

DIVERGENT SERIES

A Numeristic Approach

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[I]n the early years of this century [the 20th] the subject, while in no way mystical or unrigorous, *was* regarded as sensational, and about the present title [*Divergent Series*], now colourless, there hung an aroma of paradox and audacity.—J. E. Littlewood in his preface to [\[H49\]](#)

SUMMARY

Infinite divergent series can generate some striking results but have been controversial for centuries. The standard approaches of limits and method of summation have drawbacks which do not account for the full range of behavior of these series. A simpler approach, using **numeristics** (p. 31), is developed here, one which better accounts for divergent series and their sums.

The modern theory of divergent series can be said to begin with Euler, whose basic technique we call the *Euler extension method*. The numeric approach starts with this method and has two basic aspects:

1. *Recursion*, which is the Euler extension method coupled with the understanding that most infinite series have at least two values, one finite and one infinite. For example, using recursion, we establish that $1 + 2 + 4 + 8 + \dots = \{-1, \infty\}$.
2. *Equipoint summation*, which uses *equipoint analysis* (p. 125–301).

Equipoint analysis uses unfolding to extend real and complex numbers to unfolded numbers, functions, and relations. Equipoint summation uses unfolded terms of an infinite series and shows that the sum of a divergent series may depend on the mode of unfolding. For instance, $1 - 1 + 1 - 1 + \dots$ may have the value ∞ , $\frac{1}{2}$, an arbitrary finite number, 1, or 0, depending on the mode of unfolding the terms.

Various objections to the numeric approach are addressed. It is shown that the numeric approach enables the use of algebraic properties that are commonly used to handle finite expressions but that conventional theory claims do not hold for infinite series. By enabling the use of these properties, numeristics greatly simplifies the handling of infinite series, including divergent series, while providing a consistent and simple approach that handles a wide variety of cases.

The numeric approach to divergent series developed in this part can be summarized as follows.

1. Use only strict equality, not any form of weakened equality.

2. Use the projectively extended real numbers and numeric classes to extend arithmetic to any operation, including the exponential of infinity, which has two values.
3. Use algebraic transformations to compute the equality of a series with its sum, instead of defining the sum as a limit.
4. Denote methods of summation other than the Euler extension method as modified forms of summation, instead of modified forms of equality.
5. Use equipoint arithmetic to determine the sum of an infinite number of zeros, instead of assuming that the sum is always zero.

INTRODUCTION

In this part, we will see that we can find finite sums for infinite divergent series. Surprisingly, we find that it is controversial among mathematicians as to whether these series even have sums. We will examine this topic in some detail, and we will develop an approach which enables us to better appreciate these series and the surprising nature of the infinite which they show.

This approach is an application of *numeristics*, a number based foundational theory. Numeristics is introduced in the first part of this book.

A finite geometric series

$$\sum_{k=m}^n a^k = a^m + a^{m+1} + a^{m+2} + a^{m+3} + \dots + a^n$$

can easily be summed by through a recurrence formula. We let x be the sum:

$$x = \sum_{k=m}^n a^k = a^m + a^{m+1} + a^{m+2} + a^{m+3} + \dots + a^n,$$

and then we multiply both sides by a :

$$xa = a^{m+1} + a^{m+2} + a^{m+3} + a^{m+4} + \dots + a^{n+1},$$

and then we observe that $xa + a^m = x + a^{n+1}$. We therefore have $x - xa = a^m - a^{n+1}$, which yields

$$x = \frac{a^m - a^{n+1}}{1 - a},$$

the well-known result

$$\sum_{k=m}^n a^k = a^m + a^{m+1} + a^{m+2} + a^{m+3} + \dots + a^n = \frac{a^m - a^{n+1}}{1 - a}. \quad (1)$$

An infinite geometric series can also be summed in this way. If we have

$$\sum_{k=m}^{\infty} a^k = a^m + a^{m+1} + a^{m+2} + a^{m+3} + \dots,$$

then again we set

$$x = \sum_{k=m}^{\infty} a^k = a^m + a^{m+1} + a^{m+2} + a^{m+3} + \dots,$$

again multiply both sides by a ,

$$xa = a^{m+1} + a^{m+2} + a^{m+3} + a^{m+4} + \dots,$$

and observe that $xa + a^m = x$, yielding $x - xa = a^m$, or

$$x = \frac{a^m}{1-a},$$

and another well-known result

$$\sum_{k=m}^{\infty} a^k = a^m + a^{m+1} + a^{m+2} + a^{m+3} + \dots = \frac{a^m}{1-a}. \quad (2)$$

If $m = 0$, that is, if the first term is 1, then this result becomes

$$\sum_{k=0}^{\infty} a^k = 1 + a + a^2 + a^3 + \dots = \frac{1}{1-a}. \quad (3)$$

For example, if $a = \frac{1}{2}$ and $m = 1$, then the infinite geometric series is $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$. Equation 2 says that this sum is $\frac{1/2}{1-1/2} = 1$, that is

$$\sum_{k=1}^{\infty} \frac{1}{2^k} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots = 1. \quad (4)$$

Figure 98 shows a visualization of this sum, and the formula seems to agree with what we observe in such a diagram.

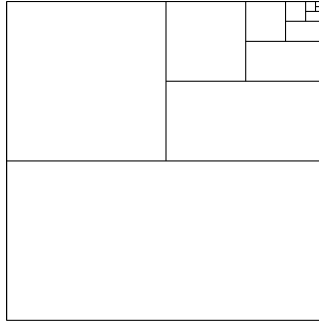


FIG. 98: Diagram of $1/2 + 1/4 + 1/8 + 1/16 + \dots = 1$

It has long been noticed that, when derived this way, Equation 2 seems to hold for almost any a . For example, if $a = 2$, we could conclude that

$$\sum_{k=0}^{\infty} 2^k = 1 + 2 + 4 + 8 + \dots = -1. \quad (5)$$

This may come as an even bigger surprise than Equation 4. Naturally, it might be wondered how Equation 5 could be true. The intermediate results, $1, 1 + 2 = 3, 1 + 2 + 4 = 7, 1 + 2 + 4 + 8 = 15$, etc., known as *partial sums*, grow without limit and are always positive, whereas in Equation 4, the partial sums $1, 1 + \frac{1}{2} = \frac{3}{2}, 1 + \frac{1}{2} + \frac{1}{4} = \frac{7}{4}, 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{15}{8}$, etc., get progressively closer to 1 but never exceed it.

Infinite series such as the one in Equation 4, in which the partial sums approach a fixed number, are known as *convergent*, while those that do not, such as the one in Equation 5, are known as *divergent*.

There are two general points of view on convergent and divergent infinite series. The conventional point of view is that divergent series are meaningless and have no sum, and only convergent series have a sum. In this view, the number that the partial sums converge to, called the *limit*, is considered the sum of the infinite series.

The alternative point of view is that divergent series are not automatically meaningless but may have a sum. Following this point of view, a theory

of divergent series has been developed. This theory is generally consistent and even has a number of applications. References [Bo, H49, Mo, Sm, Sz] are some of the important standard works of this theory, with [H49] often being regarded as the most important.

In this part, we will explore some of the results of this theory. We will use a new approach which resolves some longstanding problems. This approach is an application of *numeristics* and *equipoint analysis* (first and second parts of this book), which include an extension of arithmetic that allows us to include any values of a , m , and n in Equation 1.

The numeristic approach uses two basic techniques:

1. Recursion, presented in the first few chapters, starting with **Some results of Equation 2** (p. 311).
2. Equipoint summation, an application of equipoint analysis, presented in the **Equipoint summation** (p. 351) chapter.

Equipoint analysis includes a system of calculus which is simpler than the conventional system, but for infinite series, it adds a level of complexity which is not always necessary, so we save it for relatively difficult cases.

SOME RESULTS OF EQUATION 2

In this chapter we examine some straightforward results of Equation 2. Subtler points of the theory are examined in subsequent chapters.

$$0.999\dots = 1. \tag{6}$$

PROOF. This is a convergent series which we mention for completeness. $0.999\dots = 9(0.1 + 0.01 + 0.001 + 0.0001\dots) = 9(10^{-1} + 10^{-2} + 10^{-3} + 10^{-4} + \dots) = 9(0.1^1 + 0.1^2 + 0.1^3 + 0.1^4 + \dots) = 9 \sum_{k=1}^{\infty} .01^k = 9\left(\frac{.01}{1-.01}\right) = 9\left(\frac{.01}{.99}\right) = 9\left(\frac{1}{99}\right) = 1.$ ■

$$\dots 999 = -1. \tag{7}$$

PROOF. By the notation $\dots 999$ we mean an infinite series of digits going out to the left, just as the notation $0.999\dots$ means an infinite series of digits going out to the right. Then $\dots 999 = 9(1 + 10 + 100 + 1000 + \dots) = 9(10^0 + 10^1 + 10^2 + 10^3 + \dots) = 9 \sum_{k=0}^{\infty} 10^k = 9\left(\frac{1}{1-10}\right) = 9\left(-\frac{1}{9}\right) = -1.$ ■

Infinite left decimals of this type are explored in further detail in [Repeating Decimals](#) (p. 409–457).

$$\sum_{k=-\infty}^{\infty} a^k = \dots + a^{m-3} + a^{m-2} + a^{m-1} + a^m + a^{m+1} + a^{m+2} + a^{m+3} + \dots = 0, \tag{8}$$

$$a \neq 1.$$

PROOF.

$$\begin{aligned}
 & \dots + a^{m-3} + a^{m-2} + a^{m-1} + a^m + a^{m+1} + a^{m+2} + a^{m+3} + \dots \\
 &= (a^m + a^{m+1} + a^{m+2} + a^{m+3} + \dots) \\
 & \quad + (a^{m-1} + a^{m-2} + a^{m-3} + a^{m-4} + \dots) \\
 &= (a^m + a^{m+1} + a^{m+2} + a^{m+3} + \dots) \\
 & \quad + ([a^{-1}]^{1-m} + [a^{-1}]^{2-m} + [a^{-1}]^{3-m} + [a^{-1}]^{4-m} + \dots) \\
 &= \frac{a^m}{1-a} + \frac{a^{m-1}}{1-a^{-1}} = \frac{a^m}{1-a} + \frac{a a^{m-1}}{a(1-a^{-1})} = \frac{a^m}{1-a} + \frac{a^m}{a-1} = 0. \blacksquare
 \end{aligned}$$

ALTERNATE PROOF. If $x = \dots + a^{m-3} + a^{m-2} + a^{m-1} + a^m + a^{m+1} + a^{m+2} + a^{m+3} + \dots$, then $xa = \dots + a^{m-2} + a^{m-1} + a^m + a^{m+1} + a^{m+2} + a^{m+3} + a^{m+4} + \dots = x$, so $0 = x - xa = x(1-a)$. Then x must be 0, unless $a = 1$. \blacksquare

$$\dots 999.999 \dots = 0. \tag{9}$$

PROOF. $\dots 999.999 \dots = 9(\dots + 1000 + 100 + 10 + 1 + 0.1 + 0.01 + 0.001 + \dots) = 9(\dots + 10^3 + 10^2 + 10^1 + 10^0 + 10^{-1} + 10^{-2} + 10^{-3} + \dots) = 9 \sum_{k=-\infty}^{\infty} 10^k = 9(0) = 0$. \blacksquare

$$\sum_{k=0}^{\infty} e^{kix} = 1 + e^{ix} + e^{2ix} + \dots = \frac{1 + i \cot \frac{x}{2}}{2}. \tag{10}$$

PROOF. $e^{kix} = (e^{ix})^k$, so $1 + e^{ix} + e^{2ix} + \dots = \frac{1}{1-e^{ix}} = \frac{1}{2} + \frac{i}{2} \left(\frac{2i}{e^{ix}-1} - 1 \right) = \frac{1}{2} + \frac{i}{2} \cot \frac{x}{2}$. \blacksquare

$$\sum_{k=0}^{\infty} (-1)^k e^{kix} = 1 - e^{ix} + e^{2ix} - \dots = \frac{1 - i \tan \frac{x}{2}}{2}. \tag{11}$$

PROOF. In Equation 10, substitute x with $x + \pi$. ■

$$\begin{aligned} \sum_{k=1}^{\infty} \left[\frac{e^{kix}}{k} - (-1)^k \frac{1}{k} \right] &= e^{ix} + \frac{e^{2ix}}{2} + \frac{e^{3ix}}{3} + \dots + 1 - \frac{1}{2} + \frac{1}{3} - \dots \\ &= -\ln \sin \frac{x}{2} + i \frac{\pi - x}{2}. \end{aligned} \tag{12}$$

PROOF. Integrate Equation 10 from π to x to obtain $x - ie^{ix} - i \frac{e^{2ix}}{2} - i \frac{e^{3ix}}{3} - \dots - i + \frac{i}{2} - \frac{i}{3} + \dots = \frac{x-\pi}{2} + i \ln \sin \frac{x}{2}$. ■

$$\sum_{k=0}^{\infty} (-1)^k e^{(2k+1)ix} = e^{ix} - e^{3ix} + e^{5ix} - \dots = \frac{\sec x}{2}. \tag{13}$$

PROOF. $e^{ix} - e^{3ix} + e^{5ix} - \dots = \frac{e^{ix}}{1+e^{2ix}} = \frac{1}{2} \sec x$. ■

$$\sum_{k=1}^{\infty} \cos kx = \cos x + \cos 2x + \cos 3x + \dots = -\frac{1}{2}. \tag{14}$$

PROOF. From Equation 10, $\cos x + \cos 2x + \dots = \operatorname{Re} (1 + e^{ix} + e^{2ix} + \dots) - 1 = \frac{1}{2} - 1 = -\frac{1}{2}$. ■

$$\sum_{k=1}^{\infty} \frac{\sin kx}{k} = \sin x + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \dots = \frac{\pi - x}{2}. \tag{15}$$

PROOF. Integrate Equation 14 from π to x , or take the imaginary part of Equation 12. ■

$$\sum_{k=1}^{\infty} \sin kx = \sin x + \sin 2x + \sin 3x + \dots = \frac{\cot \frac{x}{2}}{2}. \quad (16)$$

PROOF. Take the imaginary part of Equation 10. ■

$$\begin{aligned} \sum_{k=1}^{\infty} \left[-\frac{\cos kx}{k} + (-1)^k \frac{1}{k} \right] &= -\cos x - \frac{\cos 2x}{2} - \frac{\cos 3x}{3} + \dots - 1 + \frac{1}{2} - \frac{1}{3} - \dots \\ &= \ln \sin \frac{x}{2}. \end{aligned} \quad (17)$$

PROOF. Integrate Equation 16 from π to x , or take the real part of Equation 12. ■

$$\sum_{k=1}^{\infty} (-1)^{k+1} \cos kx = \cos x - \cos 2x + \cos 3x - \dots = \frac{1}{2}. \quad (18)$$

PROOF. Starting with Equation 14, we replace x with $x + \pi$. Then $\cos nx$ remains unchanged when n is even, because we are adding $2m\pi$ to x , where $n = 2m$ and m is an integer. But when n is odd, then $n = 2m + 1$, and we are adding $2m\pi + \pi$ to x , and so $\cos nx$ becomes $-\cos nx$. This yields $-\cos x + \cos 2x - \cos 3x + \dots = -\frac{1}{2}$, or $\cos x - \cos 2x + \cos 3x - \dots = \frac{1}{2}$.

Alternatively, take the real part of Equation 11 and subtract 1. ■

$$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{\sin kx}{k} = \sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \dots = \frac{x}{2}. \tag{19}$$

PROOF. Integrate Equation 18 from 0 to x . ■

$$\sum_{k=0}^{\infty} (-1)^k \frac{1}{2k+1} = 1 - \frac{1}{3} + \frac{1}{5} - \dots = \frac{\pi}{4}. \tag{20}$$

PROOF. Evaluate Equation 19 at $x = \frac{\pi}{2}$, or evaluate Equation 15 at $x = \frac{\pi}{2}$ or $x = \frac{3\pi}{2}$. ■

This is a convergent series called the Gregory equation and is usually derived from the power series of $\tan^{-1} x$. While it can be used to approximate π , it is not very useful for this purpose, since it converges very slowly.

$$\sum_{k=1}^{\infty} (-1)^{k+1} \sin kx = \sin x - \sin 2x + \sin 3x - \dots = \frac{\tan \frac{x}{2}}{2}. \tag{21}$$

PROOF. We start with Equation 16 and use the same substitution, replacing x with $x + \pi$. Then $\sin nx$ remains unchanged for n even and becomes $-\sin nx$ for n odd. In addition, $\cot \frac{x}{2}$ becomes $\cot (\frac{x}{2} + \frac{\pi}{2}) = -\tan \frac{x}{2}$.

Alternatively, take the imaginary part of Equation 11. ■

MERCATOR'S SERIES:

$$\sum_{k=1}^{\infty} \frac{x^k}{k} = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \dots = -\ln(1-x) \tag{22}$$

$$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots = \ln(1+x) \tag{23}$$

PROOF. From Equation 2, we have $1 + x + x^2 + x^3 + \dots = \frac{1}{1-x}$ and $1 - x + x^2 - x^3 + \dots = \frac{1}{1+x}$. Integrating these from 0 to x gives the results. ■

THE HARMONIC SERIES:

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots = \infty. \quad (24)$$

PROOF. Evaluate Equation 22 for $x = 1$. ■

ALTERNATE PROOF. Let $x = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$. Then

$$\begin{aligned} x - 1 &= \frac{1}{1 \cdot 2} + \frac{2}{2 \cdot 3} + \frac{3}{3 \cdot 4} + \dots \\ &= \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \dots \\ &\quad + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \dots \\ &\quad + \frac{1}{3 \cdot 4} + \dots \\ &\quad + \dots \\ &= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \dots \\ &\quad + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \dots \\ &\quad + \left(\frac{1}{3} - \frac{1}{4}\right) + \dots \\ &\quad + \dots \\ &= 1 + \frac{1}{2} + \frac{1}{3} + \dots \\ &= x. \end{aligned}$$

The condition $x - 1 = x$ is satisfied by any infinite number, which in the projectively extended real numbers is the unique infinite value ∞ . ■

The first part of the second proof, the derivation $x - 1 = x$, is known as *Bernoulli's paradox*. In numeristics, this situation is not paradoxical at all, but simply one that is satisfied by an infinite value.

THE ALTERNATING HARMONIC SERIES:

$$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots = \ln 2. \quad (25)$$

PROOF. Evaluate Equation 23 at $x = 1$. Alternatively, evaluate Equation 17 at $x = \frac{\pi}{2}$, which yields $\frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \dots - 1 + \frac{1}{2} - \frac{1}{3} - \dots = -\frac{1}{2} (1 - \frac{1}{2} + \frac{1}{3} - \dots) = \ln \frac{1}{\sqrt{2}}$. ■

$$\sum_{k=1}^{\infty} \frac{2^k}{k} = \frac{2}{1} + \frac{4}{2} + \frac{8}{3} + \frac{16}{4} + \dots = (2\mathbb{Z} + 1)\pi i. \quad (26)$$

$(2\mathbb{Z} + 1)\pi i$ is numeric class notation for all $(2k + 1)\pi i$ where k is an integer.

PROOF. Evaluate Equation 22 for $x = 2$ and use the fact that $\ln(-1) = (2\mathbb{Z} + 1)\pi i$. ■

This result may come as an even bigger surprise than Equation 5. However, that the infinite sum of real numbers can be complex is simply an extension of the principle by which the infinite sum of positive numbers can be negative.

$$\sum_{k=0}^{\infty} (-1)^k \cos(2k + 1)x = \cos x - \cos 3x + \cos 5x - \dots = \frac{\sec x}{2}. \quad (27)$$

PROOF. From Equation 13, $\cos x - \cos 3x + \cos 5x - \dots = \operatorname{Re}(e^{ix} - e^{3ix} + e^{5ix} - \dots) = \frac{1}{2} \sec x$. ■

$$\sum_{k=0}^{\infty} (-1)^k \sin(2k+1)x = \sin x - \sin 3x + \sin 5x + \dots = 0. \quad (28)$$

PROOF. From Equation 13, $\sin x - \sin 3x + \sin 5x + \dots = \text{Im}(e^{ix} - e^{3ix} + e^{5ix} - \dots) = 0$. ■

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}. \quad (29)$$

PROOF. Integrate the negative of Equation 18 from 0 to x to obtain $\sin x - \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x - \dots = \frac{1}{2}x$. Integrate again from 0 to x to obtain $(1 - \cos x) - \frac{1}{2}(1 - \cos 2x) + \frac{1}{3^2}(1 - \cos 3x) - \dots = \frac{1}{4}x^2$. Evaluate this at $x = \pi$. ■

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots = \frac{\pi^2}{6}. \quad (30)$$

PROOF. Let $y = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots$. Then $\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots - \frac{1}{2^2} - \frac{1}{4^2} - \dots = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots - \frac{1}{4}(1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots) = y - \frac{1}{4}y = \frac{3}{4}y$, so $y = \frac{\pi^2}{6}$. ■

$$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \dots = \frac{\pi^2}{12}. \quad (31)$$

PROOF. $1 - \frac{1}{2^2} + \frac{1}{3^2} - \dots = \left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots\right) - \left(\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{8^2} + \dots\right) = \left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots\right) - \frac{1}{2^2} \left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots\right) = \frac{\pi^2}{8} - \frac{1}{4} \frac{\pi^2}{6} = \frac{\pi^2}{12}$. ■

$$\sum_{k=1}^{\infty} (-1)^{k+1} k^{2n} \cos kx = 1^{2n} \cos x - 2^{2n} \cos 2x + 3^{2n} \cos 3x - \dots$$

$$= 0, \quad n = 0, 1, 2, \dots \quad (32)$$

$$\sum_{k=1}^{\infty} (-1)^{k+1} k^{2n+1} \sin kx = 1^{2n+1} \sin x - 2^{2n+1} \sin 2x + 3^{2n+1} \sin 3x - \dots$$

$$= 0, \quad n = 0, 1, 2, \dots \quad (33)$$

PROOF. Repeated differentiation, n times, of Equation 18. ■

$$\sum_{k=1}^{\infty} (-1)^{k+1} k^{2n} \sin kx = 1^{2n} \sin x - 2^{2n} \sin 2x + 3^{2n} \sin 3x - \dots$$

$$= (-1)^n \left(\frac{d}{dx} \right)^{2n} \frac{\tan \frac{x}{2}}{2}, \quad n = 0, 1, 2, \dots \quad (34)$$

$$\sum_{k=1}^{\infty} (-1)^{k+1} k^{2n+1} \cos kx = 1^{2n+1} \cos x - 2^{2n+1} \cos 2x + 3^{2n+1} \cos 3x - \dots$$

$$= (-1)^n \left(\frac{d}{dx} \right)^{2n+1} \frac{\tan \frac{x}{2}}{2}, \quad n = 0, 1, 2, \dots \quad (35)$$

PROOF. Repeated differentiation, n times, of Equation 21. ■

$$\sum_{k=0}^{\infty} (-1)^k (2k+1)^{2n} \cos(2k+1)x =$$

$$1^{2n} \cos x - 3^{2n} \cos 3x + 5^{2n} \cos 5x - \dots = (-1)^n \left(\frac{d}{dx} \right)^{2n} \frac{\sec x}{2},$$

$$n = 0, 1, 2, \dots \quad (36)$$

$$\sum_{k=0}^{\infty} (-1)^k (2k+1)^{2n+1} \sin(2k+1)x =$$

$$1^{2n+1} \sin x - 3^{2n+1} \sin 3x + 5^{2n+1} \sin 5x - \dots = (-1)^n \left(\frac{d}{dx} \right)^{2n+1} \frac{\sec x}{2},$$

$$n = 0, 1, 2, \dots \quad (37)$$

PROOF. Repeated differentiation, n times, of Equation 27. ■

$$\sum_{k=0}^{\infty} (-1)^k (2k+1)^{2n} \sin(2k+1)x =$$

$$1^{2n} \sin x - 3^{2n} \sin 3x + 5^{2n} \sin 5x - \dots = 0, \quad n = 0, 1, 2, \dots \quad (38)$$

$$\sum_{k=0}^{\infty} (-1)^k (2k+1)^{2n+1} \cos(2k+1)x =$$

$$1^{2n+1} \cos x - 3^{2n+1} \cos 3x + 5^{2n+1} \cos 5x - \dots = 0, \quad n = 0, 1, 2, \dots \quad (39)$$

PROOF. Repeated differentiation, n times, of Equation 28. ■

$$\sum_{k=1}^{\infty} (-1)^{k+1} k^{2n} = 1^{2n} - 2^{2n} + 3^{2n} - \dots = 0, \quad n = 1, 2, 3, \dots \quad (40)$$

PROOF. Evaluate Equation 32 at $x = 0$. ■

$$\sum_{k=0}^{\infty} (-1)^k (2k+1)^{2n+1} = 1^{2n+1} - 3^{2n+1} + 5^{2n+1} - \dots = 0, \quad n = 0, 1, 2, \dots \quad (41)$$

PROOF. Evaluate Equation 33 at $x = \frac{\pi}{2}$. ■

$$\sum_{k=0}^{\infty} (2k + 1)^{2n+1} = 1^{2n+1} + 3^{2n+1} + 5^{2n+1} + \dots = 0, \quad n = 0, 1, 2, \dots \quad (42)$$

PROOF. Evaluate Equation 38 at $x = \frac{\pi}{2}$. ■

$$\sum_{k=0}^{\infty} (-1)^k (2k + 1)^{2n} = 1^{2n} - 3^{2n} + 5^{2n} - \dots = 0, \quad n = 0, 1, 2, \dots \quad (43)$$

PROOF. Evaluate Equation 39 at $x = 0$. ■

$$\begin{aligned} \sum_{k=1}^{\infty} (-1)^{k+1} k^{2n+1} &= 1^{2n+1} - 2^{2n+1} + 3^{2n+1} + \dots \\ &= \frac{2^{2n+2} - 1}{2n + 2} B_{2n+2}, \quad n = 0, 1, 2, \dots \end{aligned} \quad (44)$$

PROOF. B_k stands for the k -th Bernoulli number, which occurs in the power series $\tan x = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{2^{2k} (2^{2k} - 1)}{(2k)!} B_{2k} x^{2k-1}$.

To prove the above identity, we evaluate

$$1^{2n+1} \cos x - 2^{2n+1} \cos 2x + 3^{2n+1} \cos 3x - \dots = (-1)^n \left(\frac{d}{dx} \right)^{2n+1} \frac{1}{2} \tan \frac{x}{2}$$

for $x = 0$. We begin by computing the power series

$$\frac{1}{2} \tan \frac{x}{2} = \frac{1}{2} \sum_{k=1}^{\infty} (-1)^{k-1} \frac{2^{2k} (2^{2k} - 1)}{(2k)!} B_{2k} \frac{x^{2k-1}}{2^{2k-1}} = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{2^{2k} - 1}{(2k)!} B_{2k} x^{2k-1}.$$

We then differentiate this series at $x = 0$. We observe that

$$\left(\frac{d}{dx} \right)^n x^k = k(k-1)(k-2) \dots (k-n+1) x^{k-n}.$$

When $x = 0$, this product is nonzero only when x^{k-n} is constant, i.e. when $k = n$, at which time $\left(\frac{d}{dx}\right)^n x^k = k(k-1)(k-2)\cdots 1 = k!$. This means that all terms of the power series vanish, except for $k = n$.

Therefore, $\left(\frac{d}{dx}\right)^{2n+1} x^{2k-1} = (2n+1)!$ when $2n+1 = 2k-1$ or $k = n+1$, and is zero when $k \neq n+1$. We then have

$$\begin{aligned} (-1)^n \left(\frac{d}{dx}\right)^{2n+1} \frac{1}{2} \tan \frac{x}{2} &= (-1)^n (-1)^{n+2} \frac{(2^{2n+2}-1)(2n+1)!}{(2n+2)!} B_{2n+2} \\ &= \frac{2^{2n+2}-1}{2n+2} B_{2n+2}. \blacksquare \end{aligned}$$

Hardy and other older authors state this result in a different form, because they used an older system of indexing the Bernoulli numbers. If we let B_k^* represent the old system and B_k the new, then the relation is $B_{2k} = (-1)^{k-1} B_k^*$. We then obtain the older statement of the result,

$$1^{2n+1} - 2^{2n+1} - 3^{2n+1} + \dots = (-1)^n \frac{2^{2n+2}-1}{2n+2} B_{n+1}^*.$$

$$\sum_{k=0}^{\infty} (-1)^k (2k+1)^{2n} = 1^{2n} - 3^{2n} + 5^{2n} + \dots = \frac{1}{2} E_{2n}, \quad n = 0, 1, 2, \dots \quad (45)$$

PROOF. E_k stands for the k -th Euler number, which occurs in the power series $\sec x = \sum_{n=0}^{\infty} (-1)^k \frac{1}{(2k)!} E_{2k} x^{2k}$. To prove our identity, we evaluate

$$1^{2n} \cos x - 3^{2n} \cos 2x + 5^{2n} \cos 5x - \dots = (-1)^n \left(\frac{d}{dx}\right)^{2n} \frac{1}{2} \sec x$$

for $x = 0$.

As before, to differentiate the power series, we use the fact that for $x = 0$, $\left(\frac{d}{dx}\right)^n x^k = k!$ when $k = n$, and is zero when $k \neq n$. We then have

$$(-1)^n \left(\frac{d}{dx}\right)^{2n} \frac{1}{2} \sec x = (-1)^n (-1)^n \frac{(2n)!}{2(2n)!} E_{2n} = \frac{1}{2} E_{2n}. \blacksquare$$

As with the Bernoulli numbers, there is also an older system of indexing the Euler numbers, which leads to a different form of the result by older authors. Letting E_k^* represent the old system and E_k the new, the relation is $E_{2k} = (-1)^k E_k^*$. The older statement of the result is then

$$1^{2n} - 3^{2n} + 5^{2n} + \dots = (-1)^n \frac{1}{2} E_n^*.$$

OTHER INFINITE SERIES

Here we examine a divergent series whose sum does not depend on Equation 2 but is consistent with it.

$$\begin{aligned}
 \sum_{k=0}^{\infty} (-1)^k k! x^k &= 1 - 1!x + 2!x^2 - 3!x^3 \dots \\
 &= e^{\frac{1}{x}} \left(-\frac{\gamma}{x} + \frac{\ln x}{x} + \frac{1}{x^2} - \frac{1}{2 \cdot 2!x^3} + \frac{1}{3 \cdot 3!x^4} - \dots \right) \\
 &= \frac{e^{\frac{1}{x}}}{x} \left(-\gamma + \ln x - \sum_{k=1}^{\infty} (-1)^k \frac{1}{k \cdot k!x^k} \right) \tag{46}
 \end{aligned}$$

Here γ denotes the **Euler-Mascheroni constant**, which occurs in several contexts in analysis and has many equivalent definitions, including:

$$\begin{aligned}
 \gamma &\equiv - \int_1^{\infty} \frac{e^{-x}}{x} dx \\
 &\equiv - \int_0^{\infty} e^{-x} \ln x dx \\
 &\equiv \int_1^{\infty} \left(\frac{1}{[x]} - \frac{1}{x} \right) dx \\
 &\equiv \sum_{x=1}^{\infty} \frac{1}{x} - \int_1^{\infty} \frac{1}{x} dx
 \end{aligned}$$

The last definition uses equipoint analysis to express the same idea as the second last definition. Equipoint analysis is explained further below and in [the second part of this book](#) (p. 125–301).

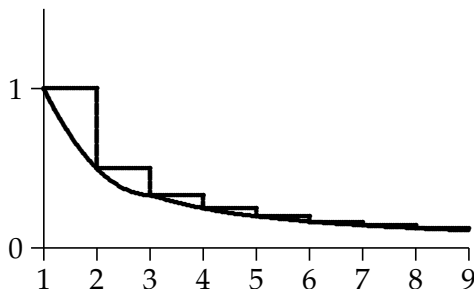


FIG. 99:
 γ as the area between the graphs
of the sum and the integral of $\frac{1}{x}$

PROOF. We set

$$f(x) \equiv 1 - 1!x + 2!x^2 - 3!x^3 + \dots$$

$$g(x) \equiv xf(x) = x - 1!x^2 + 2!x^3 - 3!x^4 + \dots$$

Then

$$x^2g'(x) + g(x) = x^2(1! - 2!x + 3!x^2 - 4!x^3 + \dots) + (x - 1!x^2 + 2!x^3 - 3!x^4 + \dots) = x.$$

The equation $x^2g'(x) + g(x) = x$ is a first order linear differential equation of the form $y'(x) + P(x)y(x) = Q(x)$. The general solution, which we will not derive here, is $y(x) = \frac{1}{I(x)} \int I(x)Q(x)dx$, with the *integrating factor* $I(x) = e^{\int P(x)dx}$. For our specific case, $P(x) = \frac{1}{x^2}$, $Q(x) = \frac{1}{x}$, $I(x) = e^{\int \frac{dx}{x^2}} = e^{-\frac{1}{x}}$, and

$$y(x) = g(x) = e^{\frac{1}{x}} \int \frac{e^{-\frac{1}{x}}}{x} dx = e^{\frac{1}{x}} \int_0^x \frac{e^{-\frac{1}{u}}}{u} du.$$

Alternatively, by Equation 2,

$$f(x) = 1 - 1!x + 2!x^2 - 3!x^3 \dots$$

$$= \int_0^\infty e^{-v} dv - x \int_0^\infty v e^{-v} dv + x^2 \int_0^\infty v^2 e^{-v} dv - x^3 \int_0^\infty v^3 e^{-v} dv + \dots$$

$$= \int_0^\infty \frac{e^{-v} dv}{1 + xv}.$$

We then substitute $u \equiv \frac{x}{1+xv}$, from which we have $v = \frac{1}{u} - \frac{1}{x}$ and

$$\begin{aligned} \frac{-e^{-v} dv}{1+xv} &= -\frac{e^{\frac{1}{x}}}{x} \frac{-xe^{-\frac{1}{x}-v} dv}{1+xv} \\ &= -\frac{e^{\frac{1}{x}}}{x} \frac{(1+xv)e^{-\frac{1}{x}-v} -x^2 dv}{1+xv} \\ &= -\frac{e^{\frac{1}{x}}}{x} \frac{e^{-\frac{1}{u}} du}{u}. \end{aligned}$$

When $v = 0$, $u = x$, and when $v = \infty$, $u = 0$, so

$$f(x) = \frac{g(x)}{x} = \int_0^\infty \frac{e^{-v} dv}{1+xv} = \frac{e^{\frac{1}{x}}}{x} \int_0^x \frac{e^{-\frac{1}{u}} du}{u},$$

the same result as above.

We now substitute $w \equiv \frac{1}{u}$ and obtain

$$\int_0^x \frac{e^{-\frac{1}{u}} du}{u} = -\int_\infty^{\frac{1}{x}} \frac{e^{-w} w}{w^2} = \int_{\frac{1}{x}}^\infty \frac{e^{-w}}{w} dw.$$

We then have

$$\begin{aligned} \int_{\frac{1}{x}}^\infty \frac{e^{-w}}{w} dw &= \int_0^\infty \frac{e^{-w}}{w} dw - \int_0^{\frac{1}{x}} \frac{e^{-w}}{w} dw \\ &= \int_0^\infty \frac{e^{-w}}{w} dw + e^{-w} \ln w \Big|_0^\infty - \int_0^{\frac{1}{x}} \left(\frac{1}{w} - 1 + \frac{w}{2!} - \frac{w^2}{3!} + \dots \right) dw \\ &= -\gamma + \ln w \Big|_0^1 - \ln w \Big|_0^{\frac{1}{x}} + w \Big|_0^{\frac{1}{x}} - \frac{w^2}{2 \cdot 2!} \Big|_0^{\frac{1}{x}} + \frac{w^3}{3 \cdot 3!} \Big|_0^{\frac{1}{x}} - \dots \\ &= -\gamma + \ln x + \frac{1}{x} - \frac{1}{2 \cdot 2!x^2} + \frac{1}{3 \cdot 3!x^3} - \dots \\ f(x) &= \frac{e^{\frac{1}{x}}}{x} \int_{\frac{1}{x}}^\infty \frac{e^{-w}}{w} dw \\ &= e^{\frac{1}{x}} \left(-\frac{\gamma}{x} + \frac{\ln x}{x} + \frac{1}{x^2} - \frac{1}{2 \cdot 2!x^2} + \frac{1}{3 \cdot 3!x^3} - \dots \right) \quad \blacksquare \end{aligned}$$

$$\sum_{k=0}^{\infty} (-1)^k k! = 1 - 1! + 2! - 3! \dots$$

$$\begin{aligned}
&= e \left(-\gamma + 2\mathbb{Z}\pi i + 1 - \frac{1}{2 \cdot 2!} + \frac{1}{3 \cdot 3!} - \dots \right) \\
&= e \left(-\gamma + 2\mathbb{Z}\pi i - \sum_{k=1}^{\infty} (-1)^k \frac{1}{k \cdot k!} \right) \tag{47}
\end{aligned}$$

PROOF. Application of Equation 46 for $x = 1$. It has one real value and an infinite number of complex values. To four decimal places, its approximate value is $0.5963 + 17.0795\mathbb{Z}i$. ■

$$\begin{aligned}
\sum_{k=0}^{\infty} k! &= 1 + 1! + 2! + 3! \dots \\
&= \frac{1}{e} \left(-\gamma + (2\mathbb{Z} + 1)\pi i + 1 + \frac{1}{2 \cdot 2!} + \frac{1}{3 \cdot 3!} + \dots \right) \\
&= \frac{1}{e} \left(-\gamma + (2\mathbb{Z} + 1)\pi i + \sum_{k=1}^{\infty} \frac{1}{k \cdot k!} \right) \tag{48}
\end{aligned}$$

PROOF. Application of Equation 46 for $x = -1$. It has no real values but an infinite number of complex values. To four decimal places, its approximate value is $0.2719 + 1.557 \cdot (2\mathbb{Z} + 1)i$. ■

INFINITE PRODUCTS

We now consider various infinite products involving prime numbers. We will first need to know how to multiply two infinite series, and how to multiply an infinite number of binomials.

$$\begin{aligned}
 \left(\sum_{j=1}^m a_j \right) \left(\sum_{k=1}^n b_k \right) &= (a_1 + a_2 + a_3 + \dots + a_m)(b_1 + b_2 + b_3 + \dots + b_n) \\
 &= \sum_{j=1}^m \sum_{k=1}^n a_j b_k = a_1 b_1 + a_1 b_2 + a_1 b_3 + \dots + a_1 b_n \\
 &\quad + a_2 b_1 + a_2 b_2 + a_2 b_3 + \dots + a_2 b_n \\
 &\quad + a_3 b_1 + a_3 b_2 + a_3 b_3 + \dots + a_3 b_n \\
 &\quad \vdots \\
 &\quad + a_m b_n + a_m b_n + a_m b_3 + \dots + a_m b_n
 \end{aligned} \tag{49}$$

PROOF. This is the product of two series. Both series are finite, and we simply multiply out each series. ■

$$\begin{aligned}
 \left(\sum_{m=1}^{\infty} a_m \right) \left(\sum_{n=1}^{\infty} b_n \right) &= (a_1 + a_2 + a_3 + \dots)(b_1 + b_2 + b_3 + \dots) \\
 &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_m b_n = a_1 b_1 + a_1 b_2 + a_1 b_3 + \dots \\
 &\quad + a_2 b_1 + a_2 b_2 + a_2 b_3 + \dots \\
 &\quad + a_3 b_1 + a_3 b_2 + a_3 b_3 + \dots \\
 &\quad \vdots \\
 &= \sum_{m,n=1}^{\infty} a_m b_n.
 \end{aligned} \tag{50}$$

PROOF. This is similar to the previous equation, but with each series now infinite. ■

$$\begin{aligned}
 \prod_{k=1}^n (1 + a_k) &= (1 + a_1)(1 + a_2)(1 + a_3) \dots (1 + a_n) \\
 &= 1 + \sum_{k=1}^n b_k = 1 + b_1 + b_2 + b_3 + \dots + b_n,
 \end{aligned}$$

where

$$\begin{aligned}
 b_1 &= \sum_{t=1}^n a_t, \\
 b_2 &= \sum_{\substack{t,u=1 \\ t \neq u}}^n a_t a_u, \\
 b_3 &= \sum_{\substack{t,u,v=1 \\ t,u,v \text{ distinct}}}^n a_t a_u a_v, \\
 &\vdots \\
 b_k &= \sum_{\substack{t_1, \dots, t_k=1 \\ t_1, \dots, t_k \text{ distinct}}}^n a_{t_1} a_{t_2} a_{t_3} \dots a_{t_k} \\
 &= \sum_{\substack{t_1, \dots, t_k=1 \\ t_1, \dots, t_k \text{ distinct}}}^n \prod_{j=1}^k a_{t_j}.
 \end{aligned} \tag{51}$$

PROOF. This is the product of a finite number of binomials, each with a first term of 1. The number of binomials is finite, and we multiply out their product to obtain 1 plus a sum of b terms.

For the b terms, b_1 is the sum of all a , b_2 is the sum of the products of any two distinct a , b_3 is the sum of the products of any three distinct a , and so on. Each b_k term consists of k factors from the set of the a terms, with the index of each a term being distinct. ■

$$\begin{aligned}
\prod_{k=1}^{\infty} (1 + a_k) &= (1 + a_1)(1 + a_2)(1 + a_3) \dots \\
&= 1 + \sum_{k=1}^{\infty} b_k = 1 + b_1 + b_2 + b_3 + \dots,
\end{aligned}$$

where

$$\begin{aligned}
b_1 &= \sum_{t=1}^{\infty} a_t, \\
b_2 &= \sum_{\substack{t,u=1 \\ t \neq u}}^{\infty} a_t a_u, \\
b_3 &= \sum_{\substack{t,u,v=1 \\ t,u,v \text{ distinct}}}^n a_t a_u a_v, \\
&\vdots \\
b_k &= \sum_{\substack{t_1, \dots, t_k=1 \\ t_1, \dots, t_k \text{ distinct}}}^{\infty} a_{t_1} a_{t_2} a_{t_3} \dots a_{t_k} \\
&= \sum_{\substack{t_1, \dots, t_k=1 \\ t_1, \dots, t_k \text{ distinct}}}^{\infty} \prod_{j=1}^k a_{t_j}.
\end{aligned} \tag{52}$$

PROOF. Similar to the previous equation, but with the number of binomials now infinite. ■

$$\begin{aligned}
\prod_{\substack{p=2 \\ p \text{ prime}}}^{\infty} \frac{p}{p-1} &= \frac{2}{1} \cdot \frac{3}{2} \cdot \frac{5}{4} \cdot \frac{7}{6} \cdot \frac{11}{10} \dots \\
&= \frac{1}{\prod_{\substack{p=2 \\ p \text{ prime}}}^{\infty} \left(1 - \frac{1}{p}\right)} = \frac{1}{\left(1 - \frac{1}{2}\right) \left(1 - \frac{1}{3}\right) \left(1 - \frac{1}{5}\right) \left(1 - \frac{1}{7}\right) \left(1 - \frac{1}{11}\right) \dots} \\
&= \sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots
\end{aligned} \tag{53}$$

PROOF. The products in the first and second lines are taken over all prime numbers. The first line becomes the second line through two identities: $\frac{1}{1-\frac{1}{p}} = \frac{p}{p-1}$, and $\frac{1}{a} \cdot \frac{1}{b} = \frac{1}{ab}$ or its extended form $\prod \frac{1}{a} = \frac{1}{\prod a}$.

To see how the second line becomes the third line, we take the first two primes, 2 and 3. From Equation 2, $\frac{1}{1-\frac{1}{2}} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$, and $\frac{1}{1-\frac{1}{3}} = 1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \dots$

By Equation 49, when we multiply these two infinite series, we get $\frac{1}{(1-\frac{1}{2})(1-\frac{1}{3})} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{12} + \dots$, where the denominators in the sum all have as their prime factors powers of 2 and 3 only.

As we continue to multiply each side by $\frac{1}{1-\frac{1}{p}} = 1 + \frac{1}{p} + \frac{1}{p^2} + \frac{1}{p^3} + \dots$ for successive prime numbers p , the denominators in the sum have as their prime factors powers of the primes up to p . When all the prime numbers have been included in the product, all the integers are included in the denominators in the sum. ■

$$\begin{aligned} \prod_{\substack{p=2 \\ p \text{ prime}}}^{\infty} \frac{p^n}{p^n - 1} &= \frac{2^n}{1^n} \cdot \frac{3^n}{2^n} \cdot \frac{5^n}{4^n} \cdot \frac{7^n}{6^n} \cdot \frac{11^n}{10^n} \cdots \\ &= \frac{1}{\prod_{\substack{p=2 \\ p \text{ prime}}}^{\infty} \left(1 - \frac{1}{p^n}\right)} = \frac{1}{\left(1 - \frac{1}{2^n}\right) \left(1 - \frac{1}{3^n}\right) \left(1 - \frac{1}{5^n}\right) \left(1 - \frac{1}{7^n}\right) \left(1 - \frac{1}{11^n}\right) \cdots} \\ &= \sum_{k=1}^{\infty} \frac{1}{k^n} = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \dots = \zeta(n), \\ &\qquad n = \dots, -2, -1, 0, 1, 2, \dots \end{aligned} \tag{54}$$

PROOF. The proof is similar to the previous equation, except that now we use a constant power of the primes instead of the primes themselves. The last equality is the definition of the Riemann zeta function. ■

$$\begin{aligned}
 \prod_{\substack{p=2 \\ p \text{ prime}}}^{\infty} (1+p) &= (1+2)(1+3)(1+5)(1+7)(1+11)\dots \\
 &= \sum_{\substack{k=1 \\ k \text{ squarefree}}}^{\infty} k = 1 + 2 + 3 + 5 + 6 + 7 + 10 + \dots, \\
 n &= \dots, -2, -1, 0, 1, 2, \dots
 \end{aligned} \tag{55}$$

PROOF. The product in the first line is taken over all prime numbers. The sum on the second line is taken over all *squarefree* integers, which are those which contain only single powers of prime factors, and so are not divisible by the square of any integer greater than 1.

By Equation 52, the product on the first line when multiplied out becomes 1, plus the sum of all p , plus the sum of the products of any two distinct p , plus the sum of the products of any three distinct p , and so on. Since each occurrence of p can be used at most once in a product, each term in the sum is squarefree. ■

$$\begin{aligned}
 \prod_{\substack{p=2 \\ p \text{ prime}}}^{\infty} \left(1 + \frac{1}{p}\right) &= \left(1 + \frac{1}{2}\right) \left(1 + \frac{1}{3}\right) \left(1 + \frac{1}{5}\right) \left(1 + \frac{1}{7}\right) \left(1 + \frac{1}{11}\right) \dots \\
 &= \sum_{\substack{k=1 \\ k \text{ squarefree}}}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{10} + \dots, \\
 n &= \dots, -2, -1, 0, 1, 2, \dots
 \end{aligned} \tag{56}$$

PROOF. Similar to the above equation, with p replaced by $\frac{1}{p}$. ■

$$\prod_{\substack{p=2 \\ p \text{ prime}}}^{\infty} \left(1 + \frac{1}{p^n}\right) = \left(1 + \frac{1}{2^n}\right) \left(1 + \frac{1}{3^n}\right) \left(1 + \frac{1}{5^n}\right) \left(1 + \frac{1}{7^n}\right) \left(1 + \frac{1}{11^n}\right) \dots$$

$$= \sum_{\substack{k=1 \\ k \text{ squarefree}}}^{\infty} \frac{1}{k^n} = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{6^n} + \frac{1}{7^n} + \frac{1}{10^n} + \dots,$$

$$n = \dots, -2, -1, 0, 1, 2, \dots \tag{57}$$

PROOF. Similar to the previous equation, with $\frac{1}{p}$ replaced by $\frac{1}{p^n}$. ■

$$\prod_{\substack{p=2 \\ p \text{ prime}}}^{\infty} (1 - p) = (1 - 2)(1 - 3)(1 - 5)(1 - 7)(1 - 11) \dots$$

$$= \sum_{\substack{k=1 \\ k \text{ squarefree}}}^{\infty} \pm k = 1 - 2 - 3 - 5 + 6 - 7 + 10 + \dots,$$

where $\pm = \begin{Bmatrix} + \\ - \end{Bmatrix}$ when k has an $\begin{Bmatrix} \text{even} \\ \text{odd} \end{Bmatrix}$ number of primes,

$$n = \dots, -2, -1, 0, 1, 2, \dots \tag{58}$$

PROOF. Similar to Equation 55, with p replaced by $-p$. ■

$$\prod_{\substack{p=2 \\ p \text{ prime}}}^{\infty} \left(1 - \frac{1}{p}\right) = \left(1 - \frac{1}{2}\right) \left(1 - \frac{1}{3}\right) \left(1 - \frac{1}{5}\right) \left(1 - \frac{1}{7}\right) \left(1 - \frac{1}{11}\right) \dots$$

$$= \sum_{\substack{k=1 \\ k \text{ squarefree}}}^{\infty} \pm \frac{1}{k} = 1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{5} + \frac{1}{6} - \frac{1}{7} + \frac{1}{10} + \dots,$$

where $\pm = \begin{Bmatrix} + \\ - \end{Bmatrix}$ when k has an $\begin{Bmatrix} \text{even} \\ \text{odd} \end{Bmatrix}$ number of primes,

$$n = \dots, -2, -1, 0, 1, 2, \dots \tag{59}$$

PROOF. Similar to the above equation, with p replaced by $\frac{1}{p}$. ■

$$\prod_{\substack{p=2 \\ p \text{ prime}}}^{\infty} \left(1 - \frac{1}{p^n}\right) = \left(1 - \frac{1}{2^n}\right) \left(1 - \frac{1}{3^n}\right) \left(1 - \frac{1}{5^n}\right) \dots$$

$$= \sum_{k=1}^{\infty} \pm \frac{1}{k^n} = 1 - \frac{1}{2^n} - \frac{1}{3^n} - \frac{1}{5^n} + \frac{1}{6^n} + \frac{1}{7^n} + \frac{1}{10^n} + \dots = \frac{1}{\zeta(n)},$$

where $\pm = \begin{Bmatrix} + \\ - \end{Bmatrix}$ when k has an $\begin{Bmatrix} \text{even} \\ \text{odd} \end{Bmatrix}$ number of primes,

$$n = \dots, -2, -1, 0, 1, 2, \dots \quad (60)$$

PROOF. Similar to the previous equation, with $\frac{1}{p}$ replaced by $\frac{1}{p^n}$. The product is the reciprocal of the product in Equation 54, which is $\zeta(n)$. ■

INFINITE ITERATIONS

In this chapter, we generalize summation and product notation with *iteration notation*. This notation defines an indexed iteration, a simple type of recursion. Two or more relations are required to define a function recursively, but iteration notation can define an iterated function with a single expression. Iteration notation is less flexible than recursion; it cannot be used to define the Fibonacci sequence, for example.

The *explicit seed form of iteration notation* explicitly declares the seed, or initial value, of the iteration:

$\prod_{k=m}^n f(\langle a \rangle, k) \equiv r_n$, the last term of the following sequence:

If $n \geq m$

$$\begin{aligned}
 r_{m-1} &= a \\
 r_m &= f(r_{m-1}, m) \\
 r_{m+1} &= f(r_m, m+1) \\
 r_{m+2} &= f(r_{m+1}, m+2) \\
 &\dots \\
 r_k &= f(r_{k-1}, k) \\
 &\dots \\
 r_{n-1} &= f(r_{n-2}, n-1) \\
 r_n &= f(r_{n-1}, n)
 \end{aligned}$$

If $n \leq m$

$$\begin{aligned}
 r_{m+1} &= a \\
 r_m &= f(r_{m+1}, m) \\
 r_{m-1} &= f(r_m, m-1) \\
 r_{m-2} &= f(r_{m-1}, m-2) \\
 &\dots \\
 r_k &= f(r_{k+1}, k) \\
 &\dots \\
 r_{n+1} &= f(r_{n+2}, n+1) \\
 r_n &= f(r_{n+1}, n)
 \end{aligned}$$

In the *implicit seed form of iteration notation*, the seed is inferred from the lower limit of the iteration:

$\prod_{k=m}^n f(\cdot, k) \equiv r_n$, the last term of the following sequence:

If $n \geq m$

$$r_m = m$$

$$r_{m+1} = f(r_m, m + 1)$$

$$r_{m+2} = f(r_{m+1}, m + 2)$$

...

$$r_k = f(r_{k-1}, k)$$

...

$$r_{n-1} = f(r_{n-2}, n - 1)$$

$$r_n = f(r_{n-1}, n)$$

If $n \leq m$

$$r_m = g(m)$$

$$r_{m-1} = f(r_m, m - 1)$$

$$r_{m-2} = f(r_{m-1}, m - 2)$$

...

$$r_k = f(r_{k+1}, k)$$

...

$$r_{n+1} = f(r_{n+2}, n + 1)$$

$$r_n = f(r_{n+1}, n)$$

Sum and product notation are easily translated to iteration notation:

$$\sum_{k=m}^n h(k) = \prod_{k=m}^n \langle 0 \rangle + h(k) = \prod_{k=m}^n \cdot + h(k)$$

$$\prod_{k=m}^n h(k) = \prod_{k=m}^n \langle 1 \rangle h(k) = \prod_{k=m}^n \cdot h(k)$$

Continued fractions are also easily converted to iteration notation:

$$\begin{aligned} \frac{h(m)}{j(m)+} \frac{h(m+1)}{j(m+1)+} \frac{h(m+2)}{j(m+2)+} \cdots \frac{h(n)}{j(n)} &= \frac{h(m)}{j(m) + \frac{h(m+1)}{j(m+1) + \frac{h(m+2)}{j(m+2) + \cdots + \frac{h(n)}{j(n)}}}} \\ &= \mathbf{K}_{k=m}^n \frac{h(k)}{j(k)+} = \mathbf{I}_{k=m}^n \frac{h(k)}{j(k)+}. \end{aligned}$$

$\mathbf{K}_{k=m}^n$ is Gaussian continued fraction notation: “K” stands for *Kettenbruch*, the German word for continued fraction.

Iteration notation is useful for expressing power series in factored form, as follows.

$$\begin{aligned} f(t) &= e^{(t-a)\sigma^{-1}} f(a) = \sum_{k=0}^{\infty} \frac{(t-a)^k}{k!} \sigma^{-k} f(a) \\ &= f(a) + (t-a)\sigma^{-1} f(a) + \frac{(t-a)^2}{2} \sigma^{-2} f(a) + \frac{(t-a)^3}{3!} \sigma^{-3} f(a) + \dots \\ &= \left(1 + (t-a)\sigma^{-1} \left(1 + \frac{t-a}{2} \sigma^{-1} \left(1 + \frac{t-a}{3} \sigma^{-1} \left(\dots \right) \right) \right) \right) f(a) \\ &= \left[\mathbf{I}_{k=\infty}^1 1 + \frac{t-a}{k} \sigma^{-1}(\cdot) \right] f(a) \end{aligned}$$

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots = \sum_{k=0}^{\infty} \frac{x^k}{k!} \\ &= 1 + x \left(1 + \frac{x}{2} \left(1 + \frac{x}{3} \left(\dots \right) \right) \right) = \mathbf{I}_{k=\infty}^1 1 + \frac{x}{k} (\cdot) \end{aligned}$$

$$\begin{aligned} \cos x &= 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} \\ &= 1 - \frac{x^2}{2} \left(1 - \frac{x^2}{3 \cdot 4} \left(1 - \frac{x^2}{5 \cdot 6} \left(\dots \right) \right) \right) = \mathbf{I}_{k=\infty}^1 1 - \frac{x^2}{(2k-1)(2k)} (\cdot) \end{aligned}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$$

$$= x \left(1 - \frac{x^2}{2 \cdot 3} \left(1 - \frac{x^2}{4 \cdot 5} \left(1 - \frac{x^2}{6 \cdot 7} \left(\dots \right) \right) \right) \right) = x \prod_{k=\infty}^1 1 - \frac{x^2}{(2k)(2k+1)} \quad (\cdot)$$

The infinite case of Euler's continued fraction formula (which we do not prove here) is as follows:

$$\begin{aligned} a_0 + a_0 a_1 + a_0 a_1 a_2 + \dots &= \sum_{k=0}^{\infty} \prod_{j=0}^k a_j \\ &= a_0 (1 + a_1 (1 + a_2 (\dots))) = \prod_{k=\infty}^0 a_k (1 + \cdot) \\ &= \frac{a_0}{1 - \frac{a_1}{1 + a_1 - \frac{a_2}{1 + a_2 - \frac{a_3}{\ddots}}}} = \frac{a_0}{1 - \prod_{k=\infty}^1 \frac{a_k}{1 + a_k}} \quad (\cdot) \end{aligned}$$

Setting $a_n = 2$, this becomes

$$\begin{aligned} 1 + 2 + 4 + 8 + \dots &= \sum_{k=0}^{\infty} 2^k \\ &= 1 + 2(1 + 2(1 + 2(\dots))) = \prod_{k=\infty}^1 1 + 2(\cdot) \\ &= \frac{1}{1 - \frac{2}{3 - \frac{2}{3 - \frac{2}{\ddots}}}} = \frac{1}{1 - \prod_{k=\infty}^1 \frac{2}{3 - (\cdot)}} \end{aligned}$$

That these three are equivalent is verified as follows. Letting x denote the sum on the first line, we obtain the recurrence relation $x = 1 + 2x$, as noted in the **Introduction** (p. 307), which in turn yields $x = -1$. Letting x denote the factorization on the second line, we also obtain $x = 1 + 2x$. Finally, letting x denote the continued fraction on the third line, we obtain

$$x = \frac{1}{1 - \prod_{k=\infty}^1 \frac{2}{3 - (\cdot)}}$$

$$y = \prod_{k=\infty}^1 \frac{2}{3 - (\cdot)} = 1 - \frac{1}{x} = \frac{2}{3 - y} = \frac{2}{3 - (1 - \frac{1}{x})}$$
$$\frac{x}{x-1} = \frac{2x+1}{2x}$$
$$2x^2 = 2x^2 - x - 1$$
$$x + 1 = 0.$$

DIVERGENT INTEGRALS

We use the above results to evaluate divergent integrals, also called improper integrals.

$$\int_0^{\infty} a^x dx = \frac{-1}{\ln a}. \quad (61)$$

PROOF. For an integer n , we have $\int_0^n a^x dx = \sum_{k=1}^n \int_{k-1}^k a^x dx = \frac{1}{\ln a} \sum_{k=1}^n (a^k - a^{k-1}) = \frac{1-1/a}{\ln a} \sum_{k=1}^n a^k$. For the infinite case, $\int_0^{\infty} a^x dx = \frac{1-1/a}{\ln a} \sum_{k=1}^{\infty} a^k = \frac{1-1/a}{\ln a} \frac{a}{1-a} = \frac{-1}{\ln a}$. ■

ALTERNATE PROOF. Let $u \equiv \int_0^{\infty} a^x dx$. Then $au = \int_0^{\infty} a^{x+1} dx = \int_1^{\infty} a^x dx$, and $u(1-a) = u - au = \int_0^1 a^x dx = \frac{a-1}{\ln a}$, so $u = \frac{-1}{\ln a}$. ■

$$\int_{-\infty}^{\infty} a^x dx = 0. \quad (62)$$

PROOF. By Equation 61, $\int_{-\infty}^{\infty} a^x dx = \int_0^{\infty} a^x dx + \int_0^{\infty} a^{-x} dx = \int_0^{\infty} a^x dx + \int_0^{\infty} (\frac{1}{a})^x dx = \frac{-1}{\ln a} + \frac{-1}{-\ln a} = 0$. ■

$$\int_0^{\infty} e^x dx = -1. \quad (63)$$

PROOF. Application of Equation 61 for $a = e$. ■

$$\int_0^{\infty} e^{-x} dx = 1. \quad (64)$$

PROOF. Application of Equation 62 for $a = e$ and Equation 67. This is a convergent integral, and the result agrees with conventional analysis. ■

$$\int_0^{\infty} \sin x dx = 1. \quad (65)$$

PROOF. Let $u \equiv \int_0^{\infty} \sin x dx$. Then $-u = \int_0^{\infty} \sin(x + \pi) dx = \int_{\pi}^{\infty} \sin x dx$, and $u = \frac{1}{2} [u - (-u)] = \frac{1}{2} \int_0^{\pi} \sin x dx = 1$. ■

$$\int_0^{\infty} \cos x dx = 0. \quad (66)$$

PROOF. Let $u \equiv \int_0^{\infty} \cos x dx$. Then $-u = \int_0^{\infty} \cos(x + \pi) dx = \int_{\pi}^{\infty} \cos x dx$, and $u = \frac{1}{2} [u - (-u)] = \frac{1}{2} \int_0^{\pi} \cos x dx = 0$. ■

$$\int_0^{\infty} e^{ix} = i. \quad (67)$$

PROOF. By Equations 65 and 66, $\int_0^{\infty} e^{ix} = \int_0^{\infty} \cos x dx + i \int_0^{\infty} \sin x dx = i$. ■

$$a^{\infty} = 0. \quad (68)$$

PROOF. By integration and Equation 62, $\int_0^\infty a^x dx = \frac{1}{\ln a} (a^\infty - 1) = \frac{1}{\ln a}$, so $a^\infty = 0$. Alternatively, for $n = \infty$ in Equation 1 and by Equation 2, $\sum_m^\infty a^k = \frac{a^m - a^\infty}{1-a} = \frac{a^m}{1-a}$, again yielding $a^\infty = 0$. ■

NOTE. The values $a = 0$ and $a = 1$ in this and all the equations above must be carefully examined. See the next chapter, **Infinite series have infinite values** (p. 343).

$$e^{i\infty} = 0. \quad (69)$$

PROOF. Application of Equation 68 for $a = e^i$. ■

ALTERNATE PROOF. By Equation 67, since $i = \int_0^\infty e^{ix} = -ie^{ix} \Big|_0^\infty = -ie^{i\infty} + i$, $e^{i\infty} = 0$. ■

INFINITE SERIES HAVE INFINITE VALUES

Basic observations

For real numbers, the numeristic theory of divergent series postulates adding a single infinite element ∞ . The resulting number system is called the *projectively extended real numbers* and is also described in detail in [Real infinite element extensions](#) (p. 68).

$$\sum_{k=1}^{\infty} 0^k = \sum_{k=1}^{\infty} 0 = 0 + 0 + 0 + \dots = \varphi. \quad (70)$$

PROOF. This is simply the numeristic identity $\infty \cdot 0 = \varphi$, where φ is the *full class*, which numeristics uses as the value of an indeterminate expression, as described in [The empty class and the full class](#) (p. 62). ■

However simple this identity may be from the numeristic point of view, it differs significantly from the conclusion of conventional analysis, which classifies this series as convergent and thus evaluates it through the limit $\lim_{k \rightarrow \infty} 0^k = 0$.

$$\sum_{k=1}^{\infty} \infty^k = \sum_{k=1}^{\infty} \infty = \infty + \infty + \infty + \dots = \infty. \quad (71)$$

PROOF. This is the identity $\infty \cdot \infty = \infty$. ■

$$\sum_{k=0}^{\infty} 1^k = \sum_{k=0}^{\infty} 1 = 1 + 1 + 1 + 1 + \dots = \infty. \quad (72)$$

PROOF. This is the identity $\infty \cdot 1 = \infty$. ■

$$\sum_{k=0}^{\infty} (-1)^k = 1 - 1 + 1 - 1 + \dots = \phi. \quad (73)$$

PROOF. $\sum_{k=0}^{\infty} (-1)^k = 1 - 1 + 1 - 1 + \dots = (1 - 1) \cdot \frac{\infty}{2} = 0 \cdot \infty = \phi$. ■

In the projectively extended real numbers, $+\infty = -\infty$, and thus e^∞ has two real values, $e^{\pm\infty} = \{0, \infty\}$. This in turn means that most convergent and divergent series have two sums. This is the numeristic resolution of Zeno's paradox: Even though a convergent series has a finite sum, it also has an infinite sum.

For example, as we show more carefully below, for the convergent geometric series diagrammed in Figure 98,

$$\sum_{n=1}^{\infty} 2^{-n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots = \{1, \infty\} = \{2^{-\infty} + 1, 2^{+\infty} + 1\}.$$

The finite value is suggested by the diagram and a limit argument, while the infinite value is suggested by comparison with $\infty \cdot 0 = \sum_{n=1}^{\infty} 0$, which may take any finite or infinite value, and yet each term of which is infinitely smaller than the corresponding term of the geometric series.

Another way of adding infinite elements to the real numbers is the *affinely extended real numbers*, also explored in **Real infinite element extensions** (p. 68). The affinely extended system adds *two* distinct infinite elements, $+\infty$ and $-\infty$. In this system, $a^{+\infty} = +\infty$ and $a^{-\infty} = 0$, so we do not have any choice between finite and infinite values of $a^{\pm\infty}$. Convergent series computed with Equation 2 can have only a finite value, even though comparison of such a series with $\infty \cdot 0$ indicates an infinite value for the series. In addition, the

results are not fully consistent with quantum renormalization. Renormalization, which has been repeatedly verified by physical experiment, is a mathematical procedure which uses assumptions similar to those of the projectively extended system. See [Quantum renormalization](#) (p. 287). In the following we use only the projectively extended real numbers.

Most of the proofs of Equations 2–69 yielded only finite values, because the steps of calculation in these proofs are only valid for finite values, and thus an infinite value was not detected. As we will see, other types of proofs may detect an infinite value. To avoid contradiction while keeping simple algebraic properties, we must generally assume that an infinite series may have both finite and infinite values.

Idempotent terms

We should also observe that the recurrence step in the proof of Equation 2 is valid when $|a|$ is not idempotent, i.e. $|a|^2 \neq |a|$. In the derivation of Equation 2, we said that if

$$x = \sum_{k=m}^n a^k = a^m + a^{m+1} + a^{m+2} + a^{m+3} + \dots,$$

then

$$xa = a^{m+1} + a^{m+2} + a^{m+3} + a^{m+4} + \dots,$$

and so

$$x = \frac{a^m}{1 - a}.$$

However, if $|a|$ is idempotent, this can lead to an indeterminate value. For example, if $a = -1$, then

$$\begin{aligned} x &= 1 - 1 + 1 - 1 + 1 - 1 + \dots \\ -x &= -1 + 1 - 1 + 1 - 1 + 1 + \dots \\ x &= -x + 1 = -x + 1 - 1 + 1 = -x + 1 - 1 + 1 - 1 + 1 - \dots \\ &= -x + (1 - 1)\infty = \varnothing \end{aligned}$$

Other idempotents also have such indeterminacies.

With the above understandings, we now revise Equation 2.

If $|a|$ is not idempotent,

$$\sum_{k=m}^{\infty} a^k = a^m + a^{m+1} + a^{m+2} + a^{m+3} + \dots = \left\{ \infty, \frac{a^m}{1-a} \right\}. \quad (74)$$

PROOF. Set

$$x \equiv a^m + a^{m+1} + a^{m+2} + a^{m+3} + \dots$$

First assume x is finite. Then

$$\begin{aligned} ax &= a^{m+1} + a^{m+2} + a^{m+3} + a^{m+4} + \dots \\ x - ax &= a^m \\ x &= \frac{a^m}{1-a}. \end{aligned}$$

Now observe that $x = \infty$ also satisfies these equations. This is confirmed by observing that each term of Equation 74 is infinitely greater than each term of Equation 70. Hence

$$x = \left\{ \infty, \frac{a^m}{1-a} \right\}. \blacksquare$$

Although the proof we gave of Equation 1 did not permit it, Equation 74 can also be obtained by setting $n = \infty$ for non-idempotent $|a|$ in Equation 1:

$$\sum_{k=m}^{\infty} a^k = \frac{a^m - a^{\infty+1}}{1-a} = \left\{ \frac{a^m - \infty}{1-a}, \frac{a^m - 0}{1-a} \right\} = \left\{ \infty, \frac{a^m}{1-a} \right\}.$$

This phenomenon is thoroughly explored in the next chapter, [Equi-point summation](#) (p. 351).

$$\sum_{k=1}^n k a^{k-1} = 1 + 2a + 3a^2 + 4a^3 + \dots + n a^{n-1} = \frac{1 + (an - n - 1)a^n}{(1-a)^2}. \quad (81)$$

PROOF. Let

$$x \equiv \sum_{k=1}^n k a^{k-1} = 1 + 2a + 3a^2 + 4a^3 + \dots + n a^{n-1}.$$

Then, using Equation 1:

$$\begin{aligned}
 ax &= a + 2a^2 + 3a^3 + 4a^4 + \dots + na^n \\
 x - ax &= 1 + a + a^2 + a^3 + \dots + a^{n-1} - na^n = \frac{1 - a^n}{1 - a} - na^n \\
 x &= \frac{1 - a^n}{(1 - a)^2} - \frac{na^n}{1 - a} = \frac{1 + (an - n - 1)a^n}{(1 - a)^2}. \blacksquare
 \end{aligned}$$

If $|a|$ is not idempotent,

$$\sum_{k=1}^{\infty} ka^{k-1} = 1 + 2a + 3a^2 + 4a^3 + \dots = \left\{ \infty, \frac{1}{(1 - a)^2} \right\}. \tag{82}$$

PROOF. Set $n = \infty$ in Equation 81. Then

$$\begin{aligned}
 \sum_{k=1}^{\infty} ka^{k-1} &= \frac{1 + (\infty[a - 1])a^\infty}{(1 - a)^2} \\
 &= \frac{1 + \infty \cdot a^\infty}{(1 - a)^2} \\
 &= \left\{ \infty, \frac{1 - \infty \cdot a^{-\infty}}{(1 - a)^2} \right\}.
 \end{aligned}$$

By L'Hôpital's rule,

$$\infty \cdot a^{-\infty} = \frac{x}{a^x} \Big|_{x=\infty} = \frac{1}{(\ln a)a^x} \Big|_{x=\infty} = 0.$$

Hence

$$\sum_{k=1}^{\infty} ka^{k-1} = \left\{ \infty, \frac{1}{(1 - a)^2} \right\}. \blacksquare$$

$$\sum_{k=1}^{\infty} k = 1 + 2 + 3 + 4 + \dots = \infty. \tag{83}$$

PROOF. We use the formula for the sum of an arithmetic series:

$$\sum_{k=1}^n k = 1 + 2 + 3 + \dots + n = \frac{n(n + 1)}{2}.$$

For Equation 83, this yields

$$\sum_{k=1}^{\infty} k = 1 + 2 + 3 + \dots + \infty = \frac{\infty(\infty + 1)}{2} = \frac{\infty \cdot \infty}{2} = \infty. \blacksquare$$

$$\sum_{k=1}^{\infty} (-1)^{k-1} k = 1 - 2 + 3 - 4 + \dots = \phi. \tag{84}$$

PROOF. Again, we use the formula for the sum of an arithmetic series:

$$\sum_{k=1}^n k = 1 + 2 + 3 + \dots + n = \frac{n(n + 1)}{2}.$$

∞ may function as an integer, since it is the sum of units (Equation 72). Moreover, it may function as an even integer, since it is twice an integer ($2 \cdot \infty = \infty$), and it may also function as an odd integer, since it is one more than an even integer ($\infty + 1 = \infty$).

When ∞ functions as an even integer,

$$\begin{aligned} 1 - 2 + 3 - 4 + \dots - \infty &= [1 + 3 + 5 + \dots + \infty - 1] - [2 + 4 + 6 + \dots + \infty] \\ &= 2 \left(1 + 2 + 3 + \dots + \frac{\infty}{2} \right) - \frac{\infty}{2} - 2 \left(1 + 2 + 3 + \dots + \frac{\infty}{2} \right) \\ &= \frac{\infty(\infty + 1)}{8} - \frac{\infty}{2} - \frac{\infty(\infty + 1)}{8} = \infty - \infty - \infty = \phi. \end{aligned}$$

When ∞ functions as an odd integer,

$$\begin{aligned} 1 - 2 + 3 - 4 + \dots + \infty &= [1 + 3 + 5 + \dots + \infty] - [2 + 4 + 6 + \dots + \infty - 1] \\ &= 2 \left(1 + 2 + 3 + \dots + \frac{\infty - 1}{2} \right) + \frac{\infty}{2} \\ &\quad - 2 \left(1 + 2 + 3 + \dots + \frac{\infty - 1}{2} \right) \\ &= \frac{(\infty - 1)\infty}{8} + \frac{\infty}{2} - \frac{(\infty - 1)\infty}{8} = \infty + \infty - \infty = \phi. \blacksquare \end{aligned}$$

The results in this and preceding chapters have used recurrence patterns to evaluate infinite sums. In the next chapter, **Equipoint summation** (p. 351), we develop an alternative approach which is consistent with and more rigorous than the recurrence approach, and which also enables us to sum the series in Equations 74 and 82 even when $|a|$ is idempotent.

Alternating series

A similar issue comes up with addition of series. As an example, we apply equation 74 to the following absolutely convergent alternating geometric series:

$$\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \dots = \left\{ \infty, \frac{1}{3} \right\} \tag{75}$$

$$\frac{1}{2} + \frac{1}{8} + \frac{1}{32} + \dots = \left\{ \infty, \frac{2}{3} \right\} \tag{76}$$

$$\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \dots = \left\{ \infty, \frac{1}{3} \right\} \tag{77}$$

If we try to derive equation 77 by subtracting equation 76 from 75, then we obtain:

$$\begin{aligned} & \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \dots \\ &= \left(\frac{1}{2} + \frac{1}{8} + \frac{1}{32} + \dots \right) - \left(\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \dots \right) \\ &= \left\{ \infty, \frac{2}{3} \right\} - \left\{ \infty, \frac{1}{3} \right\} \\ &= \left\{ \infty - \infty, \frac{2}{3} - \infty, \infty - \frac{1}{3}, \frac{2}{3} - \frac{1}{3} \right\} \\ &= \left\{ \emptyset, \infty, \infty, \frac{1}{3} \right\} = \emptyset. \end{aligned}$$

Obviously in these two classes, we should only operate on corresponding terms, i.e. ∞ with ∞ , and $\frac{2}{3}$ with $\frac{1}{3}$, giving $\left\{ \infty, \frac{1}{3} \right\}$.

Another example is to apply equation 74 to absolute geometric series:

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots = \{ \infty, 1 \} \tag{78}$$

$$\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \dots = \left\{ \infty, \frac{1}{3} \right\} \quad (79)$$

$$\frac{1}{2} + \frac{1}{8} + \frac{1}{32} + \frac{1}{128} + \dots = \left\{ \infty, \frac{2}{3} \right\} \quad (80)$$

Again we obtain an indeterminate result if we try to derive equation 80 by subtracting equations 79 from 78:

$$\begin{aligned} & \frac{1}{2} + \frac{1}{8} + \frac{1}{32} + \frac{1}{128} + \dots \\ &= \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots \right) - \left(\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \dots \right) \\ &= \{ \infty, 1 \} - \left\{ \infty, \frac{1}{3} \right\} \\ &= \left\{ \infty - \infty, 1 - \infty, \infty - \frac{1}{3}, 1 - \frac{1}{3} \right\} \\ &= \left\{ \emptyset, \infty, \infty, \frac{2}{3} \right\} = \emptyset. \end{aligned}$$

Again, we should we should only operate on corresponding terms, i.e. ∞ with ∞ , and 1 with $\frac{1}{3}$, giving $\left\{ \infty, \frac{2}{3} \right\}$.

In such cases, **Equipoint summation** (p. 351) allows us to match corresponding elements properly.

EQUIPOINT SUMMATION

Introduction

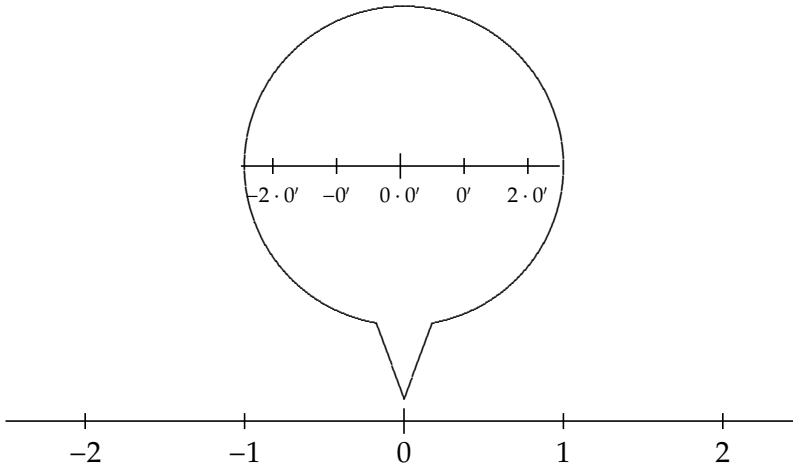


FIG. 100:
Real number line with
microscope view of unfolded 0

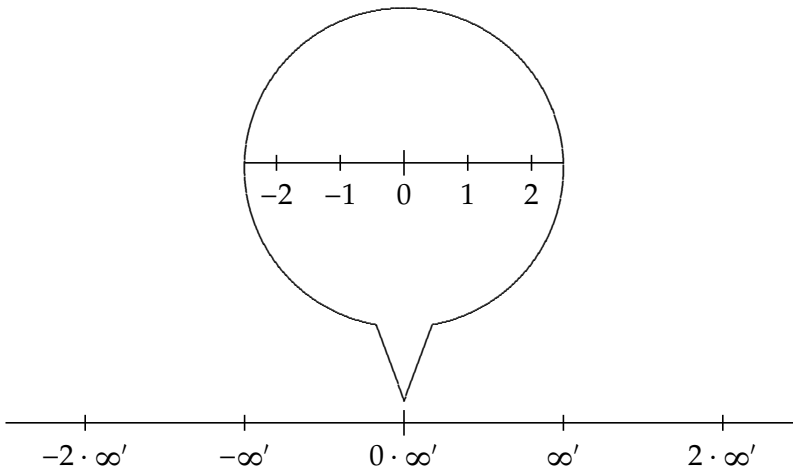


FIG. 101:
Line of infinities with microscope view of
real number line within $0 \cdot \infty'$

The general numeric approach to divergent series uses **equipoint analysis** (p. 125–301) as an application of numeristics to analysis.

Briefly, equipoint analysis uses an extension of the number system, called *unfolded numbers*, to define derivatives and integrals in simple algebraic terms. The ordinary real numbers are the *folded* real numbers. Every number becomes a multivalued class when it is unfolded, and the class of such numbers is the *unfolded* real numbers. For 0 and ∞ , we denote representative units within their unfoldings as $0'$ and ∞' . The unfolding of 0 is the class $\mathbb{R}0'$, the real multiples of $0'$, such as $2 \cdot 0'$ or $\pi \cdot 0'$.

Figure 100 shows the real number line with the unfolding of 0 in an infinitely expanded space, where the single element 0 becomes a multivalued class of unfolded multiples of $0'$, the class $\mathbb{R}0'$. The bubble in this diagram is called a *microscope*. Figure 101 shows the unfolding of ∞ into a multivalued class consisting of multiples of ∞' .

If two unfolded elements a and b are identical at the folded level, i.e. they are unfolded members of the same folded number, or one is folded and the other is a member of the first, or they are both the same folded number, we denote this $a='b$ (pronounced “ a equals prime b ”). Examples:

$$0' \neq 2 \cdot 0'$$

$$0' = 2 \cdot 0'$$

$$0^2 = 0'$$

$$0' \neq 1 + 0'.$$

We add to this the postulate of the projectively extended real numbers that $+\infty = -\infty$. The unfolding of ∞ includes both positive and negative multiples of ∞' , such as $+\infty'$ and $-\infty'$. At the folded level, such numbers are equal and both positive and negative, but at the unfolded level, they are distinct and are either positive or negative but not both, e.g.:

$$-\infty = +\infty$$

$$-\infty' \neq +\infty'$$

$$-\infty' < 0$$

$$+\infty' > 0$$

$$e^{-\infty} = e^{+\infty} = \{0, \infty\}$$

$$e^{-\infty'} = 0$$

$$e^{+\infty'} = \infty$$

$$\ln 0 = \ln \infty = \infty$$

Equipoint summation is the application of unfolded arithmetic to infinite series. It enables us to determine sums of many series that resist summation in folded arithmetic, including geometric series in a^k where $|a|$ is idempotent. We now examine several such series.

Sum of zeros

As noted in Equation 70,

$$\sum_{k=1}^{\infty} 0 = 0 + 0 + 0 + \dots = \infty \cdot 0 = \phi.$$

We now consider this sum in unfolded equipoint arithmetic. We choose an unfolded ∞' as the upper limit and an unfolded $0'$ as the summand. The sum can then be written as follows.

$$\sum_{k=0}^{\infty'} 0' = 0' + 0' + 0' + \dots = \begin{cases} \infty & \text{for } \infty' \cdot 0' \text{ infinite} \\ r \in (0, +\infty) & \text{for } \infty' \cdot 0' \text{ positive perfinite} \\ r \in (-\infty, 0) & \text{for } \infty' \cdot 0' \text{ negative perfinite} \\ 0 & \text{for } \infty' \cdot 0' \text{ infinitesimal} \end{cases} \quad (85)$$

PROOF. The sum is now $\infty' \cdot 0'$, which is a specific extended real number, which may be any infinitesimal, perfinite, or infinite value, depending on the choice of ∞' and $0'$. For instance:

- if $0' \equiv \frac{2}{\infty'}$, then $\infty' \cdot 0' = 2$;
- if $0' \equiv \frac{1}{\sqrt{\infty'}}$, then $\infty' \cdot 0' = \sqrt{\infty'} = \infty$;
- if $0' \equiv \frac{1}{\infty'^2}$, then $\infty' \cdot 0' = \frac{1}{\infty'} = 0$. ■

Since we use the projectively extended real numbers, we allow ∞' to be unfolded negative as well as positive. The series

$$\sum_{k=0}^{\infty} a_k = a_0 + a_1 + a_2 + \dots + a_{\infty}$$

can thus be unfolded into the two sums

$$\sum_{k=0}^{\infty'} a_k = a_0 + a_1 + a_2 + \dots + a_{\infty-2} + a_{\infty-1} + a_{\infty'}$$

$$\sum_{k=0}^{-\infty'} a_k = a_0 + a_1 + a_2 + \dots + a_{-\infty-2} + a_{-\infty-1} + a_{-\infty'}$$

We emphasize that the latter series is *not* the same as

$$\sum_{k=-\infty'}^0 a_k = a_{-\infty'} + a_{-\infty'+1} + a_{-\infty'+2} + \dots + a_{-2} + a_{-1} + a_0,$$

since only the first and last terms are the same. The upper limit of $-\infty'$ means that we start at $k = 0$ and *increase* k through all the finite positive integers, and then, within the infinite integers, we use *negative* infinite integers.

We can unfold Equation 85 in a more general manner by allowing each unfolded 0 to vary with k . This is just the equipoint definite integral:

$$\sum_{k=0}^{\infty'} 0_k = 0_1 + 0_2 + 0_3 + \dots + 0_{\infty'} = \int_a^b f(x) dx,$$

where $0_k \equiv f\left(a + \frac{b-a}{\infty'} k\right) \frac{b-a}{\infty'}$ for any f, a, b .

We now investigate the geometric series in which $0_k \equiv 0'^k$.

$$\sum_{k=1}^{\infty'} 0'^k = 0' + 0'^2 + 0'^3 + \dots = \begin{cases} 0 & \text{for } \infty' \cdot \ln 0' > 0 \\ \infty & \text{for } \infty' \cdot \ln 0' < 0 \end{cases} \quad (86)$$

PROOF. As in Equation 85, the value of this series depends on the relationship between $0'$ and ∞' . Table 102 shows the possible cases. ■

TABLE 102: Cases of $\sum_{k=1}^{\infty'} 0'^k$ evaluated through Equation 1

In this table:

- ∞' is an infinite integer $0'$ is an infinitesimal real
- ∞_n is a positive infinite real 0_n is a positive infinitesimal real
- r_n is a positive perfinite real M_n is a positive infinite integer
- p_n is an even positive finite integer
- q_n is an odd positive finite integer
- P_n is an even positive infinite integer
- Q_n is an odd positive infinite integer

| ∞' | $0'$ | $\ln 0'$ | $(\infty' + 1) \cdot \ln 0'$ | $0'^{\infty'+1}$ | $\frac{0' - 0'^{\infty'+1}}{1 - 0'}$ | $= \sum_{k=1}^{\infty'} 0'^k$ |
|-----------|--------|-----------------------|------------------------------|------------------|--------------------------------------|-------------------------------|
| M_1 | 0_1 | $-\infty_2$ | $-\infty_3$ | 0_4 | $\frac{0' - 0_4}{1 - 0'}$ | $= ' 0$ |
| M_1 | 0_1 | ∞_2 | ∞_3 | ∞_4 | $\frac{0' - \infty_4}{1 - 0'}$ | $= ' \infty$ |
| $-M_1$ | 0_1 | $-\infty_2$ | ∞_3 | ∞_4 | $\frac{0' - \infty_4}{1 - 0'}$ | $= ' \infty$ |
| $-M_1$ | 0_1 | ∞_2 | $-\infty_3$ | 0_4 | $\frac{0' - 0_4}{1 - 0'}$ | $= ' 0$ |
| P_1 | -0_1 | $q_1\pi i - \infty_2$ | $Q_1\pi i - \infty_3$ | -0_5 | $\frac{0' + 0_5}{1 - 0'}$ | $= ' 0$ |
| P_1 | -0_1 | $q_1\pi i + \infty_2$ | $Q_1\pi i + \infty_3$ | $-\infty_5$ | $\frac{0' + \infty_5}{1 - 0'}$ | $= ' \infty$ |
| $-P_1$ | -0_1 | $q_1\pi i - \infty_2$ | $Q_1\pi i + \infty_3$ | $-\infty_5$ | $\frac{0' + \infty_5}{1 - 0'}$ | $= ' \infty$ |
| $-P_1$ | -0_1 | $q_1\pi i + \infty_2$ | $Q_1\pi i - \infty_3$ | -0_5 | $\frac{0' + 0_5}{1 - 0'}$ | $= ' 0$ |
| Q_1 | -0_1 | $q_1\pi i - \infty_2$ | $P_1\pi i - \infty_3$ | 0_5 | $\frac{0' - 0_5}{1 - 0'}$ | $= ' 0$ |
| Q_1 | -0_1 | $q_1\pi i + \infty_2$ | $P_1\pi i + \infty_3$ | ∞_5 | $\frac{0' - \infty_5}{1 - 0'}$ | $= ' \infty$ |
| $-Q_1$ | -0_1 | $q_1\pi i - \infty_2$ | $P_1\pi i + \infty_3$ | ∞_5 | $\frac{0' - \infty_5}{1 - 0'}$ | $= ' \infty$ |
| $-Q_1$ | -0_1 | $q_1\pi i + \infty_2$ | $P_1\pi i - \infty_3$ | 0_5 | $\frac{0' - 0_5}{1 - 0'}$ | $= ' 0$ |

Sum of infinities

As noted in Equation 71, in folded arithmetic,

$$\sum_{k=1}^{\infty} \infty = \infty + \infty + \infty + \dots = \infty.$$

The unfolded version of this sum is:

$$\sum_{k=1}^{\infty'} \infty'' = \underbrace{\infty'' + \infty'' + \infty'' + \dots + \infty''}_{\infty' \text{ times}} = \infty. \quad (87)$$

PROOF. The sum is $\infty' \cdot \infty''$, which is always infinite, regardless of the choice of ∞' and ∞'' . ■

$$\sum_{k=1}^{\infty'} \infty''^k = \infty'' + \infty''^2 + \infty''^3 + \dots + \infty''^{\infty'} = \begin{cases} \infty & \text{for } \infty' \cdot \ln 0' > 0 \\ -1 & \text{for } \infty' \cdot \ln 0' < 0 \end{cases} \quad (88)$$

PROOF. This is similar to Equation 86, since $\infty = 0^{-1}$. Table 103 shows the cases for this series. ■

TABLE 103: Cases of $\sum_{k=1}^{\infty'} \infty''^k$ evaluated through Equation 1

In this table:

- ∞' is an infinite integer $0'$ is an infinitesimal real
- ∞_n is a positive infinite real 0_n is a positive infinitesimal real
- r_n is a positive perfinite real M_n is a positive infinite integer
- p_n is an even positive finite integer
- q_n is an odd positive finite integer
- P_n is an even positive infinite integer
- Q_n is an odd positive infinite integer

| ∞' | ∞'' | $\ln \infty''$ | $(\infty' + 1) \cdot \ln \infty''$ | $\infty''^{\infty'+1}$ | $\frac{\infty'' - \infty''^{\infty'+1}}{1 - \infty''}$ | $= \sum_{k=1}^{\infty'} \infty''^k$ |
|-----------|-------------|-----------------------|------------------------------------|------------------------|--|-------------------------------------|
| M_1 | ∞_2 | ∞_3 | ∞_4 | ∞_6 | $\frac{\infty'' - \infty_6}{1 - \infty''}$ | $= \infty$ |
| M_1 | ∞_2 | $-\infty_3$ | $-\infty_4$ | 0_6 | $\frac{\infty'' - 0_6}{1 - \infty''}$ | $= -1$ |
| $-M_1$ | ∞_2 | ∞_3 | $-\infty_4$ | 0_6 | $\frac{\infty'' - 0_6}{1 - \infty''}$ | $= -1$ |
| $-M_1$ | ∞_2 | $-\infty_3$ | ∞_4 | ∞_6 | $\frac{\infty'' - \infty_6}{1 - \infty''}$ | $= \infty$ |
| P_1 | $-\infty_2$ | $q_3\pi i + \infty_3$ | $Q_4\pi i + \infty_5$ | $-\infty_6$ | $\frac{\infty'' + \infty_6}{1 - \infty''}$ | $= \infty$ |
| P_1 | $-\infty_2$ | $q_3\pi i - \infty_3$ | $Q_4\pi i - \infty_5$ | -0_6 | $\frac{\infty'' + 0_6}{1 - \infty''}$ | $= -1$ |
| $-P_1$ | $-\infty_2$ | $q_3\pi i + \infty_3$ | $Q_4\pi i - \infty_5$ | -0_6 | $\frac{\infty'' + 0_6}{1 - \infty''}$ | $= -1$ |
| $-P_1$ | $-\infty_2$ | $q_3\pi i - \infty_3$ | $Q_4\pi i + \infty_5$ | $-\infty_6$ | $\frac{\infty'' + \infty_6}{1 - \infty''}$ | $= \infty$ |
| Q_1 | $-\infty_2$ | $q_3\pi i + \infty_3$ | $Q_4\pi i + \infty_5$ | ∞_6 | $\frac{\infty'' - \infty_6}{1 - \infty''}$ | $= \infty$ |
| Q_1 | $-\infty_2$ | $q_3\pi i - \infty_3$ | $Q_4\pi i - \infty_5$ | 0_6 | $\frac{\infty'' - 0_6}{1 - \infty''}$ | $= -1$ |
| $-Q_1$ | $-\infty_2$ | $q_3\pi i + \infty_3$ | $Q_4\pi i - \infty_5$ | 0_6 | $\frac{\infty'' - 0_6}{1 - \infty''}$ | $= -1$ |
| $-Q_1$ | $-\infty_2$ | $q_3\pi i - \infty_3$ | $Q_4\pi i + \infty_5$ | ∞_6 | $\frac{\infty'' - \infty_6}{1 - \infty''}$ | $= \infty$ |

Sum of non-idempotents

As stated in Equation 74, when $|a|$ is not idempotent,

$$\sum_{k=m}^{\infty} a^k = a^m + a^{m+1} + a^{m+2} + a^{m+3} + \dots = \left\{ \infty, \frac{a^m}{1-a} \right\}.$$

Equipoint summation on this series yields only the same two cases.

If $|a|$ is not idempotent,

$$\begin{aligned} \sum_{k=0}^{\infty'} a^k &= 1 + a + a^2 + a^3 + \dots \\ &= \begin{cases} \infty & \text{for } |a| > 1 \text{ and } \infty' > 0, \text{ or } |a| < 1 \text{ and } \infty' < 0 \\ \frac{1}{1-a} & \text{for } |a| > 1 \text{ and } \infty' < 0, \text{ or } |a| < 1 \text{ and } \infty' > 0 \end{cases} \quad (89) \end{aligned}$$

PROOF. We do not need to unfold a , since the sum does not yield any indeterminate values in folded arithmetic. It is sufficient to unfold ∞ into $\pm\infty'$. Table 104 shows these two cases for $|a| > 1$ and $|a| < 1$, along with the examples $a = 2$ and $a = \frac{1}{2}$. ■

TABLE 104: Cases of $\sum_{k=0}^{\infty'} a^k$ evaluated through Equation 1

In this table:

∞' is an infinite integer $0'$ is an infinitesimal real
 ∞_n is a positive infinite real 0_n is a positive infinitesimal real
 M_n is a positive infinite integer

| | | | |
|-----------|----------------------------------|-----------------------------------|------------------------------|
| ∞' | $a^{\infty'+1}$ for $ a > 1$ | $\frac{1 - a^{\infty'+1}}{1 - a}$ | $= \sum_{k=0}^{\infty'} a^k$ |
| M_1 | ∞_2 | $\frac{1 - \infty_2}{1 - a}$ | $= \infty$ |
| $-M_1$ | 0_2 | $\frac{1 - 0_2}{1 - a}$ | $= \frac{-1}{a-1}$ |
| ∞' | $2^{\infty'+1}$ | $\frac{1 - 2^{\infty'+1}}{1 - 2}$ | $= \sum_{k=0}^{\infty'} 2^k$ |

| | | | |
|-----------|----------------------------------|---|---|
| M_1 | ∞_2 | $\frac{1-\infty_2}{-1}$ | $='_\infty$ |
| $-M_1$ | 0_2 | $\frac{1-0_2}{-1}$ | $='_-1$ |
| ∞' | $a^{\infty'+1}$ for $ a < 1$ | $\frac{1 - a^{\infty'+1}}{1 - a}$ | $= \sum_{k=0}^{\infty'} a^k$ |
| M_1 | 0_2 | $\frac{1-0_2}{1-a}$ | $='_\frac{1}{1-a}$ |
| $-M_1$ | ∞_2 | $\frac{1-\infty_2}{1-a}$ | $='_\infty$ |
| ∞' | $(\frac{1}{2})^{\infty'+1}$ | $\frac{1 - (\frac{1}{2})^{\infty'+1}}{1 - \frac{1}{2}}$ | $= \sum_{k=0}^{\infty'} \left(\frac{1}{2}\right)^k$ |
| M_1 | 0_2 | $\frac{1-0_2}{\frac{1}{2}}$ | $='_2$ |
| $-M_1$ | ∞_2 | $\frac{1-\infty_2}{\frac{1}{2}}$ | $='_\infty$ |

If $|a|$ is not idempotent,

$$\sum_{k=1}^{\infty'} k a^{k-1} = 1 + 2a + 3a^2 + 4a^3 + \dots$$

$$='_\begin{cases} \infty & \text{for } |a| > 1 \text{ and } \infty' > 0, \text{ or } |a| < 1 \text{ and } \infty' < 0 \\ \frac{1}{(1-a)^2} & \text{for } |a| > 1 \text{ and } \infty' < 0, \text{ or } |a| < 1 \text{ and } \infty' > 0 \end{cases} \quad (90)$$

PROOF. We use Equation 81 to evaluate the two cases, as shown in Table 105. ■

TABLE 105: Cases of $\sum_{k=1}^{\infty'} ka^{k-1}$ evaluated through Equation 81

In this table:

∞' is an infinite integer $0'$ is an infinitesimal real
 ∞_n is a positive infinite real 0_n is a positive infinitesimal real
 M_n is a positive infinite integer

$$\frac{1}{(1-a)^2} [1 + a^{\infty'}]$$

| | | | | |
|-----------|--|------------|------------|---|
| ∞' | $a \cdot \infty' - \infty' - 1$ for $ a > 1$ | ∞_3 | ∞_3 | $(a \cdot \infty' - \infty' - 1) \Big] = \sum_{k=1}^{\infty'} ka^{k-1}$ |
|-----------|--|------------|------------|---|

| | | | | |
|--------|-------------|------------|------------------------------|-----------------------|
| M_1 | ∞_2 | ∞_3 | $\frac{1+\infty_4}{(1-a)^2}$ | $= \infty$ |
| $-M_1$ | $-\infty_2$ | 0_3 | $\frac{1-0_4}{(1-a)^2}$ | $= \frac{1}{(1-a)^2}$ |

To calculate the fourth column in the above line, we apply L'Hôpital's rule:

$$(-aM_1 + M_1 - 1)a^{-M_1} = \frac{aM_1 - M_1 - 1}{a^{M_1}} = \frac{a - 1}{(\ln a)a^{M_1}} = 0_4$$

| | | | | |
|-------|-------------|-------|-------------------------|-------------------|
| M_1 | $-\infty_2$ | 0_3 | $\frac{1-0_4}{(1-a)^2}$ | $= \frac{1}{1-a}$ |
|-------|-------------|-------|-------------------------|-------------------|

To calculate the fourth column in the above line, we apply L'Hôpital's rule as above.

| | | | | |
|--------|------------|------------|------------------------------|------------|
| $-M_1$ | ∞_2 | ∞_3 | $\frac{1+\infty_4}{(1-a)^2}$ | $= \infty$ |
|--------|------------|------------|------------------------------|------------|

Absolute sum of units

As noted in Equation 72, in folded arithmetic,

$$\sum_{k=0}^{\infty} 1^k = 1 + 1 + 1 + 1 + \dots = \infty \cdot 1 = \infty.$$

The unfolded version of this sum is:

$$\sum_{k=1}^{\infty'} 1'^k = 1' + 1'^2 + 1'^3 + \dots = \infty. \tag{91}$$

PROOF. We use an unfolded value of $1, 1' \equiv e^{0'}$, so that we can apply Equation 1 in unfolded arithmetic. Table 106 shows that all cases of unfolding yield an infinite value. ■

TABLE 106: Cases of $\sum_{k=1}^{\infty'} 1'^k$ evaluated through Equation 1

In this table:

∞' is an infinite integer $0'$ is an infinitesimal real
 ∞_n is a positive infinite real 0_n is a positive infinitesimal real
 r_n is a positive perfinite real M_n is a positive infinite integer
 $1' \equiv e^{0'}$

| ∞' | $0'$ | $(\infty' + 1) \cdot 0'$ | $1'^{\infty'+1}$ | $\frac{1' - 1'^{\infty'+1}}{1 - 1'}$ | $= \sum_{k=1}^{\infty'} 1'^k$ |
|-----------|--------|--------------------------|------------------|--------------------------------------|-------------------------------|
| M_1 | 0_1 | ∞_2 | e^{∞_2} | $\frac{1' - e^{\infty_2}}{1 - 1'}$ | $= \infty$ |
| M_1 | 0_1 | r_2 | e^{r_2} | $\frac{1' - e^{r_2}}{1 - 1'}$ | $= \infty$ |
| M_1 | 0_1 | 0_2 | e^{0_2} | $\frac{1' - e^{0_2}}{1 - 1'}$ | $= \infty$ |
| M_1 | -0_1 | $-\infty_2$ | $e^{-\infty_2}$ | $\frac{1' - e^{-\infty_2}}{1 - 1'}$ | $= \infty$ |
| M_1 | -0_1 | $-r_2$ | e^{-r_2} | $\frac{1' - e^{-r_2}}{1 - 1'}$ | $= \infty$ |
| M_1 | -0_1 | -0_2 | e^{-0_2} | $\frac{1' - e^{-0_2}}{1 - 1'}$ | $= \infty$ |
| $-M_1$ | 0_1 | $-\infty_2$ | $e^{-\infty_2}$ | $\frac{1' - e^{-\infty_2}}{1 - 1'}$ | $= \infty$ |
| $-M_1$ | 0_1 | $-r_2$ | e^{-r_2} | $\frac{1' - e^{-r_2}}{1 - 1'}$ | $= \infty$ |
| $-M_1$ | 0_1 | -0_2 | e^{-0_2} | $\frac{1' - e^{-0_2}}{1 - 1'}$ | $= \infty$ |
| $-M_1$ | -0_1 | ∞_2 | e^{∞_2} | $\frac{1' - e^{\infty_2}}{1 - 1'}$ | $= \infty$ |
| $-M_1$ | -0_1 | r_2 | e^{r_2} | $\frac{1' - e^{r_2}}{1 - 1'}$ | $= \infty$ |
| $-M_1$ | -0_1 | 0_2 | e^{0_2} | $\frac{1' - e^{0_2}}{1 - 1'}$ | $= \infty$ |

Alternating sum of units

As noted in Equation 73, in folded arithmetic,

$$\sum_{k=0}^{\infty} (-1)^k = 1 - 1 + 1 - \dots = (1 - 1) \frac{\infty}{2} = \varnothing.$$

CLAIMED THEOREM:

$$\sum_{k=0}^{\infty} (-1)^k = 1 - 1 + 1 - \dots = \frac{1}{2}. \quad (\text{X1})$$

CLAIMED PROOF. Set $a = -1$ in Equation 2. ■

REBUTTAL. This result is frequently claimed elsewhere, but it violates the condition in Equation 74 that $|a|$ be non-idempotent. As we saw there, this violation leads to an indeterminate result in folded arithmetic. ■

When we unfold the limits and the terms, we find that $\frac{1}{2}$ is only one of several possible results.

$$\sum_{k=0}^{\infty'} (-1')^k = 1 - 1' + 1'^2 - 1'^3 + \dots$$

$$= \begin{cases} \infty & \text{for } \infty' \cdot 0' \text{ positive infinite} \\ \frac{1}{2} & \text{for } \infty' \cdot 0' \text{ negative infinite} \\ r \in (1, \infty) & \text{for } \infty' \text{ even and } \infty' \cdot 0' \text{ positive perfinite} \\ r \in (-\infty, 0) & \text{for } \infty' \text{ odd and } \infty' \cdot 0' \text{ positive perfinite} \\ r \in (\frac{1}{2}, 1) & \text{for } \infty' \text{ even and } \infty' \cdot 0' \text{ negative perfinite} \\ r \in (0, \frac{1}{2}) & \text{for } \infty' \text{ odd and } \infty' \cdot 0' \text{ negative perfinite} \\ 1 & \text{for } \infty' \text{ even and } \infty' \cdot 0' \text{ infinitesimal} \\ 0 & \text{for } \infty' \text{ odd and } \infty' \cdot 0' \text{ infinitesimal,} \end{cases}$$

where $1' \equiv e^{0'}$. (92)

PROOF. As we did with the absolute sum of units, we use an unfolded value of 1, $1' \equiv e^{0'}$, so that we can apply Equation 1. The results are in Table 107. ■

TABLE 107: Cases of $\sum_{k=0}^{\infty'} (-1')^k$ evaluated through Equation 1

In this table:

- ∞' is an infinite integer $0'$ is an infinitesimal real
- ∞_n is a positive infinite real 0_n is a positive infinitesimal real
- P_n is an even positive infinite integer
- Q_n is an odd positive infinite integer
- r_n is a positive perfinite real
- $1' \equiv e^{0'}$ $2' \equiv 1 + 1'$

| ∞' | $0'$ | $\binom{\infty' + 1}{\cdot 0'}$ | $(-1)^{\infty'+1}$ | $(-1')^{\infty'+1}$ | $\frac{1 - (-1')^{\infty'+1}}{1 - (-1')} = \sum_{k=0}^{\infty'} (-1')^k$ | |
|-----------|-------|---------------------------------|--------------------|---------------------|--|--------------------------|
| P_1 | 0_2 | ∞_3 | -1 | $-e^{\infty_3}$ | $\frac{1+e^{\infty_3}}{2}$ | $= \infty$ |
| Q_1 | 0_2 | ∞_3 | 1 | e^{∞_3} | $\frac{1-e^{\infty_3}}{2}$ | $= \infty$ |
| P_1 | 0_2 | r_3 | -1 | $-e^{r_3}$ | $\frac{1+e^{r_3}}{2}$ | $= r_3 \in (1, \infty)$ |
| Q_1 | 0_2 | r_3 | 1 | e^{r_3} | $\frac{1-e^{r_3}}{2}$ | $= r_3 \in (-\infty, 0)$ |

| | | | | | | |
|--------|--------|-------------|------|------------------|------------------------------|---------------------------------|
| P_1 | 0_2 | 0_3 | -1 | $-e^{0_3}$ | $\frac{1+e^{0_3}}{2'}$ | $\neq 1$ |
| Q_1 | 0_2 | 0_3 | 1 | e^{0_3} | $\frac{1-e^{0_3}}{2'}$ | $\neq 0$ |
| P_1 | -0_2 | $-\infty_3$ | -1 | $-e^{-\infty_3}$ | $\frac{1+e^{-\infty_3}}{2'}$ | $\neq \frac{1}{2}$ |
| Q_1 | -0_2 | $-\infty_3$ | 1 | $e^{-\infty_3}$ | $\frac{1-e^{-\infty_3}}{2'}$ | $\neq \frac{1}{2}$ |
| P_1 | -0_2 | $-r_3$ | -1 | $-e^{-r_3}$ | $\frac{1+e^{-r_3}}{2'}$ | $\neq r_3 \in (\frac{1}{2}, 1)$ |
| Q_1 | -0_2 | $-r_3$ | 1 | e^{-r_3} | $\frac{1-e^{-r_3}}{2'}$ | $\neq r_3 \in (0, \frac{1}{2})$ |
| P_1 | -0_2 | -0_3 | -1 | $-e^{-0_3}$ | $\frac{1+e^{-0_3}}{2'}$ | $\neq 1$ |
| Q_1 | -0_2 | -0_3 | 1 | e^{-0_3} | $\frac{1-e^{-0_3}}{2'}$ | $\neq 0$ |
| $-P_1$ | 0_2 | $-\infty_3$ | -1 | $-e^{-\infty_3}$ | $\frac{1+e^{-\infty_3}}{2'}$ | $\neq \frac{1}{2}$ |
| $-Q_1$ | 0_2 | $-\infty_3$ | 1 | $e^{-\infty_3}$ | $\frac{1-e^{-\infty_3}}{2'}$ | $\neq \frac{1}{2}$ |
| $-P_1$ | 0_2 | $-r_3$ | -1 | $-e^{-r_3}$ | $\frac{1+e^{-r_3}}{2'}$ | $\neq r_3 \in (\frac{1}{2}, 1)$ |
| $-Q_1$ | 0_2 | $-r_3$ | 1 | e^{-r_3} | $\frac{1-e^{-r_3}}{2'}$ | $\neq r_3 \in (0, \frac{1}{2})$ |
| $-P_1$ | 0_2 | -0_3 | -1 | $-e^{-0_3}$ | $\frac{1+e^{-0_3}}{2'}$ | $\neq 1$ |
| $-Q_1$ | 0_2 | -0_3 | 1 | e^{-0_3} | $\frac{1-e^{-0_3}}{2'}$ | $\neq 0$ |
| $-P_1$ | -0_2 | ∞_3 | -1 | $-e^{\infty_3}$ | $\frac{1+e^{\infty_3}}{2'}$ | $\neq \infty$ |
| $-Q_1$ | -0_2 | ∞_3 | 1 | e^{∞_3} | $\frac{1-e^{\infty_3}}{2'}$ | $\neq \infty$ |
| $-P_1$ | -0_2 | r_3 | -1 | $-e^{r_3}$ | $\frac{1+e^{r_3}}{2'}$ | $\neq r_3 \in (1, \infty)$ |
| $-Q_1$ | -0_2 | r_3 | 1 | e^{r_3} | $\frac{1-e^{r_3}}{2'}$ | $\neq r_3 \in (-\infty, 0)$ |
| $-P_1$ | -0_2 | 0_3 | -1 | $-e^{0_3}$ | $\frac{1+e^{0_3}}{2'}$ | $\neq 1$ |
| $-Q_1$ | -0_2 | 0_3 | 1 | e^{0_3} | $\frac{1-e^{0_3}}{2'}$ | $\neq 0$ |

Alternating arithmetic series

An *alternating arithmetic series* is one whose terms form an alternating arithmetic sequence. As noted in Equation 84,

$$\sum_{k=1}^{\infty} (-1)^{k-1} k = 1 - 2 + 3 - 4 + \dots = \phi.$$

CLAIMED THEOREM:

$$\sum_{k=1}^{\infty} (-1)^{k-1} k = 1 - 2 + 3 - 4 + \dots = \frac{1}{4}. \tag{X2}$$

CLAIMED PROOF. Starting from Equation X1, we compute

$$\begin{aligned} \sum_{k=0}^{\infty} (-1)^{k+1} &= 1 - 1 + 1 - \dots \\ &= [1 - (1 - 2 + 3 - \dots)] - [1 - (2 - 3 + 4 - \dots)] \\ &= 1 - (1 - 2 + 3 - \dots) - (1 - 2 + 3 - \dots) \\ &= 1 - 2x = \frac{1}{2}, \end{aligned}$$

hence $x = 1 - 2 + 3 - \dots = \frac{1}{4}$. ■

REBUTTAL. This result is also frequently claimed elsewhere, but, as noted above, Equation X1 is a false conclusion based on a violation of the condition in Equation 74 that $|a|$ be non-idempotent. When we violate this condition, the series in Equation 74 becomes indeterminate. ■

This equation also does not satisfy the conditions of Equation 81. When we unfold only the upper limit of the summation, we obtain only infinite values for the sum.

By unfolding the terms and limits, we obtain a finite value under restricted conditions.

$$\begin{aligned} \sum_{k=1}^{\infty'} (-1')^{k-1} k &= 1 - 2 \cdot 1' + 3 \cdot 1'^2 - 4 \cdot 1'^3 + \dots \\ &= \begin{cases} \frac{1}{4} & \text{for } \infty' \cdot 0' \text{ negative infinite} \\ \infty & \text{otherwise,} \end{cases} \\ \text{where } 1' &\equiv e^{0'}. \end{aligned} \tag{93}$$

PROOF. We apply Equation 81 to the cases listed in Table 108. ■

TABLE 108: Cases of $\sum_{k=1}^{\infty'} (-1')^{k-1} k$

In this table:

∞' is an infinite integer $0'$ is an infinitesimal real

∞_n is a positive infinite real 0_n is an infinitesimal real

P_n is an even positive infinite integer

Q_n is an odd positive infinite integer

r_n is a positive perfinite real 1_n is infinitesimally close to 1

$$1' \equiv e^{0'} \quad 2' \equiv 1 + 1' \quad 4' \equiv 2'^2$$

| ∞' | $\infty \cdot 0'$ | $(-1')^{\infty'}$ | $\frac{1 + (-1'\infty' - \infty' - 1)(-1')^{\infty'}}{(1 + 1')^2}$ | $= \sum_{k=1}^{\infty'} (-1')^{k-1} k$ |
|-----------|-------------------|-------------------|--|--|
| P_1 | ∞_2 | ∞_3 | $\frac{1 + (-1'P_1 - P_1 - 1)\infty_3}{4'}$ | $= \infty$ |
| Q_1 | ∞_2 | ∞_3 | $\frac{1 - (-1'Q_1 - Q_1 - 1)\infty_3}{4'}$ | $= \infty$ |
| P_1 | $-\infty_2$ | 0_3 | $\frac{1 + 0_5}{4'}$ | $= \frac{1}{4}$ |

To calculate the fourth column in the above line, we apply L'Hôpital's rule twice:

$$\begin{aligned} (-1'P_1 - P_1 - 1)1^{P_1} &= (-1'P_1 - P_1 - 1)e^{0'P_1} \\ &= \frac{\frac{1'\infty_2}{0'} + \frac{\infty_2}{0'} - 1}{e^{\infty_2}} = \frac{\frac{1'}{0'} + \frac{1}{0'}}{e^{\infty_2}} = \frac{0_4}{e^{\infty_2}} = 0_5 \end{aligned}$$

| | | | | |
|-------|-------------|-------|----------------------|-----------------|
| Q_1 | $-\infty_2$ | 0_3 | $\frac{1 - 0_5}{4'}$ | $= \frac{1}{4}$ |
|-------|-------------|-------|----------------------|-----------------|

The fourth column in the above line is calculated as in the previous line.

| | | | | |
|-------|--------|-----------------------|--|------------|
| P_1 | r_2 | $r_3 \in (1, \infty)$ | $\frac{1 + (-1'P_1 - P_1 - 1)r_3}{4'}$ | $= \infty$ |
| Q_1 | r_2 | $r_3 \in (1, \infty)$ | $\frac{1 - (-1'Q_1 - Q_1 - 1)r_3}{4'}$ | $= \infty$ |
| P_1 | $-r_2$ | $r_3 \in (0, 1)$ | $\frac{1 + (-1'P_1 - P_1 - 1)r_3}{4'}$ | $= \infty$ |
| Q_1 | $-r_2$ | $r_3 \in (0, 1)$ | $\frac{1 - (-1'Q_1 - Q_1 - 1)r_3}{4'}$ | $= \infty$ |
| P_1 | 0_2 | 1_3 | $\frac{1 + (-1'P_1 - P_1 - 1)1_3}{4'}$ | $= \infty$ |
| Q_1 | 0_2 | 1_3 | $\frac{1 - (-1'Q_1 - Q_1 - 1)1_3}{4'}$ | $= \infty$ |
| P_1 | -0_2 | 1_3 | $\frac{1 + (-1'P_1 - P_1 - 1)1_3}{4'}$ | $= \infty$ |
| Q_1 | -0_2 | 1_3 | $\frac{1 - (-1'Q_1 - Q_1 - 1)1_3}{4'}$ | $= \infty$ |

$$\begin{array}{llll}
 -P_1 & \infty_2 & \infty_3 & \frac{1 + (1'P_1 + P_1 - 1)\infty_3}{4'} = ' \infty \\
 -Q_1 & \infty_2 & \infty_3 & \frac{1 - (1'Q_1 + Q_1 - 1)\infty_3}{4'} = ' \infty \\
 -P_1 & -\infty_2 & 0_3 & \frac{1 + 0_5}{4'} = ' \frac{1}{4}
 \end{array}$$

The fourth column in the above line is calculated as in the third line above.

$$\begin{array}{llll}
 -Q_1 & -\infty_2 & 0_3 & \frac{1 - 0_5}{4'} = ' \frac{1}{4}
 \end{array}$$

The fourth column in the above line is calculated as in the third line above.

$$\begin{array}{llll}
 -P_1 & r_2 & r_3 \in (1, \infty) & \frac{1 + (1'P_1 + P_1 - 1)r_3}{4'} = ' \infty \\
 -Q_1 & r_2 & r_3 \in (1, \infty) & \frac{1 - (1'Q_1 + Q_1 - 1)r_3}{4'} = ' \infty \\
 -P_1 & -r_2 & r_3 \in (0, 1) & \frac{1 + (1'P_1 + P_1 - 1)r_3}{4'} = ' \infty \\
 -Q_1 & -r_2 & r_3 \in (0, 1) & \frac{1 - (1'Q_1 + Q_1 - 1)r_3}{4'} = ' \infty \\
 -P_1 & 0_2 & 1_3 & \frac{1 + (1'P_1 + P_1 - 1)1_3}{4'} = ' \infty \\
 -Q_1 & 0_2 & 1_3 & \frac{1 - (1'Q_1 + Q_1 - 1)1_3}{4'} = ' \infty \\
 -P_1 & -0_2 & 1_3 & \frac{1 + (1'P_1 + P_1 - 1)1_3}{4'} = ' \infty \\
 -Q_1 & -0_2 & 1_3 & \frac{1 - (1'Q_1 + Q_1 - 1)1_3}{4'} = ' \infty
 \end{array}$$

Absolute arithmetic series

An *absolute arithmetic series* is one with positive terms forming an arithmetic sequence. As noted in Equation 83,

$$\sum_{k=1}^{\infty} k = 1 + 2 + 3 + 4 + \dots = \infty.$$

CLAIMED THEOREM:

$$\sum_{k=1}^{\infty} k = 1 + 2 + 3 + 4 + \dots = -\frac{1}{12}. \quad (\text{X3})$$

CLAIMED PROOF. Starting from Equation X2, compute

$$\begin{aligned} \sum_{k=1}^{\infty} (-1)^{k-1} k &= 1 - 2 + 3 - 4 + \dots \\ &= (1 + 2 + 3 + 4 + \dots) - 2(2 + 4 + 6 + 8 + \dots) \\ &= (1 + 2 + 3 + 4 + \dots) - 4(1 + 2 + 3 + 4 + \dots) \\ &= -3(1 + 2 + 3 + 4 + \dots) \\ &= -3x = \frac{1}{4}, \end{aligned}$$

$$\text{hence } x = 1 + 2 + 3 + 4 + \dots = -\frac{1}{12}. \blacksquare$$

REBUTTAL. As noted above, Equation X2 depends on Equation X1, which is a false result stemming from a violation of the condition in Equation 74 that $|a|$ be non-idempotent. When we violate this condition, the series in Equation 74 becomes indeterminate. ■

When we unfolded the series in Equation X2 into Equation 93, we showed that the finite value given by Equation X2 holds in Equation 93 only in certain conditions. Below we show that, when we similarly unfold the series in Equation X3, the finite value it gives does *not* hold under any conditions. In spite of this result, the value $-\frac{1}{12}$ is still *associated* with this series; see [Ramanujan summation](#) (p. 387).

$$\sum_{k=1}^{\infty'} k = 1 + 2 \cdot 1' + 3 \cdot 1'^2 + 4 \cdot 1'^3 + \dots = ' \infty,$$

where $1' \equiv e^{0'}$. (94)

PROOF. Apply Equation 81 to the cases listed in Table 109. ■

TABLE 109: Cases of $\sum_{k=1}^{\infty'} 1'^{k-1} k$

In this table:

∞' is an infinite integer $0'$ is an infinitesimal real
 ∞_n is a positive infinite real 0_n is an infinitesimal real
 r_n is a positive perfinite real M_n is a positive infinite integer
 $1' \equiv e^{0'}$ $1_n = ' 1$

| ∞' | $\infty \cdot 0'$ | $1'^{\infty'+1}$ | $x = \frac{1 - 1'^{\infty'}}{(1 - 1')^2} - \frac{\infty' 1'^{\infty'}}{1 - 1'}$ | $= \sum_{k=1}^{\infty'} 1'^{k-1} k$ |
|-----------|-------------------|-----------------------|---|-------------------------------------|
| M_1 | ∞_2 | ∞_3 | $\frac{1 - \infty_3}{0_4^2} - \frac{M_1 \infty_3}{0_4}$ | $= ' \infty$ |
| M_1 | $-\infty_2$ | 0_3 | $\frac{1 - 0_3}{0_4^2} - \frac{M_1 0_3}{-0_4}$ | $= ' \infty$ |
| M_1 | r_2 | $r_3 \in (1, \infty)$ | $\frac{1 - r_3}{0_4^2} - \frac{M_1 r_3}{0_4}$ | $= ' \infty$ |
| M_1 | $-r_2$ | $r_3 \in (0, 1)$ | $\frac{1 - r_3}{0_4^2} - \frac{M_1 r_3}{-0_4}$ | $= ' \infty$ |
| M_1 | 0_2 | $1_3 > ' 1$ | $\frac{1 - 1_3}{0_4^2} - \frac{M_1 1_3}{0_4}$ | $= ' \infty$ |
| M_1 | -0_2 | $1_3 < ' 1$ | $\frac{1 - 1_3}{0_4^2} - \frac{M_1 1_3}{-0_4}$ | $= ' \infty$ |
| $-M_1$ | ∞_2 | ∞_3 | $\frac{1 - \infty_3}{0_4^2} - \frac{-M_1 \infty_3}{-0_4}$ | $= ' \infty$ |
| $-M_1$ | $-\infty_2$ | 0_3 | $\frac{1 - 0_3}{0_4^2} - \frac{-M_1 0_3}{0_4}$ | $= ' \infty$ |
| $-M_1$ | r_2 | $r_3 \in (1, \infty)$ | $\frac{1 - r_3}{0_4^2} - \frac{-M_1 r_3}{-0_4}$ | $= ' \infty$ |

| | | | | |
|--------|--------|------------------|---|--------------|
| $-M_1$ | $-r_2$ | $r_3 \in (0, 1)$ | $\frac{1-r_3}{0_4^2} - \frac{-M_1 r_3}{0_4}$ | $= ' \infty$ |
| $-M_1$ | 0_2 | $1_3 > ' 1$ | $\frac{1-1_3}{0_4^2} - \frac{-M_1 1_3}{-0_4}$ | $= ' \infty$ |
| $-M_1$ | -0_2 | $1_3 < ' 1$ | $\frac{1-1_3}{0_4^2} - \frac{-M_1 1_3}{0_4}$ | $= ' \infty$ |

Riemann series theorem

The Riemann series theorem is a theorem of conventional analysis which states that any conditionally convergent series can be rearranged to give any value. Rearrangement here means a possibly infinite application of commutativity and associativity of addition.

Such a series must be alternating. If it is split into two series, one with all positive terms and the other with all negative, then both of these series diverge.

A classic example is the alternating harmonic series, equation 25. In [Some results of Equation 2](#) (p. 311) is a proof using the canonical arrangement of this series:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots = \ln 2. \quad (25)$$

Now consider the following rearrangement of this series:

$$\begin{aligned} & \left(1 - \frac{1}{2}\right) - \frac{1}{4} + \left(\frac{1}{3} - \frac{1}{6}\right) - \frac{1}{8} + \left(\frac{1}{5} - \frac{1}{10}\right) - \frac{1}{12} + \dots \\ &= \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \dots \\ &= \frac{1}{2} \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots\right) \\ &= 2 \ln 2. \end{aligned}$$

More generally, in any arrangement of this series, if there are k terms in a given partial sum, of which p_k are positive and n_k are negative, and $\lim_{k \rightarrow \infty} \frac{p_k}{n_k} = r$, then the limit of the series is $\ln(2\sqrt{r})$.

In summation notation, the first series, the original arrangement, is

$$\sum_{k=1}^{\infty} \frac{1}{n} = \sum_{k=1}^{\infty} \frac{1}{4k-3} - \frac{1}{4k-2} + \frac{1}{4k-1} - \frac{1}{4k},$$

while the second series, the rearrangement, is

$$\sum_{k=1}^{\infty} \frac{1}{2k-1} - \frac{1}{4k-2} - \frac{1}{4k}.$$

If $T(k)$ is the k -th term of the first series and $T(A(k))$ is the k -th term of the second series, then $A(k)$ maps the original indexes to their new positions as an automorphism of \mathbb{N}^* , which maps as follows:

$$\begin{aligned} 4k-3 &\mapsto 6k-5 \\ 4k-2 &\mapsto 3k-1 \\ 4k-1 &\mapsto 6k-2 \\ 4k &\mapsto 3k \end{aligned}$$

This type of inequality is usually ascribed to the failure of the commutative and associative properties of addition in infinite series, but equipoint summation shows that this is an overbroad characterization and not the real issue. Rather the problem is a rearrangement which changes the distribution of positive and negative terms.

In the equipoint version of this series, we unfold the upper summation limit. Then $A(k)$ is an automorphism of $\mathbb{N}_{\infty'}^* \equiv \{1, 2, \dots, \infty'\}$ rather than \mathbb{N}^* . The first series then becomes

$$\sum_{k=1}^{4\infty'} \frac{1}{n} = \sum_{k=1}^{\infty'} \frac{1}{4k-3} - \frac{1}{4k-2} + \frac{1}{4k-1} - \frac{1}{4k},$$

and the second series becomes

$$\sum_{k=1}^{\infty'} \frac{1}{2k-1} - \frac{1}{4k-2} - \frac{1}{4k}.$$

Now if $k > \frac{2\infty'}{3}$, then $6k > 4\infty'$, and the $6k-5$ and $6k-2$ terms are in the first series but not the second, while the $3k-1$ and $3k$ terms are in the second series but not the first. So the two series do not actually have the same terms.

It is sufficient if the automorphism maps in continuous blocks, i.e. if $(\forall n \in N^*)(\exists \text{ contiguous finite class } S \subset N^*)A|_S$ is an automorphism.

An example of this condition is the following mapping of each contiguous block of 4 indexes within itself:

$$4k - 3 \mapsto 4k - 3$$

$$4k - 2 \mapsto 4k - 1$$

$$4k - 1 \mapsto 4k - 2$$

$$4k \mapsto 4k$$

or in summation notation

$$\sum_{k=1}^{\infty'} \frac{1}{4k-3} - \frac{1}{4k-2} + \frac{1}{4k-1} - \frac{1}{4k} = \sum_{k=1}^{\infty'} \frac{1}{4k-3} + \frac{1}{4k-1} - \frac{1}{4k-2} - \frac{1}{4k}.$$

A PRACTICAL APPLICATION

It can be argued that the formula $1+2+4+8+\dots = \{\infty, -1\}$ has an application in the way that computers store integers. Computers store an *unsigned integer* (zero or positive only) as a simple binary. An unsigned binary integer of L bits can denote any integer N in the range $0 \leq N \leq 2^L - 1$. Denoting the lowest, rightmost position as position 0 and the highest, leftmost position as position $L - 1$, and the bit at position k as $b_k \in \mathbb{B} \equiv \{0, 1\}$, the value of N is given by

$$N = \sum_{k=0}^{L-1} b_k 2^k.$$

For $L = 8$, the unsigned integer conversions are as follows. We denote an unsigned 8-bit binary integer with the subscript $u8$.

$$11111111_{u8} = 255$$

$$11111110_{u8} = 254$$

...

$$10000010_{u8} = 130$$

$$10000001_{u8} = 129$$

$$10000000_{u8} = 128$$

$$01111111_{u8} = 127$$

$$01111110_{u8} = 126$$

...

$$00000010_{u8} = 2$$

$$00000001_{u8} = 1$$

$$00000000_{u8} = 0$$

A *signed integer* (negative, zero, or positive) uses a convention known as *twos-complement*, in which the highest-order bit is a *sign bit*: 1 means a negative integer and 0 non-negative. The negative of a positive number is generated by switching all the bits (ones-complement or bitwise not) and then adding 1 (twos-complement). Here we use the subscript $s8$ for a signed 8-bit binary integer.

Example: $+20 = 00010100_{s8}$. To represent -20 , we transform the 0's into 1's and the 1's into 0's to get the ones-complement representation 11101011 , and then we add 1 to get the twos-complement representation 11101100_{s8} .

A signed integer of L bits can encode any integer N within the range $-2^{L-1} \leq N < 2^{L-1} - 1$. Again denoting the lowest position as position 0 and the highest position as position $L - 1$, and the bit at position k as $b_k \in \mathbb{B}$, the value of N is given by

$$N = \left[\sum_{k=0}^{L-2} b_k 2^k \right] - b_{L-1} 2^{L-1}.$$

Thus $01111111_{s8} = 1 + 2 + 4 + 8 + 16 + 32 + 64 = 127$, and $11111111_{s8} = 1 + 2 + 4 + 8 + 16 + 32 + 64 - 128 = -1$.

$$01111111_{s8} = +127$$

$$01111110_{s8} = +126$$

...

$$00000010_{s8} = +2$$

$$00000001_{s8} = +1$$

$$00000000_{s8} = 0$$

$$11111111_{s8} = -1$$

$$11111110_{s8} = -2$$

...

$$10000010_{s8} = -126$$

$$10000001_{s8} = -127$$

$$10000000_{s8} = -128$$

The great advantage of the twos-complement system is that the same rules can be used for arithmetic operations on both positive and negative numbers. This in turn helps to make processors faster.

Regardless of the bit length, the twos-complement representation of -1 is always all 1's:

$$111 \dots 111_{sL} = \left[\sum_{k=0}^{L-2} 2^k \right] - 2^{L-1} = (2^{L-1} - 1) - 2^{L-1} = -1.$$

Now consider the theoretical cases of all 1's in signed and unsigned integers with an infinite number of bits. The binary representations are infinite

but can be abbreviated $\dots 111_{u\infty}$ and $\dots 111_{s\infty}$, and the integers they represent are both given by the series $1 + 2 + 4 + 8 + \dots$. Using equipoint analysis as we did in the previous chapter, we have

$$\begin{aligned}
 \dots 111_{s\infty'} &= 1 + 2 + 4 + \dots + 2^{\infty'-2} - \{0, 2^{\infty'-1}\} \\
 &= \left[\sum_{k=0}^{\infty'-2} 2^k \right] - \{0, 2^{\infty'-1}\} \\
 &= 2^{\infty'-1} - 1 - \{0, 2^{\infty'-1}\} \\
 &= \{2^{\infty'-1} - 1, -1\} \\
 &= \{ \infty, -1 \} \\
 &= 1 + 2 + 4 + \dots \\
 &= \dots 111_{u\infty'}.
 \end{aligned}$$

Thus the infinite signed and unsigned representations are equivalent.

Further,

$$\begin{aligned}
 -2 &= \dots 111101_{s\infty} \\
 -3 &= \dots 111100_{s\infty} \\
 -4 &= \dots 111011_{s\infty} \\
 &\dots
 \end{aligned}$$

This scheme can denote any integer. An infinite left binary can denote any negative integer, while a finite binary can denote any positive integer. Infinite left representations such as these are explored in further detail in [Repeating Decimals](#) (p. 409–457).

COMPARISON TO CONVENTIONAL THEORY

Three approaches

We will consider the following three approaches to divergent series.

1. **Limits.** In this approach, any infinite series is considered to be the limit of its partial sums:

$$\sum_{k=1}^{\infty} a_k \equiv \lim_{n \rightarrow \infty} \sum_{k=1}^n a_k.$$

If the limit exists, the series is said to be convergent, and otherwise divergent. A divergent series has no limit and therefore, strictly speaking, no sum. More informally we may say that the sum is ∞ , but this symbol is usually not defined as a number.

2. **Methods.** In this approach, a convergent series is still considered to be a limit, but a divergent series may be said to have a finite sum in a restricted sense. It has this sum if some calculational procedure based on the terms of the series yields a finite sum. The procedure is called a *method of summation*. Ideally, a method applied to a convergent series should yield the same sum as the limit, in which case the method is said to be *regular*. When it is applied to at least some divergent series, it may yield a finite sum. Since the sum of a divergent series may vary from one method to another, the equality of the series with its sum is said to exist only in the sense of the method.
3. **Numeristic.** In this approach, any infinite series is considered simply as a purely infinite arithmetic sum, without regard to a limit. It regards infinite sums as actual. It uses two primary techniques: (1) recursion and (2) an analytic approach called *equipoint summation*.

The limits approach was developed in the early nineteenth century and became nearly universal as direct calculation with the infinite and the infinitesimal was replaced with limits and set theoretical notions. Today, many mathematicians are unaware of any alternatives.

The methods approach developed in parallel to the limit approach but remained much less popular. It is today known as the theory of divergent series. By far the most comprehensive treatment of this approach is [Mo], now considered a standard reference.

The numeristic approach is the one we develop in this book. As we see in **The Euler extension method of summation** (p. 391), the recursion technique of the numeristic approach is not really new, but a better understanding and enhancement of an old technique.

Hardy [H49, p. 6–7] defines the notation $\sum a_m = s (P)$ to mean that the series $\sum a_m$ has a sum s in the sense of a method P . This symbolism means that equality holds only in a certain sense, because changing the sense, the method of determining the sum, can change the value s that we define as the sum. This weakens the meaning of equality, since it is relative to a method of computation, and not, as is normally the case, independent of it.

The methods approach has several weaknesses:

1. An unworkable conception of weak equality.
2. The failure of all methods, except the **Euler extension method** (p. 391), to account for Equation 2.
3. A faulty understanding of the Euler extension method, including a circular definition in [H49], and failure to realize this method as an extension of arithmetic.

We further examine these points in later chapters and show how the numeristic approach avoids all of them.

Other approaches

A few other approaches to divergent series deserve consideration.

Transformation to convergent series. In this approach, a divergent series is considered as an encoded form of a convergent series, using the transformation

$$\frac{1}{1-a} = \left(-\frac{1}{a}\right) \frac{1}{1-\frac{1}{a}}.$$

This approach has the disadvantage of conceptualizing simple arithmetic statements as something different from what they state, and therefore of introducing extra steps of calculation. Another disadvantage is that an expression which combines convergent and divergent series, such as $\sum_{-\infty}^{+\infty} a^n$, must be considered as the sum of two separate series.

Congruence. In this approach, we observe that the partial sums $\sum_0^j 2^n$ are each congruent to $-1 \pmod{2^{j+1}}$. The drawback of this approach is that it requires us to define an equality through a congruence. It is also not immediately clear how to extend it to any other case than $\sum_0^j 2^n$, since for integral bases greater than two, the sum is a fraction.

Adjusted series. Another approach is to return to the twos-complement arithmetic described above, and generalize its sign bit mechanism. This seems at first to be a combination of the limit and numeric approach, but it actually ends up being essentially just the numeric approach.

If we define

$$f(k) = \begin{cases} a^k, & k < n \\ \frac{a^k}{1-a}, & k = n \end{cases}$$

then

$$\sum_{k=0}^n f(k) = \left(\sum_{k=0}^{n-1} a^k\right) + \frac{a^n}{1-a} = \frac{1-a^n}{1-a} + \frac{a^n}{1-a} = \frac{1}{1-a}.$$

This value does not depend on n —for all values of n , the sum is constant. In the infinite case, the last term does not exist. Hence we say that

$$\sum_{k=0}^{\infty} f(k) = \sum_{k=0}^{\infty} a^k = \frac{1}{1-a}.$$

This satisfies the limits approach, because the infinite case is the limit of the finite case. But a limits approach alone does not suggest how to modify the definition of $\sum_{k=0}^{\infty} a^k$ so that the result is constant. The numeric approach supplies the crucial last term and thus satisfies this requirement.

Objections

Approaches to the theory of divergent series historically have faced various objections which did not seem to allow simple algebraic treatment. Here we examine some of these objections and their numeric resolutions.

Incorrect sum of absolute arithmetic series

It is sometimes claimed that the absolute arithmetic series $1 + 2 + 3 + \dots$ has the sum $-\frac{1}{12}$. Equipoint summation does not support this claim, finding instead that this series has only an infinite value. See [Absolute arithmetic series](#) (p. 368). Nevertheless, $-\frac{1}{12}$ is *associated* with this series, as is shown in [Ramanujan summation](#) (p. 387).

Apparent incompatibility of linearity and stability

In the method approach to divergent series, we say that a summation method is *linear* if

$$\sum_{n=0}^{\infty} a_n + \sum_{n=0}^{\infty} b_n = \sum_{n=0}^{\infty} (a_n + b_n)$$

for any two series a_n and b_n , and we say that the method is *stable* if

$$a_0 + \sum_{n=1}^{\infty} a_n = \sum_{n=0}^{\infty} a_n,$$

i.e. adding a term to the beginning of the series increases the sum of the series by the same amount.

The method approach claims that a summation method that is both linear and stable cannot sum the series $1 + 2 + 3 + \dots$. The argument is that if

$$1 + 2 + 3 + \dots = x,$$

then by stability

$$0 + 1 + 2 + \dots = 0 + x = x.$$

By linearity, one may subtract the second equation from the first to give

$$1 + 1 + 1 + \dots = x - x = 0.$$

Again by stability,

$$0 + 1 + 1 + 1 + \dots = 0,$$

and subtracting the last two series gives

$$1 + 0 + 0 + \dots = 0,$$

contradicting stability.

Numeristically, the problem with this argument is that the first and second series both have only one value, ∞ , as shown in [Absolute arithmetic series](#) (p. 368). Subtracting the second from the first yields \emptyset rather than zero. The third series, as shown in [Absolute sum of units](#) (p. 360), also has only the value ∞ , which is an element of \emptyset and is thus consistent with the previous subtraction. Likewise the fourth series has only the value ∞ , so subtracting it from the third again yields \emptyset , which includes 0 and 1.

Apparent multiple values

Hardy [H49, p. 16] points out several examples of divergent series which appear to give multiple values. For example, it appears that

$$x + (2x^2 - x) + (3x^3 - 2x^2) + (4x^4 - 3x^3) + \dots = 0$$

and

$$x + (3x^2 - x) + (7x^4 - 3x^2) + (15x^8 - 7x^4) + \dots = 0$$

for $0 \leq x < 1$, but for $x = 1$ these give $1 + 1 + 1 + \dots = 0$ and $1 + 2 + 4 + \dots = 0$.

But he does not recognize that

$$\begin{aligned} & x + (2x^2 - x) + (3x^3 - 2x^2) + (4x^4 - 3x^3) + \dots \\ &= (x - x) + (2x^2 - 2x^2) + (3x^3 - 3x^3) + (4x^4 - 4x^4) + \dots \\ &= 0 + 0 + 0 + \dots = 0 \cdot \infty, \end{aligned}$$

and similarly

$$\begin{aligned} & x + (3x^2 - x) + (7x^4 - 3x^2) + (15x^8 - 7x^4) + \dots \\ &= (x - x) + (3x^2 - 3x^2) + (7x^4 - 7x^4) + (15x^8 - 15x^8) + \dots \\ &= 0 + 0 + 0 + \dots = 0 \cdot \infty, \end{aligned}$$

which of course are indeterminate, not merely zero.

Apparently incorrect sum of the Grandi series

Using Equation 2, we have shown that we can calculate that $1 - 1 + 1 - 1 + \dots = \frac{1}{2}$. It is frequently objected that this cannot be the only correct result, since we could also have $1 - 1 + 1 - 1 + \dots = (1 - 1) + (1 - 1) + \dots = 0 + 0 + \dots = 0$. This objection comes from one or both of two points of view: (1) an infinite sum is a limit; or (2) the sum of an infinite number of zeros is zero. The first point of view overlooks the important fact that the type of sum we are considering here is not a limit. The second point of view overlooks the fact that an infinite sum of zeros is $\infty \times 0$, which is indeterminate. See also [Alternating sum of units](#) (p. 362), where we explore the Grandi series with equipoint summation.

Callet's objection

Euler was the first to systematically use Equation 2 for divergent series and was the first to state this case of it. Some time afterwards, Jean-Charles Callet challenged this view, citing the series

$$1 - a^2 + a^3 - a^5 + a^6 - a^7 + a^8 - \dots$$

Setting x to this series, we have

$$\begin{aligned} x &= 1 - a^2 + a^3 - a^5 + a^6 - a^8 + a^9 - \dots \\ &= 1 - a^2 + a^3(1 - a^2 + a^3 - a^5 + a^6 - \dots) \\ &= 1 - a^2 + a^3x. \end{aligned}$$

Then $x - a^3x = 1 - a^2$, and

$$x = \frac{1 - a^2}{1 - a^3} = \frac{1 + a}{1 + a + a^2}.$$

When $a = 1$, we would then have

$$1 - 1 + 1 - 1 + \dots = \frac{2}{3}.$$

Lagrange replied that actually

$$\frac{1 - a^2}{1 - a^3} = 1 + 0a - a^2 + a^3 + 0a^4 - a^5 + a^6 + 0a^7 - a^8 + a^9 + \dots,$$

which for $a = 1$ becomes

$$1 + 0 - 1 + 1 + 0 - 1 + 1 + 0 - 1 + 1 + \dots = \frac{2}{3}.$$

This differs from $1 - 1 + 1 - 1 + \dots$ by the addition of an infinite number of zeroes, which, as we have seen above, is not zero but indeterminate. Besides this, for $a = 1$, $\frac{1+a}{1+a+a^2} = \frac{2}{3}$, but $\frac{1-a^2}{1-a^3}$ is indeterminate.

Indeterminacy of $0 \cdot \infty$

A common thread in these considerations is the indeterminacy of $0 \cdot \infty$. From Equation 3 we have

$$\sum_{k=0}^{\infty} 0a^k = 0 + 0a + 0a^2 + 0a^3 + \dots = 0(1 + a + a^2 + a^3 + \dots) = \frac{0}{1-a},$$

which is zero except for $a = 1$, where it is indeterminate. Thus the sum of an infinite number of zeros may be zero or indeterminate. It is indeterminate when the ratio between the zeros is constant, and this is the case in the above examples. An example where this is not the case is a terminating decimal, which is also a repeating decimal with repetend 0:

$$0.5 = 0.5000\dots = \frac{1}{2} + \frac{0}{100} \left(1 + \frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \dots \right) = \frac{1}{2} + \frac{0}{1 - \frac{1}{10}} = \frac{1}{2}.$$

For further information on infinite sums of zeros, see [Sum of zeros](#) (p. 353) above. For further information on terminating decimals, see [Repeating Decimals](#) (p. 409–457).

Apparent inconsistency with Riemann zeta function

Hardy [[H49](#), p. 16] states that $1+1+1+\dots = -\frac{1}{2}$, because the well-known Riemann zeta function, which he states as

$$\zeta(x) = \sum_{k=1}^{\infty} k^x,$$

has the value $\zeta(0) = -\frac{1}{2}$. The problem with this value is that it is calculated from the definition

$$\zeta(x) = \frac{1}{\Gamma(x)} \int_0^{\infty} \frac{u^{x-1}}{e^u - 1} du,$$

which is equivalent to the first expression for $x > 1$, but not elsewhere.

Paradox of comparison of series and integrals

In equipoint analysis, a **definite integral** (p. 165) is an infinite series in which each term is zero, while in a convergent or divergent series, an infinite number of terms are nonzero. When the terms can be directly compared, this may lead to a paradoxical condition wherein both a series and an integral yield a finite result. For example, consider that

$$\sum_{n=1}^{\infty} 2^{-n} = 1 < \int_0^1 2 \, dx = \sum_{n=1}^{\infty'} \frac{2}{\infty'} = 2,$$

even though, if we look at individual terms,

$$2^{-n} \geq \frac{2}{\infty}$$

for all n , with equality holding only for infinite n .

We have already seen that the first series actually has two values, one infinite and one finite. See **Infinite series have infinite values** (p. 343).

The conventional theory of divergent series

For several centuries there has been a theory of divergent series, which attempts to show how and why we can find a sum for many types of divergent series. It is often said that the theory of divergent series started with Euler, whose findings on the subject are probably best represented in [Eu55] and [Eu60]. This theory has attracted other well known names, including Poisson, Abel, and Hardy. Hardy's posthumous book [H49] of 1949 is generally considered a standard work.

Euler and other mathematicians of the 18th century generally either took a basically numeric approach to divergent series, or rejected such series, or vacillated. In the 19th century, the limits approach developed and gradually came to dominate the approach to infinite arithmetic. In the 20th century, the methods approach developed and attracted some attention, but still remained poorly known and did not come to dominate over the limits approach.

In recent years, the methods approach seems to be attracting much more attention. One example of this transition occurred between two editions of the *Encyclopedic Dictionary of Mathematics*, a standard general mathematics reference work. In the first edition [ED77], published in 1977, the article

Summability describes the methods approach to divergent series, while the article *Infinite series* gives only the conventional limits approach without even a reference the summability article, and the index does not reference the summability article under *series*, only under *summability*. In the second edition [ED87] of 1987, the summability material has been merged into the *Infinite series* article. Both [ED77] and [ED87], in their succinct but formalistic way, define a method as a linear transformation and thereby exclude the Euler extension method, described below. They also use weak equality, the assertion that equality established through a method is relative to the method.

Hardy does not define method in general but does define and use many individual methods which imply weak equality, including the Euler extension method. This method is best suited to power series, which is the key to the whole theory.

Methods of summation

Hardy [H49] defines several dozen methods of summation. We will now examine a significant cross section of these methods, and a few that have been developed since [H49] was published. This includes all the methods commonly encountered in the literature and many other more obscure methods. We will apply each one to the series

$$\sum_{k=0}^{\infty} a^k, \quad |a| > 1. \quad (95)$$

Surprisingly, we will find that only *one* of these methods is capable of summing this series. That method is the **Euler extension method** (p. 391), discussed below.

A key criterion for assessing the validity of any method is *regularity*. A method P is regular if, for any convergent series $\sum a_m$, $\sum_{(P)} a_m = \lim \sum a_m$, i.e. if the method sums any convergent series to its ordinary value as a limit. Most, but not all, methods defined in [H49] are regular.

For each of these methods:

- We give a page reference to [H49], except for two methods which are not in [H49].
- We indicate whether the method is regular.

- We give a definition of the method when it is fairly simple. Otherwise, we express our conclusions in the same notation that [H49] uses.
- We apply each method to Equation 95.
- We denote partial sums as $s_n \equiv \sum^n a_k$.

Euler method (\mathfrak{E}), which we call the *Euler extension method* to distinguish it from another Euler method: [H49, p. 7]. For a full discussion, see **The Euler extension method of summation** (p. 391). Regular. Sums (95) to $\frac{a^n}{1-a}$.

Cesàro mean ($C, 1$): [H49, p. 7]. Defined as

$$\sum_{(C,1)} a_n = \lim_{n \rightarrow \infty} \frac{s_0 + s_1 + s_2 + \dots + s_n}{n+1}.$$

Regular. Diverges for (95).

Abel sum (A): [H49, p. 7]. Defined as $\sum_{(A)} a_n = \lim_{x \rightarrow 1^-} \sum a_n x^n$ if $\sum a_n x^n$ is convergent for $0 \leq x < 1$. Regular. Does not apply to (95) since $\sum a_n x^n = \sum a^n x^n = \sum (ax)^n$ is not convergent for all $x \in [0, 1)$.

Euler's polynomial method ($E, 1$): [H49, p. 7]. Regular. Diverges for (95), since $b_n = (a+1)^n > 2^n$.

Hutton's method (Hu, k): [H49, p. 21]. Regular. A limit of positive terms for (95), and thus diverges.

Ramanujan's method (\mathfrak{R}, a): [H49, p. 327]. Hardy misrepresents Ramanujan's original method. This is examined in detail in the next chapter, **Ramanujan summation** (p. 387).

Borel integral method (B'): [H49, p. 83]. Regular. Sums (95) to $\frac{1}{1-a}$ only when $\text{Re } a < 1$.

Borel exponential method (B): [H49, p. 80]. Regular. Sums (95) to $\frac{1}{1-a}$ only when $\text{Re } a < 1$.

Nörlund means (N, p_n): [H49, p. 64]. Scheme requiring choice of $\{p_n\}$. Regular for some $\{p_n\}$. The partial quotients for (95) t_m are always positive, and thus cannot yield $\frac{1}{1-a}$, which is negative.

Abel means (A, λ_n) : [H49, p. 71]. Scheme requiring choice of $\{\lambda_n\}$. Regular. For (95), yields only positive terms, so the method yields a positive limit or no limit.

Lindelöf method (L) : [H49, p. 99]. Regular. Same as $(A, n \ln n)$ for (95), which fails in all cases.

Mittag-Leffler method (M) : [H49, p. 79]. Regularity not stated in [H49]. Similar to Lindelöf method. Hardy shows that L and M methods take (95) to $\frac{1}{1-a}$, but only on a region Δ in the complex plane, called the *Mittag-Leffler star* of a^n for $a \in \mathbb{C}$. This region does not include any point in $(1, \infty)$.

Riemann method (R, k) : [H49, p. 89]. Regular for $k > 1$. For (95), a^n eventually overwhelms $\left(\frac{\sin nh}{nh}\right)^k$, so the limit diverges.

Euler's general polynomial method (E, q) : [H49, p. 178]. Regular. Requires $q > 0$ and sums (95) to $\frac{1}{1-a}$ only within a circle with center at $-q$ and radius $q + 1$, which excludes $a > 1$.

General Cesàro means (C, k) : [H49, p. 96]. Regular for $k > 0$. Limit of the quotient of two positive terms for (95); diverges.

Hölder means (H, k) : [H49, p. 94]. Regularity not stated in [H49]. Limit of positive terms for (95); diverges.

Ingham's method (I) : [H49, p. 399]. Not regular. Limit of the sum of positive terms for (95); diverges.

Second Nörlund method (\bar{N}, p_n) : [H49, p. 57]. Regular. Yields a positive fraction for (95) and any set of positive $\{p_n\}$, and thus cannot yield $\frac{1}{1-a}$, which is negative.

De la Vallée-Poussin's method (VP) : [H49, p. 88]. Regular. Limit of the sum of positive terms for (95), which is never negative.

Bernoulli summability (Be) : Not in [H49]. Defined by

$$\sum_{(Be)} a_n = \lim_{N \rightarrow \infty} \sum_{n=0}^N s_n \binom{N}{n} p^n (1-p)^{N-n},$$

where $s_n = \sum^n a_k$ and $0 < p < 1$. Always yields a positive result for (95) and thus cannot assume a negative value.

Dirichlet summability (D): Not in [H49]. Defined by

$$\sum_{(D)} a_n = \lim_{x \rightarrow 1^+} \sum_{n=1}^{\infty} \frac{s_n}{n^x},$$

where $s_n = \sum^n a_k$. Always yields a positive result for (95) and thus cannot assume a negative value.

Ramanujan summation

Hardy defines a Ramanujan method of summation, but Hardy’s definition differs significantly from Ramanujan’s original definition. Hardy also states that Ramanujan’s definition is a method of summing a series, whereas Ramanujan’s claim is more modest. Since Hardy somewhat misrepresents Ramanujan’s work, we now examine the discrepancies in detail.

Ramanujan’s original method is described in [Be, p. 133-136]. This posthumous edition of Ramanujan’s notebooks by B. Berndt includes much commentary by Berndt.

This section functions as a correction of Hardy. Hardy ([H49, p. 327]) defines a Ramanujan sum (\mathfrak{R}, a) of a series, which depends on a , and says there is a natural value of a for every series but does not describe a general procedure for finding this a . Hardy’s definition is reproduced below.

Hardy says that Ramanujan’s work with divergent series was based on this definition, but (1) Ramanujan uses a simpler definition, also given below, that does not depend on a parameter, and (2) Ramanujan calls the number he computes the “constant” of a series and denotes it as an operator on a function. He relates it to but does not equate it with a sum of a divergent series.

Both of these definitions are based on the Euler-Maclaurin formula, which relates a sum with finite limits and an integral over a finite interval. This formula uses the Bernoulli numbers B_n and periodic Bernoulli polynomials P_n . Hardy uses a definition of Bernoulli number that is now generally considered obsolete and is now written B_k^* . These two are related by $B_k^* = (-1)^{k+1} B_{2k}$. Here we express Hardy’s definitions in the newer notation.

The Euler-Maclaurin formula is usually given as:

$$\sum_{k=0}^x f(k) = \int_0^x f(t)dt + \frac{1}{2} [f(x) + f(0)]$$

$$+ \sum_{k=1}^p \frac{B_{2k}}{(2k)!} [f^{(2k+1)}(x) - f^{(2k+1)}(0)] + \int_0^x \frac{P_{2p+1}(t)}{(2p+1)!} f^{(2p+1)}(t) dt.$$

Subtracting $f(0)$ gives the form that Ramanujan/Berndt uses:

$$\sum_{k=1}^x f(k) = \int_0^x f(t) dt + \frac{1}{2} [f(x) - f(0)] + \sum_{k=1}^p \frac{B_{2k}}{(2k)!} [f^{(2k+1)}(x) - f^{(2k+1)}(0)] + \int_0^x \frac{P_{2p+1}(t)}{(2p+1)!} f^{(2p+1)}(t) dt.$$

In both cases, p is arbitrary. The last term is a remainder which is often denoted $R_p f(x)$, while the second last term is denoted $S_p f(x) - S_p f(0)$.

When $x = \infty$, both the sum and the integral have infinite limits. The sum is convergent if and only if the integral is also convergent.

Ramanujan/Berndt sets $p = \infty$ and examines two cases, $f(x) = 1$ and $f(x) = x$. We examine two additional cases, $f(x) = e^x$ and its general case $f(x) = b^x$ for a perfinite nonunit constant b .

First we compute R_∞ indirectly by evaluating

$$\sum_{k=1}^x f(k) - \int_0^x f(t) dt - \frac{1}{2} [f(x) - f(0)] - S_\infty f(x) + S_\infty f(0).$$

We then have:

For $f(x) = 1$:

$$R_\infty = x - x - 0 - 0 + 0 = 0$$

For $f(x) = x$:

$$R_\infty = \frac{x^2 + x}{2} - \frac{x^2}{2} + \frac{x}{2} - 1 + 1 = 0$$

For $f(x) = e^x$:

$$\begin{aligned} R_\infty &= \frac{e^{x+1} - e^x}{e - 1} - e^x - 1 - \frac{1}{2}(e^x - 1) - \sum_{k=1}^\infty \frac{B_{2k}}{(2k)!} (e^x - 1) \\ &= (e^x - 1) \frac{e}{e - 1} - (e^x - 1) - (e^x - 1) \frac{1}{2} - (e^x - 1) \sum_{k=2}^\infty \frac{B_k}{k!} \\ &= (e^x - 1) \left[\frac{e}{e - 1} - 1 - \frac{1}{2} - \left(\frac{1}{e - 1} - 1 + \frac{1}{2} \right) \right] \end{aligned}$$

$$\begin{aligned}
 &= (e^x - 1) \frac{e - e + 1 - 1}{e - 1} \\
 &= 0
 \end{aligned}$$

For $f(x) = b^x$:

$$\begin{aligned}
 R_\infty &= \frac{b^{x+1} - b^x}{b - 1} - \frac{b^x - 1}{\ln b} - \frac{1}{2}(b^x - 1) - \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} (\ln b)^{2k-1} (b^x - 1) \\
 &= (b^x - 1) \frac{b}{b - 1} - (b^x - 1) \frac{1}{\ln b} - (b^x - 1) \frac{1}{2} - (b^x - 1) \frac{1}{\ln b} \sum_{k=2}^{\infty} \frac{B_k}{k!} (\ln b)^k \\
 &= (b^x - 1) \left[\frac{b}{b - 1} - \frac{1}{\ln b} - \frac{1}{2} - \frac{1}{\ln b} \left(\frac{1}{b - 1} - 1 + \frac{\ln b}{2} \right) \right] \\
 &= (b^x - 1) \frac{2b \ln b - 2(b - 1) - (b - 1) \ln b - 2 + 2(b - 1) - (b - 1) \ln b}{2(b - 1) \ln b} \\
 &= (b^x - 1) \frac{\ln b - 1}{(b - 1) \ln b}
 \end{aligned}$$

From the Euler-Maclaurin formula, Ramanujan/Berndt extracts the portion of the right side which is independent of x and calls it the constant C of the series:

$$C = -\frac{1}{2}f(0) - \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} f^{(2k+1)}(0).$$

We then have:

$$\text{For } f(x) = 1 : \quad C = -\frac{1}{2} - 0 = -\frac{1}{2}$$

$$\text{For } f(x) = x : \quad C = -0 - \frac{1}{12} = -\frac{1}{12}$$

$$\begin{aligned}
 \text{For } f(x) = e^x : \quad C &= -\frac{1}{2} - \left(\frac{1}{e - 1} - 1 + \frac{1}{2} \right) \\
 &= \frac{-1}{e - 1}
 \end{aligned}$$

$$\begin{aligned}
 \text{For } f(x) = b^x : \quad C &= -\frac{1}{2} - \frac{1}{\ln b} \left(\frac{1}{b - 1} - 1 + \frac{\ln b}{2} \right) \\
 &= \frac{b - 2}{(b - 1) \ln b} - 1
 \end{aligned}$$

Hardy refines Ramanujan's C by adding a parameter a . Berndt notes this (p. 135) and gives Hardy's C_a as

$$C_a = \int_0^a f(t)dt - \frac{1}{2}f(0) - \sum_{k=1}^p \frac{B_{2k}}{(2k)!} f^{(2k-1)}(0) + \int_0^\infty P_{2p+1}(t) f^{(2p+1)}(t)dt.$$

C_a is independent of p , since p is arbitrary in the Euler-Maclaurin formula.

Hardy's actual definition is somewhat different (p. 326). To distinguish it from Berndt's, we call it C' :

$$C'_a = \int_1^a f(t)dt + \frac{1}{2}f(1) - \sum_{k=1}^p \frac{(-1)^{k-1} B_k}{k!} f^{(2k-1)}(1) - \frac{1}{(2n+2)!} \int_0^\infty \psi_{2p+2}(t) f^{(2p+2)}(t)dt,$$

where $\psi_n(x) = \psi_n(x - [x])$ and $t \frac{e^{xt}-1}{e^t-1} = \sum_{n=0}^\infty \phi_n(x) \frac{t^n}{n!}$.

Hardy compares \sum_1 to \int_1 , whereas Berndt compares \sum_1 to \int_0 . This means we must transpose f so that Hardy's $f(x)$ is Berndt's $f(x - 1)$. It also means that Hardy uses the first form of the Euler-Maclaurin formula, whereas Berndt uses the second form. After transposing f , we must add $f(0)$ to Berndt's formula to get Hardy's formula. We ignore discrepancies in the remainder, since we calculate it indirectly as above.

We can therefore express C_a and C'_a as:

$$C_a = \int_0^a f(t)dt - \frac{1}{2}f(0) - \sum_{k=1}^p \frac{B_{2k}}{(2k)!} f^{(2k-1)}(0) + R_\infty$$

$$C'_a = \int_0^a f(t)dt + \frac{1}{2}f(0) - \sum_{k=1}^p \frac{B_{2k}}{(2k)!} f^{(2k-1)}(0) + R_\infty$$

$$= C_a + f(0)$$

and we compute:

For $f(x) = 1$: $C_0 = -\frac{1}{2} - 0 = -\frac{1}{2}$
 $C'_0 = \frac{1}{2}$
 For $f(x) = x$: $C_0 = -\frac{1}{12} - 0 = -\frac{1}{12}$

$$C'_0 = -\frac{1}{12}$$

$$\text{For } f(x) = e^x : C_0 = \frac{-1}{e-1} + 0 = \frac{-1}{e-1}$$

$$C'_0 = \frac{-1}{e-1} + 1 = \frac{e-2}{e-1}$$

$$\text{For } f(x) = b^x : C_0 = \frac{b-2}{(b-1)\ln b} - 1 - \frac{\ln b - 1}{(b-1)\ln b} = \frac{-b\ln b + b - 1}{(b-1)\ln b}$$

$$C'_0 = \frac{-b\ln b + b - 1}{(b-1)\ln b} + 1 = \frac{-\ln b + b - 1}{(b-1)\ln b} + 1$$

Claiming that this is a sum is not true to Ramanujan's work because:

1. Ramanujan never claimed that his constant was a sum.
2. The constant formula only represents a portion of the difference between a sum and an integral, not the sum itself.
3. The portion it represents does not depend on the upper bound of the sum, which may be finite or infinite, whereas a sum of course generally depends on its upper limit.

The Euler extension method of summation

We now examine in detail the Euler extension method of summation, which Hardy denotes as \mathfrak{E} . We have seen how every other method commonly used in the current theory of divergent series fails to derive $\sum a^n = \frac{1}{1-a}$ for $|a| \geq 1$. Strangely, even though this is the only such method that can derive this sum, it is usually omitted from works on divergent series. For instance, [ED87, p. 1415] defines a method as a linear transformation and thus defines away this method.

We will see that this method is poorly understood. We will develop a better understanding of it and see how this yields a satisfactory theory of divergent series. We start with Hardy's definition [H49, p. 7]:

If $\sum a_n x^n$ is convergent for small x , and defines a function $f(x)$ of the complex variable x , one-valued and regular in an open and connected region containing the origin and the point $x = 1$; and $f(1) = s$; then we call s the \mathfrak{E} sum of $\sum a_n$. The value of s may naturally depend on the region chosen.

Unfortunately, this definition is circular. If the series defines f , then $f(x) = \sum a_n x^n$ within some region, and if this region contains $x = 1$, then $f(1) = \sum a_n$ *by definition*. This does not extend the definition of the series from the convergent case to the divergent case, but instead assumes that we already have a definition of the divergent case.

Euler's own definition of his method is quite different. He identifies an infinite series, convergent or divergent, with a finite expression from which it is "expanded," and the sum of the series with the value of the expression. He defends such an assignment of a sum to a divergent series as consistent and useful:

Let us say, therefore, that the *sum* of any infinite series is the finite expression, by the expansion of which the series is generated. In this sense, the sum of the infinite series $1 + x + x^2 - x^3 + \dots$ will be $\frac{1}{1-x}$, because the series arises from the expansion of the fraction, whatever number is put in place of x . If this is agreed, the new definition of the word *sum* coincides with the ordinary meaning when a series converges; and since divergent series have no sum, in the proper sense of the word, no inconvenience can arise from this new terminology. Finally, by means of this definition, we can preserve the utility of divergent series and defend their use from all objections.

[Eu55, Ch. 3, Sect. 111, p. 78–79] tr. [BL, p. 142]

The above passage appears in a book on analysis which Euler wrote in 1755. Five years later, Euler wrote a paper on divergent series, in which he appears to be assuming the role of mediator between two opposing factions, one which opposes assigning any sum to a divergent series, and one which supports it. After carefully considering the arguments of both sides, he comes down firmly on the side of supporting the assignment and gives several reasons for his decision. The following three quotes from this paper are highlights of his reasoning. The first quote makes it clear that his method of expanding $\frac{1}{1-x}$ into $1 + x + x^2 - x^3 + \dots$ is polynomial division:

Of the second type [in his list of types of infinite series, the second type being oscillating series] is this series, $1 - 1 + 1 - 1 + \dots$, first considered by Leibniz, whose sum he gave as equal to $\frac{1}{2}$, with the support of the following fairly sound reasoning: first, this series appears if the fraction $\frac{1}{1+a}$ is expanded in the usual way by continued division into the following series $1 - a + a^2 - a^3 + a^4 - a^5 + \dots$, and the value of the letter a is taken equal to unity.

[Eu60, §3, p. 207] tr. [BL, p. 145]

In the next passage, Euler addresses the objection that using the above procedure on a series such as $1+2+4+8+\dots$, consisting entirely of positive terms, leads to a negative sum, -1 . He points out that this is not as unreasonable as it may appear, since a transition from positive to negative can occur through the infinite as well as through zero:

However, it seems in accord with the truth if we say that the same quantities which are less than zero can be considered to be greater than infinity. For not only from algebra but also from geometry, we learn that there are two jumps from positive quantities to negative ones, one through nought or zero, the other through infinity, and that quantities whether increasing from zero or decreasing come back on themselves and return to the same destination $0 \dots$

[Eu60, §8, p. 210] tr. [BL, p. 147]

This is in accord with the projectively extended real numbers system that we have used here and defined in **Real infinite element extensions** (p. 68). In this system, ∞ is both positive and negative, and, depending on the mapping method, maps to either one or two points on a unit circle that is tangent to the real number line.

In the following quote from his paper, Euler repeats his contention that his method of assigning a sum to a divergent series is meaningful and useful:

But I think all this wrangling can be easily ended if we should carefully attend to what follows. Whenever in analysis we arrive at a rational or transcendental expression, we customarily convert it into a suitable series on which the subsequent calculation can more easily be performed. Therefore infinite series find a place in analysis inasmuch as they arise from the expansion of some closed expression, and accordingly in a calculation it is valid to substitute in place of the infinite series that formula from which the series came. Just as with great profit rules are usually given for converting expressions closed but awkward in form into infinite series, so likewise the rules, by whose help the closed expression, from which a proposed infinite series arises, can be investigated, are to be thought highly useful. Since

this expression can always be substituted without error for the infinite series, both must have the same value: it follows that there is no infinite series for which the closed expression equivalent to it cannot be conceived.

[Eu60, §11, p. 211–212] tr. [BL, p. 148]

Euler emphasizes the equivalence of a divergent series with its finite form. If the finite form is not known, his method may not appear to be useful. However, when he indicates that $1 + a + a^2 + a^3 + \dots$ is obtained from $\frac{1}{1-a}$ by polynomial division, he seems to imply that the arithmetic of divergent series is consistent with the rest of algebra. Therefore, any algebraic technique which transforms a series to a finite expression should be valid.

The recursion approach we have used here to obtain finite values for infinite series is thus essentially equivalent to the extension method Euler proposes. The recursion approach enhances Euler's method by showing that, whenever an infinite series is algebraically equivalent to an expression that has a finite value, that expression also has an infinite value. We have seen that this approach, when properly applied, does not yield any known inconsistencies. This method is clearly regular, since it does not involve any transformation of a divergent series to a convergent one.

Euler's extension method is essentially the extension of meaning in a consistent way to formulas which were previously considered meaningless. This extension is similar in principle to the extension of the number system to irrational, negative, and imaginary numbers, all of which initially met with opposition.

- In the 5th century BC, the Greek mathematician Hippasus's proof of the existence of irrational numbers led to his being persecuted by fellow Pythagoreans. It took another century for Greek mathematicians to accept irrational numbers.
- Negative numbers were rejected as meaningless by ancient Greek mathematicians. They were used in ancient and medieval times in India, China, and Islamic countries, but they were not accepted in Europe until the 16th and 17th centuries.
- Imaginary numbers became known in Europe in the 16th century, but they were usually regarded as meaningless until the 18th century.

Euler's extension method involves no limits and no transformation of terms or partial sums. It can calculate the sum of series that no other method can, and there is no known series that any other method can sum that this method cannot. The numeric approach starts with Euler's extension method and adds an infinite value to whatever finite value Euler's method may find, and yields an infinite value alone for those for which that Euler's methods finds no value.

There is no need for a weakened form of equality, since we assume that a divergent series is strictly equal to its algebraic sum. The results of this approach can be applied without ambiguity, since the equalities it establishes are as strong as all other equalities.

In this approach, if we wish to use other methods of summation, we denote them as modified summation rather than weakened equality. Szász [Sz, p. 2] uses the notation $P(\sum a_m) = s$. This leaves equality strong, but it also implies that we first compute the value of $\sum a_m$ and then apply P after we compute the sum, whereas the real situation is that a method P operates first on the terms of the series, and then either sums the modified terms or performs some other operation on them. This means that P is really a modified form of summation. The notation $\sum_{(P)} a_m = s$ properly conveys that a method P sums a series with terms a_m to the value s .

Analytic continuation

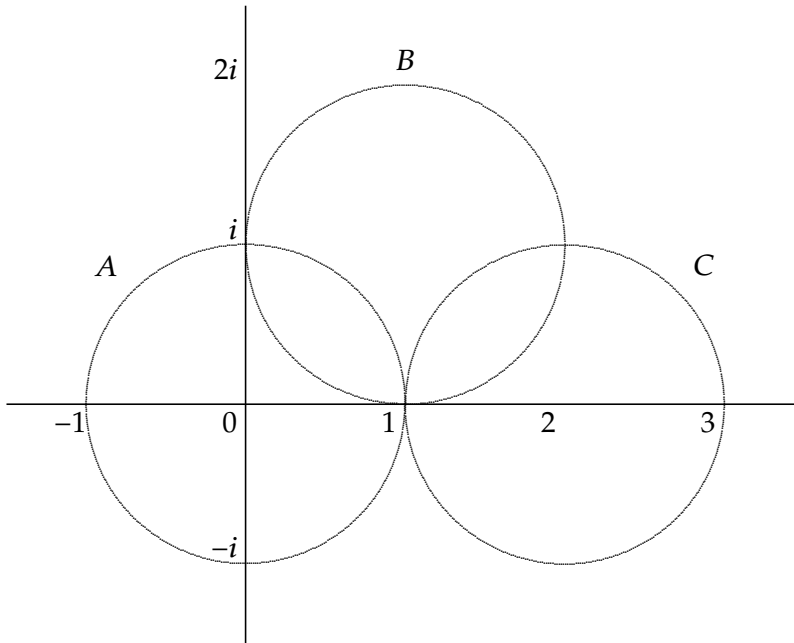


FIG. 110:
Analytic extension of $f(z) = \frac{1}{1-z}$
into neighborhoods of $z = 0, 1 + i, 2$

The Euler extension method bears some resemblance to analytic continuation, but has an important difference from it. In complex analysis, given an analytic function, analytic continuation is a procedure of deriving multiple convergent series in overlapping neighborhoods, in each of which the series converges to the same analytic function.

Figure 110 shows this process for $f(z) = \frac{1}{1-z}$. We start with the interior of circle A , the neighborhood $|z| < 1$, where the series

$$\sum_{k=0}^{\infty} z^k = 1 + z + z^2 + z^3 + \dots \quad (A)$$

converges to $f(z)$. Then we move to the interior of circle B , the neighborhood $|z - 1 - i| < 1$, which overlaps the interior of A . Here the convergent power

series for $f(z)$ is

$$\sum_{k=0}^{\infty} (z - 1 - i)^k = 1 + (z - 1 - i) + (z - 1 - i)^2 + (z - 1 - i)^3 + \dots \quad (B)$$

By a well known theorem of complex analysis, if $f(z)$ is analytic in the interior of A , and the interiors of A and B overlap, then both series (A) and series (B) converge to the same function, $f(z)$, within the overlapping area.

In this way, one convergent series after another for $f(z)$ can then be developed. In Figure 110, the interior of circle C is $|z - 2| < 1$, where the series

$$\sum_{k=0}^{\infty} (z - 2)^k = 1 + (z - 2) + (z - 2)^2 + (z - 2)^3 + \dots \quad (C)$$

converges to $f(z)$. Again, series (B) and series (C) both converge to the same function, $f(z)$, in the overlap of the interiors of B and C . However, the interiors of A and C do not overlap, which means that series (A) diverges everywhere that series (B) converges and conversely.

We have thus analytically extended $f(z)$, but to do so, we must proceed in steps, and we derive a different power series at every step.

This contrasts with the Euler extension method, which deals with *divergent* series, as well as convergent series, and does not proceed in steps. Instead, the Euler extension method, as we have developed it here, recognizes that infinite series, and the numeric arithmetic and algebra used to handle them, are valid whether the series are convergent or divergent. In this approach, series (A), (B), and (C) are valid for $f(z)$ throughout the complex plane, whether they are convergent or not.

THE NUMERISTIC CONTEXT

One important aspect of numeristics is the view that numbers and arithmetic stand on their own and do not require definitions to exist. Although our discussions require that we define our terms, definitions are not a substitute for mathematical truth. The truth of mathematical expressions must ultimately be verified by observation of nature, just as in any other science. In other words, mathematical truth cannot simply be defined into existence.

In this light, we now examine Hardy's remarks in [Mo, p. 5–6] concerning the role of definitions in the methods approach to divergent series. In the numeristic view, these remarks contain a mixture of both valid and excessive claims about definitions.

[I]t does not occur to a modern mathematician that a collection of mathematical symbols should have a 'meaning' until one has been assigned by definition. It was not a triviality even to the greatest mathematicians of the eighteenth century. They had not the habit of definition; it was not natural to them to say, in so many words, 'by X we mean Y '. There are reservations to be made, . . . but it is broadly true to say that mathematicians before Cauchy asked not 'How shall we *define* $1 - 1 + 1 - \dots$?' but 'What *is* $1 - 1 + 1 - \dots$?' and that this habit of mind led them into unnecessary perplexities and controversies which were often really verbal. [Emphasis in the original.]

While Hardy further acknowledges that the value of $\frac{1}{2}$ seems "natural," he says that assigning this as the sole value of the series actually has problems, which he attributes to lack of proper definition. His solution is to use methods of summation, with its concept of weak equality. However, as we saw in **The Euler extension method of summation** (p. 391), his definition of the Euler extension method is quite different from Euler's, and it is circular. He seems to miss its essence as a simple extension of arithmetic.

The numeristic approach gains support from the recent development of Maharishi Vedic Mathematics, a formulation of ancient Vedic philosophies and technologies of consciousness in mathematical terms [M96, CI] by Maharishi Mahesh Yogi, the founder of the Transcendental Meditation program. In this approach, pure consciousness is the basic experience and governing intelligence of life. The technologies of Maharishi Vedic Mathematics, including Transcendental Meditation, give the experience of pure consciousness. Pure

consciousness is equated with zero, which is characterized as the *point of infinity* and the *Absolute Number*, the support of all number systems, and thereby all of natural law. Number thereby becomes a natural experience of self-referral available to everyone.

In the author's experience, Maharishi Vedic Mathematics gives both experience and understanding that makes one at home with the infinite. The infinite sums of the numeristic approach to divergent series are an example of the surprising and beautiful nature of the infinite.

APPENDIX: MATHEMATICIANS' OPINIONS ABOUT DIVERGENT SERIES

The theory of divergent series makes startling claims about elementary facts of arithmetic which it often does not prove in a convincing way. It therefore elicits controversy even from well known mathematicians. Here is a sampling.

Gottfried Wilhelm Leibniz (1646–1716)

Porro hoc argumentandi genus, etsi Metaphysicum magis quam Mathematicum videatur, tamen firmum est: et alioqui Canonum Verae Metaphysicae major est usus in Mathesi, in Analysisi, in ipsa Geometria, quam vulgo putatur.

Again, this kind of argument, although it appears more metaphysically magical than mathematical, nevertheless is well founded; and besides, the true canon of the metaphysics of our forefathers is used in mathematics, in analysis, in its geometry, for ordinary reckoning.

Leibnitz, quoted in [\[H49, p. 14\]](#)

Leonhard Euler (1707–1783)

Summa cujusque seriei est valor expressionis illius finitae, ex cujus evolutione illa series oritur.

The sum of every series is the value of an expression which is defined by the process from which that series arises.

Euler, quoted in [\[H49, p. 8\]](#)

Darüber hat er zwar kein Exempel gegeben, ich glaube aber gewiß zu sein, daß nimmer eben dieselbe series aus der Evolution zweier wirklich verschiedenen expressionum finitorum entstehen könne.

He has given no example about it; however I believe it is certain that the same series can never develop from the evolution of two really different finite expressions.

Euler, quoted in [H49, p. 14]

Per rationes metaphysicas . . . quibus in analysi acquiescere queamus.

By metaphysical reasoning . . . which is able to submit to analysis.

Euler, quoted in [H49, p. 14]

Ich glaube, daß jede series einen bestimmten Wert haben müsse. Um aber allen Schwierigkeiten, welche dagegen gemacht worden, zu begegnen, so sollte dieser Wert nicht mit dem Namen der Summe belegt werden, weil man mit diesem Wort gemeiniglich einen solchen Begriff zu verknüpfen pflegt, als wenn die Summe durch eine wirkliche Summierung herausgebracht würde: welche Idee bei den seriebus divergentibus nicht stattfindet.

I believe that every series must have a certain value. However, in order to meet all difficulties which could be made against it, then this value should not be assigned the name of sum, because one tends to commonly link such a term with this word, as if the sum was brought about by a real summation, which idea does not take place with divergent series.

Euler, quoted in [H49, p. 15]

Quemadmodum autem iste dissensus realis videatur, tamen neutra pars ab altera ullius erroris argui potest, quoties in analysi hujusmodi serierum usus occurrit: quod gravi argumento esse debet, neutram partem in errore versari, sed totum dissidium in solis verbis esse positum.

However, whatever that disagreement seems to give rise to, still neither party can prove any error by the other, as often in analysis this kind of series resists being used; because serious evidence must exist, neither party is in error, but everyone disagrees only in words of expression.

Euler, quoted in [H49, p. 15]

Dicamus ergo seriei cuiusque infinitae summam esse expressionem finitam, ex cuius evolutione illa series nascatur. Hocque sensu seriei infinitae $1 + x + x^2 + x^3 + \text{\&c.}$ summa revera erit $= \frac{1}{1-x}$, quia illa series ex huius fractionis evolutione oritur: quicumque numerus loco x substituatur. Hoc pacto, si series fuerit convergens, ista nova vocis summae definitio, cum consueta congruet; & quia divergenes nullas habent summas proprie sic dictas, hinc nullum incommodum ex nova hac appellatione orietur. Denique ope huius definitionis utilitatem serierum divergentium tueri, atque ab omnibus iniuriis vindicare poterimus.

Let us say, therefore, that the *sum* of any infinite series is the finite expression, by the expansion of which the series is generated. In this sense, the sum of the infinite series $1 + x + x^2 + x^3 + \dots$ will be $\frac{1}{1-x}$, because the series arises from the expansion of the fraction, whatever number is put in place of x . If this is agreed, the new definition of the word *sum* coincides with the ordinary meaning when a series converges; and since divergent series have no sum, in the proper sense of the word, no inconvenience can arise from this new terminology. Finally, by means of this definition, we can preserve the utility of divergent series and defend their use from all objections.

Euler, [Eu55, Ch. 3, Sect. 111, p. 78-79] tr. [BL, p. 142]

§. 3. *Ex specie secunda Leibniti primus hanc contemplatus est seriem: $1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + \text{\&c.}$ cuius summas valere $= \frac{1}{2}$ statuerat, his satis firmis rationibus innixus; Primum enim haec series prodit, si fractio haec $\frac{1}{1+a}$ per diuisionem continuas more solito in hanc seriem $1 - a + a^2 - a^3 + a^4 - a^5 + \text{\&c.}$ resoluat, et valor litterae a vnitatem aequalis sumatur.*

3. Of the second type [oscillating series] is this series, $1 - 1 + 1 - 1 + \dots$, first considered by Leibniz, whose sum he gave as equal to $\frac{1}{2}$, with the support of the following fairly sound reasoning: first, this series appears if the fraction $\frac{1}{1+a}$ is expanded in the usual way by continued division into the following series $1 - a + a^2 - a^3 + a^4 - a^5 + \dots$, and the value of the letter a is taken equal to unity.

Euler, [Eu60, §3, p. 207] tr. [BL, p. 145]

Interim tamen veritati consentaneum videtur, si dicamus easdem quantitates, quae sint nihilo minores, simul infinito maiores censi posse. Non solum enim ex algebra, sed etiam ex geometria discimus, duplicem dari saltum a quantitatibus positiuis ad negativas, alterum per cyphram, seu nihilum, alterum per infinitum: atque adeo quantitates a cyphra, tam crescendo, quam decrescendo, in se redire, et ad eundem terminum 0 reuerti; ita vt quantitates infinito maiores eadem perinde sint nihilo minores, ac quantitates infinito minores conueniunt cum quantitatibus nihilo maioribus.

However, it seems in accord with the truth if we say that the same quantities which are less than zero can be considered to be greater than infinity. For not only from algebra but also from geometry, we learn that there are two jumps from positive quantities to negative ones, one through nought or zero, the other through infinity, and that quantities whether increasing from zero or decreasing come back on themselves and return to the same destination 0, so that quantities greater than infinity are thereby less than zero and quantities less than infinity coincide with quantities greater than zero.

Euler, [Eu60, §8, p. 210] tr. [BL, p. 147]

§. 11. *Puto autem, totam hanc litem facile compositum iri, si ad sequentia sedulo attendere velimus. Quoties in analysi ad expressionem vel fractam, vel transcendentem, pertingimus; toties eam in idoneam seriem conuertere solemus, ad quam sequens calculus commodius applicare queat. Eatenus ergo tantum series infinitae in analysi locum inueniunt, quatenus ex euolutione cuiuspiam expressionis finitae sunt ortae; et hanc ob rem in calculo semper loco cuiusque seriei infinitae eam formulam, ex cuius euolutione est nata, substituere licet. Hinc quemadmodum summo cum fructu regulae tradi solent, expressiones finitas, sed forma minus idonea praeditas, in series infinitas conuertendi, ita vicissim vtilissimae sunt censendae regulae, quarum ope, si proposita fuerit series infinita quaecunque, ea expressio finita inuestigari queat, ex qua ea resultet; et cum haec expressio, semper sine errore loco seriei infinitae substitui possit, necesse est, vt vtriusque idem sit valor; ex quo efficitur, nullam dari seriem infinitam, quin simul expressio finita illi aequiualens concipi queat.*

11. But I think all this wrangling can be easily ended if we should carefully attend to what follows. Whenever in analysis we arrive at a rational or transcendental expression, we customarily convert it into a suitable series on which the subsequent calculation can more easily be performed. Therefore infinite series find a place in analysis inasmuch as they arise from the expansion of some closed expression, and accordingly in a calculation it is valid to substitute in place of the infinite series that formula from which the series came. Just as with great profit rules are usually given for converting expressions closed but awkward in form into infinite series, so likewise the rules, by whose help the closed expression, from which a proposed infinite series arises, can be investigated, are to be thought highly useful. Since this expression can always be substituted without error for the infinite series, both must have the same value: it follows that there is no infinite series for which the closed expression equivalent to it cannot be conceived.

Euler, [Eu60, §11, p. 211-212] tr. [BL, p. 148]

Jean le Rond d'Alembert (1717–1783)

Pour moi j'avoue que tous les raisonnements et les calculs fondés sur des séries que ne sont pas convergents . . . me paraîtront toujours très suspects, même quand les résultats de ces raisonnements s'accorderaient avec des vérités connues d'ailleurs.

For my part, I acknowledge that all the reasoning and the calculations based on series that are not convergent . . . will always appear very suspect to me, even when the results of this reasoning would agree with truths known elsewhere.

D'Alembert, quoted in [H49, p. 17]

Joseph-Louis Lagrange (1736–1813)

Les géomètres doivent savoir gré au cit. Jean-Charles Callet d'avoir appelé leur attention sur l'espèce de paradoxe que présentent les séries dont il s'agit, et d'avoir cherché à les prémunir contre l'application des raisonnements métaphysiques aux questions qui, n'étant que de pure analyse, ne peuvent être décidées que par les premiers principes et les règles fondamentales du calcul.

The geometricians must agree with the cited text. Callet has drawn their attention to the species of paradox that present the series as it really is, and to have sought to secure them against the application of metaphysical reasoning on questions which, not being that of pure analysis, can be decided only by first principles and the fundamental rules of calculation.

Lagrange, quoted in [H49, p. 14]

Pierre-Simon Laplace (1749–1827)

Je mets encore au rang des illusions l'application que Leibniz et Dan. Bernoulli ont faite du calcul des probabilités.

I still put at the level of illusion the application that Leibniz and Dan. Bernoulli have made of the theory of probability.

Laplace, quoted in [H49, p. 17]

Siméon Denis Poisson (1781–1840)

Cette série n'est ni convergente ni divergente et ce n'est qu'en la considérant ainsi que nous la faisons comme la limite d'une série convergente, qu'elle peut avoir une valeur déterminée. . . . Nous admettrons avec Euler que les sommes de ces séries considérées en elles-mêmes n'ont pas de valeurs déterminées; mais nous ajouterons que chacune d'elles a une valeur unique et qu'on peut les employer dans l'analyse, lorsqu'on les regarde comme les limites des séries convergentes, c'est à dire quand on suppose implicitement leurs termes successifs multipliés par les puissances d'une fraction infiniment peu différent de l'unité.

This series is neither convergent nor divergent, and it is not that by considering it thus that we make it like the limit of a convergent series, that it can have a given value. . . . We will admit with Euler that the sums of these series considered in themselves do not have definite values; but we will add that each one of them has a single value and that one can employ them in analysis, when one looks upon them like the limits of convergent series, this being said when one implicitly supposes their successive terms multiplied by the powers of a fraction differing infinitesimally from unity.

Poisson, quoted in [H49, p. 17]

Niels Henrik Abel (1802–1829)

The divergent series are the invention of the devil, and it is a shame to base on them any demonstration whatsoever.

Abel, quoted in [G84, p. 170–171]

Augustus De Morgan (1806–1871)

[Divergent series is] the only subject yet remaining, of an elementary character, on which serious schism exists among mathematicians as to absolute correctness or incorrectness of results. . . . The moderns seem to me to have made a similar confusion in regard to their rejection of divergent series; meaning sometimes that they cannot safely be used under existing ideas as to their meaning and origin, sometimes that the mere idea of anyone applying them at all, under any circumstances, is an absurdity. We must admit that many series

are such as we cannot safely use, except as means of discovery, the results of which are to be subsequently verified. . . . But to say that what we cannot use no others ever can . . . seems to me a departure from all rules of prudence.

De Morgan, quoted in [H49, p. 19]

Oliver Heaviside (1850–1925)

I must say a few words on the subject of . . . divergent series. . . . It is not easy to get up any enthusiasm after it has been artificially cooled by the wet blankets of rigorists. . . . I have stated the growth of my views about divergent series. . . . I have avoided defining the meaning of equivalence. The definitions will make themselves in time. . . . My first notion of a series was that to have a finite value it must be convergent. . . . A divergent series also, of course, has an infinite value. Solutions of physical problems must always be in finite terms or convergent series, otherwise nonsense is made. . . . Then came a partial removal of ignorant blindness. In some physical problems divergent series are actually used, notably by Stokes, referring to the divergent formula for the oscillating function $J_n(x)$. He showed that the error was less than the last term included. . . . Equivalence does not mean identity. . . . But the numerical meaning of divergent series still remains obscure. . . . There will have to be a theory of divergent series, or say a larger theory of functions than the present, including convergent and divergent series in one harmonious whole.

Heaviside, quoted in [H49, p. 36]

Ernesto Cesàro (1859–1906)

Lorsque $s(n)$, sans tendre vers une limite, admet une valeur moyenne s finie et déterminée, nous dirons que la série $a(0) + a(1) + a(2) + \dots$ est simplement indéterminée, et nous conviendrons de dire que s est la somme de la série.

When $s(n)$, without tending towards a limit, admits an average value s which is finite and determinate, we will say that the series $a(0) + a(1) + a(2) + \dots$ is simply indeterminate, and we will agree to say that s is the sum of the series.

Cesàro, quoted in [H49, p. 8]

Il résulte de là une classification des séries indéterminées, qui est sans doute incomplète et pas assez naturelle.

This results in a classification of the indeterminate series, which is undoubtedly incomplete and not natural enough.

Cesàro, quoted in [H49, p. 8]

G. H. Hardy (1877–1947)

$[1 + 2 + 4 + 8 + \dots = -1]$ has an air of paradox, since it does not seem natural to attribute a negative sum to a series of positive terms.

Hardy, [H49, p. 10]

-1 and ∞ are the only “natural” sums [of $1 + 2 + 4 + 8 + \dots$].

Hardy, [H49, p. 19]

Would analysis ever have developed as it has done if Euler and others had refused to use $\sqrt{-1}$?

Hardy, [H49, p. 19], in reply to De Morgan quote above

