+0}^{x+0} f(u)du - \int_c^x f(u)du}{0} \\
&= \frac{\int_{x+0}^{x+0} f(u)du}{0} \\
&= \frac{f(x+0)0}{0} \\
&= f(x+0) \\
&= f(x).
\end{aligned}$$

We then have

$$\int_c^b f(u)du = F(b) + k,$$

where $F(x)$ is any function such that

$$\frac{dF(x)}{dx} = f(x),$$

and k is a constant that depends on c. Then we have

$$\begin{aligned}\int_a^b f(x)dx &= \int_a^c f(x)dx + \int_c^b f(x)dx \\ &= \int_c^b f(x)dx - \int_c^a f(x)dx \\ &= F(b) - F(a).\end{aligned}$$

Numeristics and equinfinitesimal analysis allow us to apply these definitions and theorems to a very wide range of functions. A function that is conventionally considered discontinuous has an infinite equinfinitesimal derivative at the point of discontinuity. A similarly wide net is cast for integration. Abscissas and ordinates may be finite or infinite, single valued or multivalued. The only restriction on f in either case is that it be defined at all points concerned.

OTHER EQUINFINITESIMAL PROOFS

Chain rule

Because equinfinitesimal derivatives are arithmetic fractions of differentials, the chain rule is almost trivial.

$$\begin{aligned}\frac{d}{dx}f(g(x)) &= \frac{f(g(x+0)) - f(g(x))}{0} \\ &= \frac{\left[f(g(x+0)) - f(g(x))\right]\left[g(x+0) - g(x)\right]}{0\left[g(x+0) - g(x)\right]} \\ &= \frac{f(g(x+0)) - f(g(x))}{g(x+0) - g(x)}\,\frac{g(x+0) - g(x)}{0} \\ &= \left[\frac{d}{d\,g(x)}\right]\left[\frac{d}{dx}g(x)\right].\end{aligned}$$

Product rule

To derive the product rule, we use the fact that f and g at the infinitesimal level are straight lines, and $f(x+0)g(x+0)$ can be considered the area of a rectangle with sides $f(x+0) = f(x) + [f(x+0) - f(x)]$ and $g(x+0) = g(x) + [g(x+0) - g(x)]$. We also use the fact that $[f(x+0) - f(x)][g(x+0) - g(x)] = 0^2 \operatorname{sen} 0$, which vanishes from the result.

$$\begin{aligned}
\frac{d}{dx} f(x)g(x) &= \frac{f(x+0)g(x+0) - f(x)g(x)}{0} \\
&= \frac{1}{0}\Big(f(x)g(x) \\
&\quad + f(x)[g(x+0) - g(x)] \\
&\quad + [f(x+0) - f(x)]g(x) \\
&\quad + [f(x+0) - f(x)][g(x+0) - g(x)] \\
&\quad - f(x)g(x)\Big) \\
&= \frac{f(x)[g(x+0) - g(x)] + g(x)[f(x+0) - f(x)]}{0} \\
&= f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x).
\end{aligned}$$

Derivatives of sine and cosine

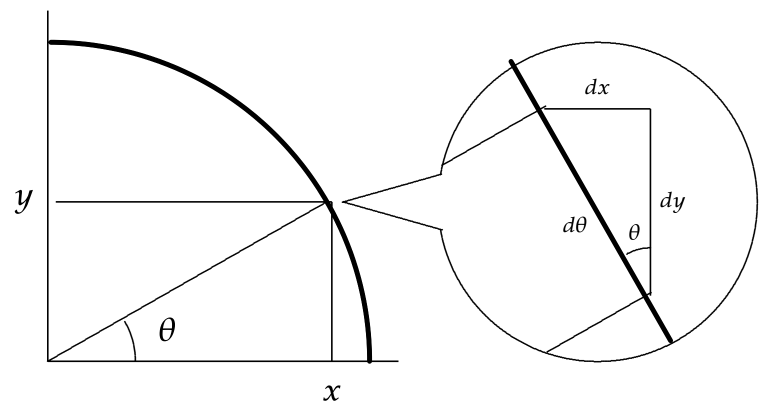

FIG. 7:
Calculation of derivatives
of sine and cosine

Figure 7 depicts the calculation of the sine and cosine functions. In this figure, we have

$$x = \cos\theta,$$
$$y = \sin\theta$$

and a microscope picture of the point (x, y).

In the microscope, the circle has become a straight line, coincident with the tangent to the circle at (x, y). Outside the microscope, the radius is a single line, but within the microscope, the radius is the class of all lines normal to the tangent. We show two such radius lines that are separated by the distance $d\theta$. The units of this distance must match the units in the tangent and radius, so we must measure $d\theta$, and thus θ itself, in radians.

The line segment along the tangent bounded by the two radii forms a triangle with legs dx and dy and hypotenuse $d\theta$. We then have

$$\begin{aligned}\frac{y}{x} &= -\frac{dx}{dy},\\ \frac{dx}{d\theta} &= -\sin\theta,\\ \frac{dy}{d\theta} &= \cos\theta.\end{aligned}$$

Derivative of exponential function

We start with an equinfinitesimal definition of e and compute e^x.

$$\begin{aligned}e &= (1+0)^{\infty}\\ &= \left(1+\frac{1}{\infty}\right)^{\infty},\\ e^x &= \left(1+\frac{1}{\infty}\right)^{\infty x}\\ &= \left(1+\frac{x}{\infty}\right)^{\infty}.\end{aligned}$$

The last line comes from the substitution $\infty x \to \infty$, or $\infty \to \dfrac{\infty}{x}$. We then have

$$\begin{aligned}e^{0x} &= (e^x)^0\\ &= \left(1+\frac{x}{\infty}\right)^{\infty 0}\\ &= \left(1+\frac{x}{\infty}\right)^{1}\\ &= \left(1+\frac{x}{\infty}\right)\\ &= \left(1+0x\right).\end{aligned}$$

Solving for x we have

$$\begin{aligned} x &= \frac{e^{0x}-1}{0}, \\ 1 &= \frac{e^{0}-1}{0}, \\ e^{x} &= e^{x}\frac{e^{0}-1}{0} \\ &= \frac{e^{x+0}-e^{x}}{0} \\ &= \frac{d}{dx}e^{x}dx. \end{aligned}$$

Natural logarithm as a polynomial

The following derivation uses equinfinitesimal analysis to show that $\int_1^t t^{-1}dt = \ln t$ is an instance of the general law $\int_0^t t^n dt = \frac{t^{n+1}}{n+1}$ and not an exception. We start with a result from the previous section and substitute $x = \ln y$.

$$\begin{aligned} x &= \frac{e^{0x}-1}{0} \\ \ln y &= \frac{y^{0}-1}{0}. \end{aligned}$$

We then integrate t^{-1} with $\int_0^t t^n dt = \frac{t^{n+1}}{n+1}$ and obtain

$$\begin{aligned} \int_1^t t^{-1}\,dt &= \frac{t^0}{0} - \frac{1}{0} \\ &= \frac{t^0-1}{0} \\ &= \ln t. \end{aligned}$$

This result can be verified with L'Hopital's rule, proved below.

DIFFERENTIAL AND INTEGRAL OPERATORS

Definition

As Leibnitz notation $\dfrac{d}{dx}$ indicates, a derivative is an arithmetic quotient of differentials. Differentials are members of zero that are exactly equal to zero at sensitivity level 1 and unequal at sensitivity level 0, and whose arithmetic transfers from the real numbers.

The differential is an operator on a function with respect to a member of zero. To put this more formally, we define

$$df(x) = (d_0 f)(x) := f(x+0) - f(x),$$

from which follows

$$dx = d_0 x = (d_0 1)(x) = x + 0 - x = 0.$$

We then obtain this definition of the derivative:

$$f'(x) := \frac{(d_0 f)(x)}{d_0 x} = \frac{f(x+0) - f(x)}{0}.$$

For $f(x) = x^n$,

$$(d_0 f)(x) = nx^{n-1}0 + \sum_{k=2}^{n} \binom{n}{k} x^{n-k} 0^k,$$

so $\dfrac{d_0 f(x)}{d_0 x}$ is independent of 0 for analytic functions.

Similarly we define the definite integral as an operator:

$$
\begin{aligned}
\int_a^b f(x) &:= \int_{\substack{a\\ \infty}}^b f(x) \\
&= \sum_{k=1}^{\infty} f\left(a + k\frac{b-a}{\infty}\right) \\
\int_a^b f(x)dx &= \int_{\substack{a\\ \infty}}^b f(x)d_{(b-a)/\infty}x \\
&= \int_{\substack{a\\ \infty}}^b f(x)\frac{b-a}{\infty} \\
&= \sum_{k=1}^{\infty} f\left(a + k\frac{b-a}{\infty}\right)\frac{b-a}{\infty}.
\end{aligned}
$$

A definite integral is thereby an infinite arithmetic sum of differentials.

We can define the indefinite integral operator in terms of the definite integral, either as a definite integral plus an arbitrary constant,

$$\int f(x) := \int_0^x f(t) + \mathbb{R} = \left\{\int_0^x f(t) + a \mid a \in \mathbb{R}\right\},$$

or as a definite integral with an arbitrary lower limit,

$$\int f(x) := \int_{\mathbb{R}}^x f(t) = \left\{\int_a^x f(t) \mid a \in \mathbb{R}\right\}.$$

Higher order derivatives

The simple form of the equinfinitesimal derivative lends itself to direct calculaton of higher order derivatives. These derivatives are are also simple quotients, with $dx^n = (dx)^n$ in the denominator.

Computing higher order derivatives is therefor mainly a matter of computing the numerator $d^n f(x)$, which is an iterated application of differential operator. For example,

$$\begin{aligned}
f''(x) &= \frac{d^2}{dx^2} f(x) \\
&= \frac{d[d[f(x)]]}{0^2} \\
&= \frac{d[f(x+0) - f(x)]}{0^2} \\
&= \frac{[f(x+2\cdot 0) - f(x+0)] - [f(x+0) - f(x)]}{0^2} \\
&= \frac{f(x+2\cdot 0) - 2f(x+0) + f(x)}{0^2},
\end{aligned}$$

The expansion of these operators is similar to expansion of the binomial $(a+b)^n$, with $f(x+0k)$ corresponding to a^k and $(-1)^k f(x)$ to b^k. The n-th derivative is thus given by

$$f^{(n)} = \frac{d^n}{dx^n} f(x) = \frac{\sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} f(x+0k)}{0^n}.$$

L'Hôpital's rule

Compared with conventional analysis, in equinfinitesimal analysis the statement of L'Hôpital's rule is different: For functions f and g and $q(x) \equiv \frac{f(x)}{g(x)}$, if f and g are continuous at c and $f(c) = g(c) = 0$, then $q(c) = \frac{f(c)}{g(c)} = \frac{f'(c)}{g'(c)}$, and $q(x)$ is continuous at c if $\frac{f'(c)}{g'(c)}$ is single valued.

The equinfinitesimal version of this rule simply evaluates the function $q(x)$ at c, since $f(c) = g(c) = 0$ is insufficient to compute $\frac{f(c)}{g(c)}$ as a single value.

Proof: If $d \neq 0$, and $f(c+d)$ or $g(c+d)$ or both are perfinite, or one is zero and the other infinite, then $q(c+d) = \frac{f(c+d)}{g(c+d)} = \frac{f(c+d)-f(c)}{g(c+d)-g(c)}$ is single valued. We then consider an infinitesimal, $d = 0 \operatorname{sen} 1$ and $d \neq 0^2 \operatorname{sen} 0$. Since f is continuous at c, we have $f(c+d) = f(c+0) = f(c) = 0 = f(d) - f(c)$, and similarly for g. For q to be continuous at c, we must also have $q(c) = q(c+0) = q(c+d)$. Then

$$\begin{aligned}
q(c) &= q(c+0) \\
&= \frac{f(c+0)}{g(c+0)} \\
&= \frac{f(c+0)-f(c)}{g(c+0)-g(c)} \\
&= \frac{\frac{f(c+0)-f(c)}{0}}{\frac{g(c+0)-g(c)}{0}} \\
&= \frac{f'(c)}{g'(c)}.
\end{aligned}$$

Since f and g are continuous, $f'(c)$ and $g'(c)$ are finite. If both $f'(c)$ and $g'(c)$ are zero, then we can iterate the rule until we find some n for which either $f^{(n)}(c)$ or $g^{(n)}(c)$ or both are nonzero. If both derivatives are zero for all n, then the rule does not give a single value for $q(c)$.

Since numeristic division and logarithms are unrestricted, it is easy to extend the rule to other indeterminate forms.

- If $q(c) = f(x)g(x)$ is of the form $0 \cdot \infty$, then use the rule on $\frac{f(x)}{\frac{1}{g(c)}}$.

- If $q(c) = \dfrac{f(c)}{g(c)}$ is of the form $\dfrac{\infty}{\infty}$, then use the rule on $\dfrac{1}{q(c)} = \dfrac{\frac{1}{f(x)}}{\frac{1}{g(c)}}$.
- If $q(c) = f(c) - g(c)$ is of the form $\infty - \infty$, then use the rule on $e^{q(c)} = \dfrac{e^{f(c)}}{e^{g(c)}}$.
- If $q(c) = f(c)^{g(c)}$ is of the form 0^0, 1^∞, or ∞^0, then use the rule on $\ln q(c) = g(c) \ln f(c)$.

Power series

In the following, we define σ as an integration operator with a fixed lower bound but an indefinite upper bound:

$$\sigma_a f(t) \equiv \int_a^t f(t)\, dt$$

and

$$\sigma f(t) \equiv \sigma_0 f(t) = \int_0^t f(t)\, dt$$

Powers of σ denote repeated integration or differentiation:

$$\begin{aligned}
\sigma^n f(t) &= \int_0^t \int_0^{u_n} \int_0^{u_{n-1}} \cdots \int_0^{u_3} \int_0^{u_2} f(u)\, du_1\, du_2\, \ldots\, du_{n-2}\, du_{n-1}\, du_n \\
\sigma^{-n} f(t) &= \frac{d^n}{du^n} f(u) \Big|_{u=t} \\
\sigma^0 f(t) &= f(t).
\end{aligned}$$

The power series for an analytic function f is derived as follows. We start by integrating and differentiating f repeatedly.

$$
\begin{aligned}
\sigma_a\sigma^{-1}f(t) &= f(t) - f(a)\\
\sigma_a\sigma^{-2}f(t) &= \sigma^{-1}f(t) - \sigma^{-1}f(a)\\
\sigma_a^2\sigma^{-2}f(t) &= f(t) - f(a) - (t-a)\sigma^{-1}f(a)\\
\sigma_a^2\sigma^{-3}f(t) &= \sigma^{-1}f(t) - \sigma^{-1}f(a) - (t-a)\sigma^{-2}f(a)\\
\sigma_a^3\sigma^{-3}f(t) &= f(t) - f(a) - (t-a)\sigma^{-1}f(a) - \frac{1}{2}(t-a)^2\sigma^{-2}f(a)\\
&\dots\\
\sigma_a^n\sigma^{-n}f(t) &= f(t) - f(a) - (t-a)\sigma^{-1}f(a) - \frac{1}{2}(t-a)^2\sigma^{-2}f(a) - \dots\\
&\quad - \frac{1}{n!}(t-a)^n\sigma^{-n}f(a).
\end{aligned}
$$

We then take the infinite case of this series and regard it as an operator ψ on f. We do similar operations on this series and find that it leaves the series unchanged.

$$
\begin{aligned}
\psi f(t) &\equiv \sigma_a^\infty\sigma^{-\infty}f(t)\\
&= f(t) - f(a) - (t-a)\sigma^{-1}f(a) - \frac{1}{2}(t-a)^2\sigma^{-2}f(a) - \dots\\
\psi\sigma f(t) &= \sigma f(t) - \sigma f(a) - (t-a)\sigma^{-2}f(a) - \frac{1}{2}(t-a)^2\sigma^{-3}f(a) - \dots\\
\sigma_a\psi\sigma f(t) &= f(t) - f(a) - (t-a)\sigma^{-1}f(a) - \frac{1}{2}(t-a)^2\sigma^{-2}f(a) - \dots\\
&= \psi f(t).
\end{aligned}
$$

For all infinitely differentiable f and all a, we now have $\psi f(t) = \sigma_a\psi\sigma f(t)$, or $\psi\sigma f(t) = \sigma_a\psi f(t)$. Since $\sigma_a\sigma f(t) = \sigma\sigma_a f(t)$, by the definition of ψ we have $\psi\sigma f(t) = \sigma\psi f(t) = \sigma_a\psi f(t) = \sigma\psi f(t) - \sigma\psi f(a)$. Subtracting, $\sigma\psi f(a) = 0$ for all a, i.e. $\sigma\psi f$ is the zero function. Hence $\psi f(t)$ must also be the zero function for all f, i.e. ψ is the zero operator. So

$$
\begin{aligned}
f(t) &= f(a) - (t-a)\sigma^{-1}f(a) - \frac{1}{2}(t-a)^2\sigma^{-2}f(a) - \dots\\
&= \sum_{n=0}^{\infty}\frac{(t-a)^n}{n!}\sigma^{-n}f(a)\\
&= e^{(t-a)\sigma^{-1}}f(a).
\end{aligned}
$$

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