

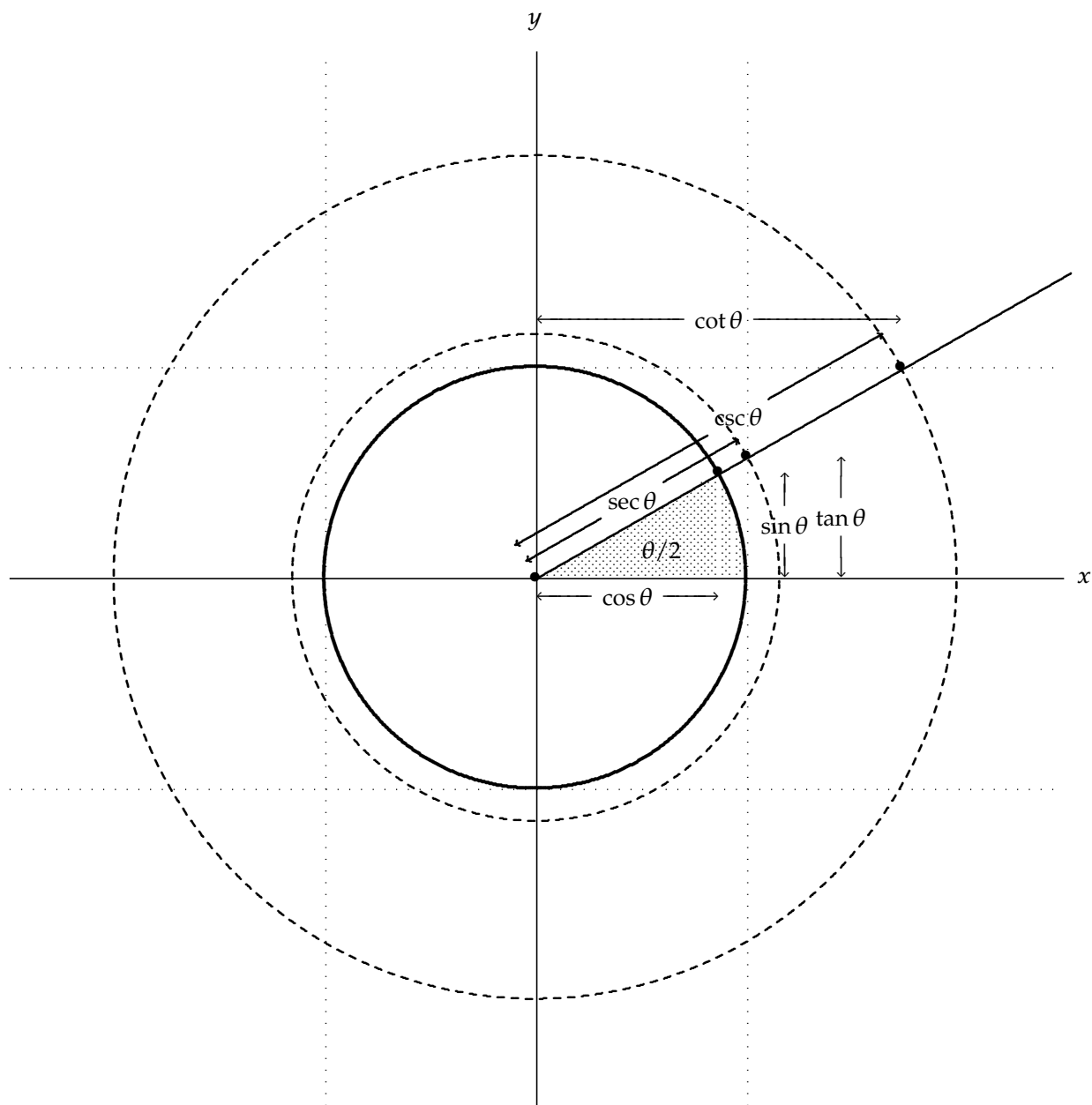
Beautiful Trigonometry

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The unit circle

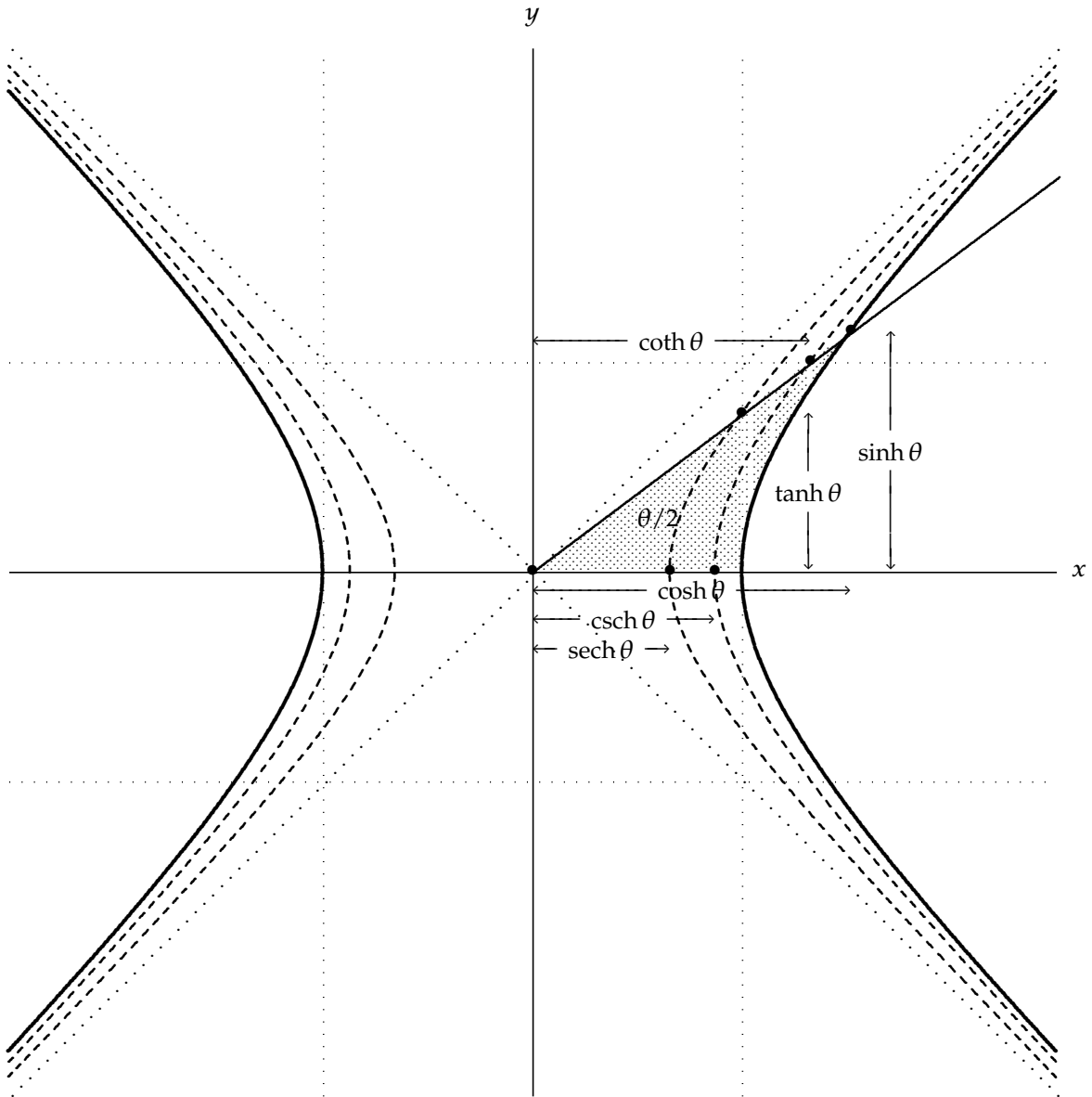


The equation $x^2 + y^2 = 1$ is a circle of radius 1. This circle is called the unit circle.

For any point on the circle, twice the shaded area is the circular angle θ .

The indicated lengths are the circular trigonometric functions of θ .

The unit hyperbola

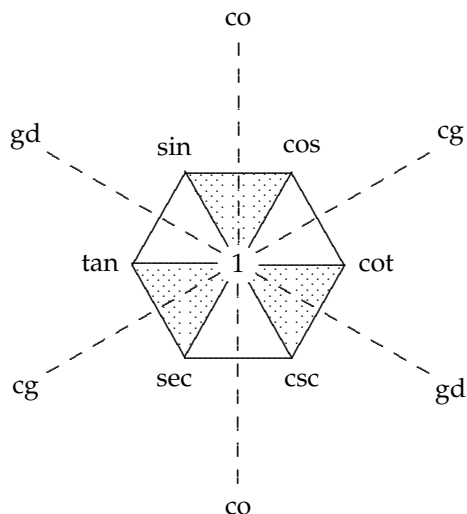


The equation $x^2 - y^2 = 1$ is a hyperbola bounded by a square and asymptotes on the corners of the square. This hyperbola is called the unit hyperbola.

For any point on the hyperbola, twice the shaded area is the hyperbolic angle θ .

The indicated lengths are the hyperbolic trigonometric functions of θ .

The trigonometric hexagon



The trigonometric hexagon relates elementary properties of the six circular and hyperbolic trigonometric functions. The points labelled \sin , \cos , \tan , etc. are valid also for \sinh , \cosh , \tanh , etc. The constant 1 at the center is treated as a function.

Each function is the product of its two neighbors: e.g. $\sin \theta = \tan \theta \cos \theta$ and $\sinh \theta = \tanh \theta \cosh \theta$.

Each function is the quotient of the two functions to the left and also of the two functions to the right: e.g. $\sin \theta = \frac{\cos \theta}{\cot \theta} = \frac{\tan \theta}{\sec \theta}$.

The product of two opposite functions is 1, which means that opposite functions are reciprocals: e.g. $\sin \theta \csc \theta = 1$, or $\sin \theta = \frac{1}{\csc \theta}$.

The three shaded triangles give the Pythagorean identities. For circular functions, the sum of the squares of the top two functions is equal to the square of the bottom function; for hyperbolic functions, the sum of the squares of the left and the bottom functions is equal to the square of the right function. For example, $\sin^2 \theta + \cos^2 \theta = 1$; $\sinh^2 \theta + 1 = \cosh^2 \theta$.

The three dashed axes give the inner transformations. Inner transformations reflect functions from one side of the axis to the other as follows:

The circular functions, when reflected across the co axis, become the circular functions on the other side, e.g. $\sin \text{co } \theta = \cos \theta$.

The hyperbolic functions, when reflected across the cg axis, become the hyperbolic functions on the other side, e.g. $\sinh \text{cg } \theta = \text{csch } \theta$.

Either circular or hyperbolic functions, when reflected across the gd axis, become the function on the other side of the opposite type, i.e. circular becomes hyperbolic, and hyperbolic becomes circular. Examples: $\sinh \text{gd } \theta = \tanh \theta$; $\sin \text{gd}^{-1} \theta = \tan \theta$.

Euler identities

$$e^\theta = \cosh \theta + \sinh \theta = \exp \theta$$
$$e^{-\theta} = \cosh \theta - \sinh \theta = \frac{1}{\exp \theta}$$

$$e^{i\theta} = \cos \theta + i \sin \theta = \operatorname{cis} \theta$$
$$e^{-i\theta} = \cos \theta - i \sin \theta = \frac{1}{\operatorname{cis} \theta}$$

Exponential definitions

$$\cosh \theta = \frac{e^\theta + e^{-\theta}}{2}$$
$$\sinh \theta = \frac{e^\theta - e^{-\theta}}{2}$$
$$\tanh \theta = \frac{e^\theta - e^{-\theta}}{e^\theta + e^{-\theta}}$$
$$\operatorname{coth} \theta = \frac{e^\theta + e^{-\theta}}{e^\theta - e^{-\theta}}$$
$$\operatorname{sech} \theta = \frac{2}{e^\theta + e^{-\theta}}$$
$$\operatorname{csch} \theta = \frac{2}{e^\theta - e^{-\theta}}$$

$$\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$
$$\sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$
$$\tan \theta = -i \frac{e^{i\theta} - e^{-i\theta}}{e^{i\theta} + e^{-i\theta}}$$
$$\cot \theta = i \frac{e^{i\theta} + e^{-i\theta}}{e^{i\theta} - e^{-i\theta}}$$
$$\sec \theta = \frac{2}{e^{i\theta} + e^{-i\theta}}$$
$$\csc \theta = \frac{2i}{e^{i\theta} - e^{-i\theta}}$$

Inverse function log formulae

$$\cosh^{-1} u = \ln \left(u + \sqrt{u^2 - 1} \right)$$
$$\sinh^{-1} u = \ln \left(u + \sqrt{u^2 + 1} \right)$$
$$\tanh^{-1} u = \frac{1}{2} \ln \left(\frac{1+u}{1-u} \right)$$
$$\operatorname{coth}^{-1} u = \frac{1}{2} \ln \left(\frac{u+1}{u-1} \right)$$
$$\operatorname{sech}^{-1} u = \ln \left(\frac{1 + \sqrt{1-u^2}}{u} \right)$$
$$\operatorname{csch}^{-1} u = \ln \left(\frac{1 + \sqrt{1+u^2}}{u} \right)$$
$$\exp^{-1} u = \ln u$$

$$\cos^{-1} u = i \ln \left(u - i\sqrt{1-u^2} \right)$$
$$\sin^{-1} u = i \ln \left(\sqrt{1-u^2} - iu \right)$$
$$\tan^{-1} u = \frac{i}{2} \ln \left(\frac{1-iu}{1+iu} \right)$$
$$\cot^{-1} u = \frac{i}{2} \ln \left(\frac{iu+1}{iu-1} \right)$$
$$\sec^{-1} u = i \ln \left(\frac{1 - i\sqrt{u^2-1}}{u} \right)$$
$$\csc^{-1} u = i \ln \left(\frac{\sqrt{u^2-1} - i}{u} \right)$$
$$\operatorname{cis}^{-1} u = -i \ln u$$

Special values

θ	$-\infty$	0	$+\infty$
$\cosh \theta$	$+\infty$	$+1$	$+\infty$
$\sinh \theta$	$-\infty$	0	$+\infty$
$\tanh \theta$	-1	0	$+1$
$\coth \theta$	-1	$\pm\infty$	$+1$
$\operatorname{sech} \theta$	0	$+1$	0
$\operatorname{csch} \theta$	0	$\pm\infty$	0
$\exp \theta$	-1	$+1$	$+\infty$

θ	0	$\frac{\pi}{2}$	$\frac{\pi}{4}$	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{10}$	$\frac{\pi}{5}$	$\frac{3\pi}{10}$	$\frac{2\pi}{5}$
$\cos \theta$	1	0	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\frac{\sqrt{2+\phi}}{2}$	$\frac{\phi}{2}$	$\frac{\sqrt{3-\phi}}{2}$	$\frac{\phi^{-1}}{2}$
$\sin \theta$	0	1	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\phi^{-1}}{2}$	$\frac{\sqrt{3-\phi}}{2}$	$\frac{\phi}{2}$	$\frac{\sqrt{2+\phi}}{2}$
$\tan \theta$	0	$\pm\infty$	1	$\frac{1}{\sqrt{3}}$	$\sqrt{3}$	$\sqrt{7-4\phi}$	$\sqrt{7+4\phi}$	$\sqrt{\frac{3+4\phi}{5}}$	$\sqrt{3+4\phi}$
$\cot \theta$	$\pm\infty$	0	1	$\sqrt{3}$	$\frac{1}{\sqrt{3}}$	$\sqrt{3+4\phi}$	$\sqrt{\frac{3+4\phi}{5}}$	$\sqrt{7+4\phi}$	$\sqrt{7-4\phi}$
$\sec \theta$	1	$\pm\infty$	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	2	$2\sqrt{3-\phi}$	$2\phi^{-1}$	$2\sqrt{\frac{2+\phi}{5}}$	2ϕ
$\csc \theta$	$\pm\infty$	1	$\sqrt{2}$	2	$\frac{2}{\sqrt{3}}$	2ϕ	$2\sqrt{\frac{2+\phi}{5}}$	$2\phi^{-1}$	$2\sqrt{3-\phi}$

Signs

$\theta \in$	$(-\infty, 0)$	$(0, +\infty)$
$\cosh \theta$	$+$	$+$
$\sinh \theta$	$-$	$+$
$\tanh \theta$	$-$	$+$
$\coth \theta$	$-$	$+$
$\operatorname{sech} \theta$	$+$	$+$
$\operatorname{csch} \theta$	$-$	$+$
$\exp \theta$	$+$	$+$

$\theta \in$	$(0, \frac{\pi}{2})$	$(\frac{\pi}{2}, \pi)$	$(\pi, \frac{3\pi}{2})$	$(\frac{3\pi}{2}, 2\pi)$
$\cos \theta$	$+$	$-$	$-$	$+$
$\sin \theta$	$+$	$+$	$-$	$-$
$\tan \theta$	$+$	$-$	$+$	$-$
$\cot \theta$	$+$	$-$	$+$	$-$
$\sec \theta$	$+$	$-$	$-$	$+$
$\csc \theta$	$+$	$+$	$-$	$-$

Ranges

$\theta \in R \Rightarrow$
$+1 \leq \cosh \theta < +\infty$
$-\infty < \sinh \theta < +\infty$
$-1 < \tanh \theta < +1$
$\coth \theta < -1$ or $+1 < \coth \theta$
$0 < \operatorname{sech} \theta \leq +1$
$-\infty < \operatorname{csch} \theta < +\infty$, but $\operatorname{csch} \theta \neq 0$
$0 < \exp \theta$

$\theta \in R \Rightarrow$
$-1 \leq \cos \theta \leq +1$
$-1 \leq \sin \theta \leq +1$
$-\infty < \tan \theta < +\infty$
$-\infty < \cot \theta < +\infty$
$\sec \theta \leq -1$ or $+1 \leq \sec \theta$
$\csc \theta \leq -1$ or $+1 \leq \csc \theta$
$ \operatorname{cis} \theta = 1$

Product formulae

$$\begin{aligned}
 \cosh \theta &= \sinh \theta \coth \theta \\
 \sinh \theta &= \cosh \theta \tanh \theta \\
 \tanh \theta &= \sinh \theta \operatorname{sech} \theta \\
 \coth \theta &= \cosh \theta \operatorname{csch} \theta \\
 \operatorname{sech} \theta &= \tanh \theta \operatorname{csch} \theta \\
 \operatorname{csch} \theta &= \coth \theta \operatorname{sech} \theta \\
 \cosh \theta \operatorname{sech} \theta &= \sinh \theta \operatorname{csch} \theta \\
 &= \tanh \theta \coth \theta = \exp \theta \exp(-\theta) = 1
 \end{aligned}$$

$$\begin{aligned}
 \cos \theta &= \sin \theta \cot \theta \\
 \sin \theta &= \cos \theta \tan \theta \\
 \tan \theta &= \sin \theta \sec \theta \\
 \cot \theta &= \cos \theta \csc \theta \\
 \sec \theta &= \tan \theta \csc \theta \\
 \csc \theta &= \cot \theta \sec \theta \\
 \cos \theta \sec \theta &= \sin \theta \csc \theta \\
 &= \tan \theta \cot \theta = \operatorname{cis} \theta \operatorname{cis}(-\theta) = 1
 \end{aligned}$$

Ratio formulae

$$\begin{aligned}
 \cosh \theta &= \frac{1}{\operatorname{sech} \theta} = \frac{\sinh \theta}{\tanh \theta} = \frac{\coth \theta}{\operatorname{csch} \theta} \\
 \sinh \theta &= \frac{1}{\operatorname{csch} \theta} = \frac{\cosh \theta}{\coth \theta} = \frac{\operatorname{sech} \theta}{\tanh \theta} \\
 \tanh \theta &= \frac{1}{\coth \theta} = \frac{\sinh \theta}{\cosh \theta} = \frac{\operatorname{sech} \theta}{\operatorname{csch} \theta} \\
 \coth \theta &= \frac{1}{\tanh \theta} = \frac{\cosh \theta}{\sinh \theta} = \frac{\operatorname{csch} \theta}{\operatorname{sech} \theta} \\
 \operatorname{sech} \theta &= \frac{1}{\cosh \theta} = \frac{\sinh \theta}{\tanh \theta} = \frac{\coth \theta}{\operatorname{csch} \theta} \\
 \operatorname{csch} \theta &= \frac{1}{\sinh \theta} = \frac{\cosh \theta}{\coth \theta} = \frac{\operatorname{sech} \theta}{\tanh \theta} \\
 \exp(-\theta) &= \frac{1}{\exp \theta}
 \end{aligned}$$

$$\begin{aligned}
 \cos \theta &= \frac{1}{\sec \theta} = \frac{\sin \theta}{\tan \theta} = \frac{\cot \theta}{\csc \theta} \\
 \sin \theta &= \frac{1}{\csc \theta} = \frac{\cos \theta}{\cot \theta} = \frac{\sec \theta}{\tan \theta} \\
 \tan \theta &= \frac{1}{\cot \theta} = \frac{\sin \theta}{\cos \theta} = \frac{\sec \theta}{\csc \theta} \\
 \cot \theta &= \frac{1}{\tan \theta} = \frac{\cos \theta}{\sin \theta} = \frac{\csc \theta}{\sec \theta} \\
 \sec \theta &= \frac{1}{\cos \theta} = \frac{\sin \theta}{\tan \theta} = \frac{\cot \theta}{\csc \theta} \\
 \csc \theta &= \frac{1}{\sin \theta} = \frac{\cot \theta}{\cos \theta} = \frac{\sec \theta}{\tan \theta} \\
 \operatorname{cis}(-\theta) &= \frac{1}{\operatorname{cis} \theta}
 \end{aligned}$$

Inverse function reciprocal formulae

$$\begin{aligned}
 \cosh^{-1} u &= \operatorname{sech}^{-1} \frac{1}{u} \\
 \sinh^{-1} u &= \operatorname{csch}^{-1} \frac{1}{u} \\
 \tanh^{-1} u &= \coth^{-1} \frac{1}{u} \\
 \coth^{-1} u &= \tanh^{-1} \frac{1}{u} \\
 \operatorname{sech}^{-1} u &= \cosh^{-1} \frac{1}{u} \\
 \operatorname{csch}^{-1} u &= \sinh^{-1} \frac{1}{u} \\
 \exp^{-1} u &= -\exp^{-1} \frac{1}{u}
 \end{aligned}$$

$$\begin{aligned}
 \cos^{-1} u &= \sec^{-1} \frac{1}{u} \\
 \sin^{-1} u &= \csc^{-1} \frac{1}{u} \\
 \tan^{-1} u &= \cot^{-1} \frac{1}{u} \\
 \cot^{-1} u &= \tan^{-1} \frac{1}{u} \\
 \sec^{-1} u &= \cos^{-1} \frac{1}{u} \\
 \csc^{-1} u &= \sin^{-1} \frac{1}{u} \\
 \operatorname{cis}^{-1} u &= -\operatorname{cis}^{-1} \frac{1}{u}
 \end{aligned}$$

Pythagorean identities

$$\begin{aligned}
 1 &= \cosh^2 \theta - \sinh^2 \theta \\
 &= \tanh^2 \theta + \operatorname{sech}^2 \theta = \operatorname{coth}^2 \theta - \operatorname{csch}^2 \theta \\
 \cosh^2 \theta &= \sinh^2 \theta + 1 \\
 \sinh^2 \theta &= \cosh^2 \theta - 1 \\
 \tanh^2 \theta &= 1 - \operatorname{sech}^2 \theta \\
 \operatorname{coth}^2 \theta &= \operatorname{csch}^2 \theta + 1 \\
 \operatorname{sech}^2 \theta &= 1 - \tanh^2 \theta \\
 \operatorname{csch}^2 \theta &= \operatorname{coth}^2 \theta - 1 \\
 (\operatorname{coth} \theta - \tanh \theta)^2 &= \operatorname{sech}^2 \theta - \operatorname{csch}^2 \theta \\
 \operatorname{coth} \theta - \tanh \theta &= \operatorname{sech} \theta \operatorname{csch} \theta \\
 \operatorname{sech}^2 \theta - \operatorname{csch}^2 \theta &= \operatorname{sech}^2 \theta \operatorname{csch}^2 \theta
 \end{aligned}$$

$$\begin{aligned}
 1 &= \cos^2 \theta + \sin^2 \theta \\
 &= \sec^2 \theta - \tan^2 \theta = \operatorname{csc}^2 \theta - \cot^2 \theta \\
 \cos^2 \theta &= 1 - \sin^2 \theta \\
 \sin^2 \theta &= 1 - \cos^2 \theta \\
 \tan^2 \theta &= \sec^2 \theta - 1 \\
 \cot^2 \theta &= \operatorname{csc}^2 \theta - 1 \\
 \sec^2 \theta &= 1 + \tan^2 \theta \\
 \operatorname{csc}^2 \theta &= \cot^2 \theta + 1 \\
 (\cot \theta + \tan \theta)^2 &= \sec^2 \theta + \operatorname{csc}^2 \theta \\
 \cot \theta + \tan \theta &= \sec \theta \operatorname{csc} \theta \\
 \sec^2 \theta + \operatorname{csc}^2 \theta &= \sec^2 \theta \operatorname{csc}^2 \theta
 \end{aligned}$$

Pythagorean conversions

$$\begin{aligned}
 \cosh \theta &= \sqrt{1 + \sinh^2 \theta} = \frac{1}{\sqrt{1 - \tanh^2 \theta}} \\
 &= \frac{\operatorname{coth} \theta}{\sqrt{\operatorname{coth}^2 \theta - 1}} = \frac{1}{\operatorname{csch} \theta} \\
 &= \frac{\exp^2 \theta + 1}{2 \exp \theta} \\
 \sinh \theta &= \sqrt{\cosh^2 \theta - 1} = \frac{\tanh \theta}{\sqrt{1 - \tanh^2 \theta}} \\
 &= \frac{1}{\sqrt{\operatorname{coth}^2 \theta - 1}} = \frac{1}{\operatorname{sech} \theta} \\
 &= \frac{\exp^2 \theta - 1}{2 \exp \theta} \\
 \tanh \theta &= \frac{\sqrt{\cosh^2 \theta - 1}}{\cosh \theta} = \frac{\sinh \theta}{\sqrt{1 + \sinh^2 \theta}} \\
 &= \frac{\sqrt{1 - \operatorname{sech}^2 \theta}}{1} = \frac{1}{\sqrt{\operatorname{csch}^2 \theta + 1}} \\
 &= \frac{\exp^2 \theta - 1}{\exp^2 \theta + 1} \\
 \operatorname{coth} \theta &= \frac{\cosh \theta}{\sqrt{\cosh^2 \theta - 1}} = \frac{\sqrt{1 + \sinh^2 \theta}}{\sinh \theta} \\
 &= \frac{1}{\sqrt{1 - \operatorname{sech}^2 \theta}} = \sqrt{\operatorname{csch}^2 \theta + 1} \\
 &= \frac{\exp^2 \theta + 1}{\exp^2 \theta - 1}
 \end{aligned}$$

$$\begin{aligned}
 \cos \theta &= \sqrt{1 - \sin^2 \theta} = \frac{1}{\sqrt{1 + \tan^2 \theta}} \\
 &= \frac{\cot \theta}{\sqrt{1 + \cot^2 \theta}} = \frac{1}{\operatorname{csc} \theta} \\
 &= \frac{\operatorname{cis}^2 \theta + 1}{2 \operatorname{cis} \theta} \\
 \sin \theta &= \sqrt{1 - \cos^2 \theta} = \frac{\tan \theta}{\sqrt{1 + \tan^2 \theta}} \\
 &= \frac{1}{\sqrt{1 + \cot^2 \theta}} = \frac{1}{\sec \theta} \\
 &= \frac{\operatorname{cis}^2 \theta - 1}{2i \operatorname{cis} \theta} \\
 \tan \theta &= \frac{\sqrt{1 - \cos^2 \theta}}{\cos \theta} = \frac{\sin \theta}{\sqrt{1 - \sin^2 \theta}} \\
 &= \frac{\sqrt{\sec^2 \theta - 1}}{1} = \frac{1}{\sqrt{\operatorname{csc}^2 \theta - 1}} \\
 &= -i \frac{\operatorname{cis}^2 \theta - 1}{\operatorname{cis}^2 \theta + 1} \\
 \cot \theta &= \frac{\cos \theta}{\sqrt{1 - \cos^2 \theta}} = \frac{\sqrt{1 - \sin^2 \theta}}{\sin \theta} \\
 &= \frac{1}{\sqrt{\sec^2 \theta - 1}} = \sqrt{\operatorname{csc}^2 \theta - 1} \\
 &= i \frac{\operatorname{cis}^2 \theta + 1}{\operatorname{cis}^2 \theta - 1}
 \end{aligned}$$

$$\begin{aligned}
\operatorname{sech} \theta &= \frac{1}{\sqrt{1 + \sinh^2 \theta}} = \sqrt{1 - \tanh^2 \theta} \\
&= \frac{\sqrt{\coth^2 \theta - 1}}{\coth \theta} = \frac{\operatorname{csch} \theta}{\sqrt{\operatorname{csch}^2 \theta - 1}} \\
&= \frac{2 \exp \theta}{\exp^2 \theta + 1} \\
\operatorname{csch} \theta &= \frac{1}{\sqrt{\cosh^2 \theta - 1}} = \frac{\sqrt{1 - \tanh^2 \theta}}{\tanh \theta} \\
&= \frac{1}{\sqrt{\coth^2 \theta - 1}} = \frac{\operatorname{sech} \theta}{\sqrt{1 - \operatorname{sech}^2 \theta}} \\
&= \frac{2 \exp \theta}{\exp^2 \theta - 1} \\
\exp \theta &= \cosh \theta + \sqrt{\cosh^2 \theta - 1} \\
&= \sqrt{\sinh^2 \theta + 1} + \sinh \theta \\
&= \frac{1 + \tanh \theta}{\sqrt{1 - \tanh^2 \theta}} = \frac{\coth \theta + 1}{\sqrt{\coth^2 \theta - 1}} \\
&= \frac{1 + \sqrt{1 - \operatorname{sech}^2 \theta}}{\operatorname{sech} \theta} = \frac{\sqrt{\operatorname{csch}^2 \theta - 1} + 1}{\operatorname{csch} \theta}
\end{aligned}$$

$$\begin{aligned}
\sec \theta &= \frac{1}{\sqrt{1 - \sin^2 \theta}} = \sqrt{1 + \tan^2 \theta} \\
&= \frac{\sqrt{1 + \cot^2 \theta}}{\cot \theta} = \frac{\operatorname{csc} \theta}{\sqrt{\operatorname{csc}^2 \theta - 1}} \\
&= \frac{2 \operatorname{cis} \theta}{\operatorname{cis}^2 \theta + 1} \\
\operatorname{csc} \theta &= \frac{1}{\sqrt{1 - \cos^2 \theta}} = \frac{\sqrt{1 + \tan^2 \theta}}{\tan \theta} \\
&= \sqrt{1 + \cot^2 \theta} = \frac{\operatorname{sec} \theta}{\sqrt{\operatorname{sec}^2 \theta - 1}} \\
&= \frac{2i \operatorname{cis} \theta}{\operatorname{cis}^2 \theta - 1} \\
\operatorname{cis} \theta &= \cos \theta + i\sqrt{1 - \cos^2 \theta} \\
&= \sqrt{1 - \sin^2 \theta} + i \sin \theta \\
&= \frac{1 + i \tan \theta}{\sqrt{1 + \tan^2 \theta}} = \frac{\cot \theta + i}{\sqrt{1 + \cot^2 \theta}} \\
&= \frac{1 + i\sqrt{\operatorname{sec}^2 \theta - 1}}{\operatorname{sec} \theta} = \frac{\sqrt{\operatorname{csc}^2 \theta - 1} + i}{\operatorname{csc} \theta}
\end{aligned}$$

Inverse function Pythagorean conversions

$$\begin{aligned}
\cosh^{-1} u &= \sinh^{-1} \sqrt{u^2 - 1} = \tanh^{-1} \frac{\sqrt{u^2 - 1}}{u} \\
&= \coth^{-1} \frac{u}{\sqrt{u^2 - 1}} = \operatorname{csch}^{-1} \frac{1}{\sqrt{u^2 - 1}} \\
&= \exp^{-1} \left(u + \sqrt{u^2 - 1} \right) \\
\sinh^{-1} u &= \cosh^{-1} \sqrt{1 + u^2} = \tanh^{-1} \frac{u}{\sqrt{1 + u^2}} \\
&= \coth^{-1} \frac{\sqrt{1 + u^2}}{u} = \operatorname{sech}^{-1} \frac{1}{\sqrt{1 + u^2}} \\
&= \exp^{-1} \left(u + \sqrt{1 + u^2} \right) \\
\tanh^{-1} u &= \cosh^{-1} \frac{1}{\sqrt{1 - u^2}} = \sinh^{-1} \frac{u}{\sqrt{1 - u^2}} \\
&= \operatorname{sech}^{-1} \sqrt{1 - u^2} = \operatorname{csch}^{-1} \frac{\sqrt{1 - u^2}}{u} \\
&= \exp^{-1} \sqrt{\frac{1 + u}{1 - u}} \\
\coth^{-1} u &= \cosh^{-1} \frac{u}{\sqrt{u^2 - 1}} = \sinh^{-1} \frac{1}{\sqrt{u^2 - 1}} \\
&= \operatorname{sech}^{-1} \frac{\sqrt{u^2 - 1}}{u} = \operatorname{csch}^{-1} \sqrt{u^2 - 1} \\
&= \exp^{-1} \sqrt{\frac{u - 1}{u + 1}}
\end{aligned}$$

$$\begin{aligned}
\cos^{-1} u &= \sin^{-1} \sqrt{1 - u^2} = \tan^{-1} \frac{\sqrt{1 - u^2}}{u} \\
&= \cot^{-1} \frac{u}{\sqrt{1 - u^2}} = \operatorname{csc}^{-1} \frac{1}{\sqrt{1 - u^2}} \\
&= \operatorname{cis}^{-1} \left(u + i\sqrt{1 - u^2} \right) \\
\sin^{-1} u &= \cos^{-1} \sqrt{1 - u^2} = \tan^{-1} \frac{u}{\sqrt{1 - u^2}} \\
&= \cot^{-1} \frac{\sqrt{1 - u^2}}{u} = \operatorname{sec}^{-1} \frac{1}{\sqrt{1 - u^2}} \\
&= \operatorname{cis}^{-1} \left(\sqrt{1 - u^2} + iu \right) \\
\tan^{-1} u &= \cos^{-1} \frac{1}{\sqrt{1 + u^2}} = \sin^{-1} \frac{u}{\sqrt{1 + u^2}} \\
&= \operatorname{sec}^{-1} \sqrt{1 + u^2} = \operatorname{csc}^{-1} \frac{\sqrt{1 + u^2}}{u} \\
&= \operatorname{cis}^{-1} \sqrt{\frac{1 + iu}{1 - iu}} \\
\cot^{-1} u &= \cos^{-1} \frac{u}{\sqrt{1 + u^2}} = \sin^{-1} \frac{1}{\sqrt{1 + u^2}} \\
&= \operatorname{sec}^{-1} \frac{\sqrt{1 + u^2}}{u} = \operatorname{csc}^{-1} \sqrt{1 + u^2} \\
&= \operatorname{cis}^{-1} \sqrt{\frac{iu - 1}{iu + 1}}
\end{aligned}$$

$$\begin{aligned}
\operatorname{sech}^{-1} u &= \sinh^{-1} \frac{\sqrt{1-u^2}}{u} = \tanh^{-1} \sqrt{1-u^2} \\
&= \operatorname{coth}^{-1} \frac{1}{\sqrt{1-u^2}} = \operatorname{csch}^{-1} \frac{u}{\sqrt{1-u^2}} \\
&= \exp^{-1} \frac{1 + \sqrt{u^2-1}}{u} \\
\operatorname{csch}^{-1} u &= \cosh^{-1} \frac{\sqrt{u^2+1}}{u} = \tanh^{-1} \frac{1}{\sqrt{u^2+1}} \\
&= \operatorname{coth}^{-1} \sqrt{u^2+1} = \operatorname{sech}^{-1} \frac{u}{\sqrt{u^2+1}} \\
&= \exp^{-1} \frac{1 + \sqrt{1+u^2}}{u} \\
\exp^{-1} u &= \cosh^{-1} \frac{u^2+1}{2u} = \sinh^{-1} \frac{u^2-1}{2u} \\
&= \tanh^{-1} \frac{u^2-1}{u^2+1} = \operatorname{coth}^{-1} \frac{u^2+1}{u^2-1} \\
&= \operatorname{sech}^{-1} \frac{2u}{u^2+1} = \operatorname{csch}^{-1} \frac{2u}{u^2-1}
\end{aligned}$$

$$\begin{aligned}
\sec^{-1} u &= \sin^{-1} \frac{\sqrt{u^2-1}}{u} = \tan^{-1} \sqrt{u^2-1} \\
&= \cot^{-1} \frac{1}{\sqrt{u^2-1}} = \operatorname{csc}^{-1} \frac{u}{\sqrt{u^2-1}} \\
&= \operatorname{cis}^{-1} \frac{1 + i\sqrt{u^2-1}}{u} \\
\operatorname{csc}^{-1} u &= \cos^{-1} \frac{\sqrt{u^2-1}}{u} = \tan^{-1} \frac{1}{\sqrt{u^2-1}} \\
&= \cot^{-1} \sqrt{u^2-1} = \sec^{-1} \frac{u}{\sqrt{u^2-1}} \\
&= \operatorname{cis}^{-1} \frac{\sqrt{u^2-1} + i}{u} \\
\operatorname{cis}^{-1} u &= \cos^{-1} \frac{u^2+1}{2u} = \sin^{-1} \frac{u^2-1}{2iu} \\
&= \tan^{-1} \frac{u^2-1}{i(u^2+1)} = \cot^{-1} \frac{i(u^2+1)}{u^2-1} \\
&= \sec^{-1} \frac{2u}{u^2+1} = \operatorname{csc}^{-1} \frac{2iu}{u^2-1}
\end{aligned}$$

Imaginary angle formulae

$$\begin{aligned}
\cosh i\theta &= \cos \theta \\
\sinh i\theta &= i \sin \theta \\
\tanh i\theta &= i \tan \theta \\
\operatorname{coth} i\theta &= -i \cot \theta \\
\operatorname{sech} i\theta &= \sec \theta \\
\operatorname{csch} i\theta &= -i \operatorname{csc} \theta \\
\exp i\theta &= \operatorname{cis} \theta \\
\exp \theta &= (\operatorname{cis} \theta)^{-i}
\end{aligned}$$

$$\begin{aligned}
\cos i\theta &= \cosh \theta \\
\sin i\theta &= i \sinh \theta \\
\tan i\theta &= i \tanh \theta \\
\cot i\theta &= -i \operatorname{coth} \theta \\
\sec i\theta &= \operatorname{sech} \theta \\
\operatorname{csc} i\theta &= -i \operatorname{csch} \theta \\
\operatorname{cis} i\theta &= \frac{1}{\exp \theta} \\
\operatorname{cis} \theta &= (\exp \theta)^i
\end{aligned}$$

Inverse function imaginary angle formulae

$$\begin{aligned}
i \cosh^{-1} u &= \cos^{-1} u \\
i \sinh^{-1} u &= \sin^{-1} iu \\
i \tanh^{-1} u &= \tan^{-1} iu \\
i \operatorname{coth}^{-1} u &= -\cot^{-1} iu \\
i \operatorname{sech}^{-1} u &= \sec^{-1} u \\
i \operatorname{csch}^{-1} u &= -\operatorname{csc}^{-1} iu
\end{aligned}$$

$$\begin{aligned}
i \cos^{-1} u &= \cosh^{-1} u \\
i \sin^{-1} u &= \sinh^{-1} iu \\
i \tan^{-1} u &= \tanh^{-1} iu \\
i \cot^{-1} u &= -\operatorname{coth}^{-1} iu \\
i \sec^{-1} u &= \operatorname{sech}^{-1} u \\
i \operatorname{csc}^{-1} u &= -\operatorname{csch}^{-1} iu
\end{aligned}$$

Negative angle formulae

$$\begin{aligned}\cosh(-\theta) &= \cosh \theta \\ \sinh(-\theta) &= -\sinh \theta \\ \tanh(-\theta) &= -\tanh \theta \\ \coth(-\theta) &= -\coth \theta \\ \operatorname{sech}(-\theta) &= \operatorname{sech} \theta \\ \operatorname{csch}(-\theta) &= -\operatorname{csch} \theta \\ \exp(-\theta) &= \frac{1}{\exp \theta}\end{aligned}$$

$$\begin{aligned}\cos(-\theta) &= \cos \theta \\ \sin(-\theta) &= -\sin \theta \\ \tan(-\theta) &= -\tan \theta \\ \cot(-\theta) &= -\cot \theta \\ \sec(-\theta) &= \sec \theta \\ \csc(-\theta) &= -\csc \theta \\ \operatorname{cis}(-\theta) &= \frac{1}{\operatorname{cis} \theta}\end{aligned}$$

Inverse function negative angle formulae

$$\begin{aligned}\cosh^{-1}(-u) &= \pi i - \cosh^{-1} u \\ \sinh^{-1}(-u) &= -\sinh^{-1} u \\ \tanh^{-1}(-u) &= -\tanh^{-1} u \\ \coth^{-1}(-u) &= -\coth^{-1} u \\ \operatorname{sech}^{-1}(-u) &= \pi i - \operatorname{sech}^{-1} u \\ \operatorname{csch}^{-1}(-u) &= -\operatorname{csch}^{-1} u \\ \exp^{-1}(-u) &= \pi i + \exp^{-1} u\end{aligned}$$

$$\begin{aligned}\cos^{-1}(-u) &= \pi - \cos^{-1} u \\ \sin^{-1}(-u) &= -\sin^{-1} u \\ \tan^{-1}(-u) &= -\tan^{-1} u \\ \cot^{-1}(-u) &= -\cot^{-1} u \\ \sec^{-1}(-u) &= \pi - \sec^{-1} u \\ \csc^{-1}(-u) &= -\csc^{-1} u \\ \operatorname{cis}^{-1}(-u) &= \pi + \operatorname{cis}^{-1} u\end{aligned}$$

Double angles

$$\begin{aligned}\cosh 2\theta &= \cosh^2 \theta + \sinh^2 \theta \\ &= 2 \cosh^2 \theta - 1 = 1 + 2 \sinh^2 \theta \\ \sinh 2\theta &= 2 \cosh \theta \sinh \theta \\ &= i \left(1 - (\cosh \theta + i \sinh \theta)^2 \right) \\ \tanh 2\theta &= \frac{2 \tanh \theta}{1 + \tanh^2 \theta} \\ \coth 2\theta &= \frac{\coth^2 \theta + 1}{2 \coth \theta} \\ \operatorname{sech} 2\theta &= \frac{\operatorname{csch}^2 \theta \operatorname{sech}^2 \theta}{\operatorname{csch}^2 \theta + \operatorname{sech}^2 \theta} \\ \operatorname{csch} 2\theta &= \frac{\operatorname{csch} \theta \operatorname{sech} \theta}{2} \\ \exp 2\theta &= \exp^2 \theta\end{aligned}$$

$$\begin{aligned}\cos 2\theta &= \cos^2 \theta - \sin^2 \theta \\ &= 2 \cos^2 \theta - 1 = 1 - 2 \sin^2 \theta \\ \sin 2\theta &= 2 \cos \theta \sin \theta \\ &= (\cos \theta + \sin \theta)^2 - 1 \\ \tan 2\theta &= \frac{2 \tan \theta}{1 - \tan^2 \theta} \\ \cot 2\theta &= \frac{\cot^2 \theta - 1}{2 \cot \theta} \\ \sec 2\theta &= \frac{\csc^2 \theta \sec^2 \theta}{\csc^2 \theta - \sec^2 \theta} \\ \csc 2\theta &= \frac{\csc \theta \sec \theta}{2} \\ \operatorname{cis} 2\theta &= \operatorname{cis}^2 \theta\end{aligned}$$

Half angles

$$\begin{aligned}
 \cosh \frac{\theta}{2} &= \sqrt{\frac{\cosh \theta + 1}{2}} & \cos \frac{\theta}{2} &= \sqrt{\frac{1 + \cos \theta}{2}} \\
 \sinh \frac{\theta}{2} &= \sqrt{\frac{\cosh \theta - 1}{2}} & \sin \frac{\theta}{2} &= \sqrt{\frac{1 - \cos \theta}{2}} \\
 \tanh \frac{\theta}{2} &= \frac{\sinh \theta}{\cosh \theta + 1} = \frac{\cosh \theta - 1}{\sinh \theta} = \coth \theta - \operatorname{csch} \theta & \tan \frac{\theta}{2} &= \frac{\sin \theta}{1 + \cos \theta} = \frac{1 - \cos \theta}{\sin \theta} = \operatorname{csc} \theta - \cot \theta \\
 \coth \frac{\theta}{2} &= \frac{\sinh \theta}{\cosh \theta - 1} = \frac{\cosh \theta + 1}{\sinh \theta} = \coth \theta + \operatorname{csch} \theta & \cot \frac{\theta}{2} &= \frac{\sin \theta}{1 - \cos \theta} = \frac{1 + \cos \theta}{\sin \theta} = \operatorname{csc} \theta + \cot \theta \\
 \operatorname{sech} \frac{\theta}{2} &= \frac{\sqrt{2} \operatorname{sech} \theta}{1 + \operatorname{sech} \theta} & \sec \frac{\theta}{2} &= \frac{\sqrt{2} \sec \theta}{\sec \theta + 1} \\
 \operatorname{csch} \frac{\theta}{2} &= \frac{\sqrt{2} \operatorname{sech} \theta}{1 - \operatorname{sech} \theta} & \operatorname{csc} \frac{\theta}{2} &= \frac{\sqrt{2} \sec \theta}{\sec \theta - 1} \\
 \exp \frac{\theta}{2} &= \sqrt{\exp \theta} & \operatorname{cis} \frac{\theta}{2} &= \sqrt{\operatorname{cis} \theta}
 \end{aligned}$$

Multiple-value identities

$$\begin{aligned}
 \cosh(\theta + 2\pi i) &= \cosh \theta & \cos(\theta + 2\pi) &= \cos \theta \\
 \sinh(\theta + 2\pi i) &= \sinh \theta & \sin(\theta + 2\pi) &= \sin \theta \\
 \tanh(\theta + 2\pi i) &= \tanh \theta & \tan(\theta + 2\pi) &= \tan \theta \\
 \coth(\theta + 2\pi i) &= \coth \theta & \cot(\theta + 2\pi) &= \cot \theta \\
 \operatorname{sech}(\theta + 2\pi i) &= \operatorname{sech} \theta & \sec(\theta + 2\pi) &= \sec \theta \\
 \operatorname{csch}(\theta + 2\pi i) &= \operatorname{csch} \theta & \operatorname{csc}(\theta + 2\pi) &= \operatorname{csc} \theta \\
 \exp(\theta + 2\pi i) &= \exp \theta & \operatorname{cis}(\theta + 2\pi) &= \operatorname{cis} \theta
 \end{aligned}$$

Straight-angle translations

$$\begin{aligned}
 \cosh(\theta + \pi i) &= \cosh(\theta - \pi i) = -\cosh \theta & \cos(\theta + \pi) &= \cos(\theta - \pi) = -\cos \theta \\
 \sinh(\theta + \pi i) &= \sinh(\theta - \pi i) = -\sinh \theta & \sin(\theta + \pi) &= \sin(\theta - \pi) = -\sin \theta \\
 \tanh(\theta + \pi i) &= \tanh(\theta - \pi i) = \tanh \theta & \tan(\theta + \pi) &= \tan(\theta - \pi) = \tan \theta \\
 \coth(\theta + \pi i) &= \coth(\theta - \pi i) = \coth \theta & \cot(\theta + \pi) &= \cot(\theta - \pi) = \cot \theta \\
 \operatorname{sech}(\theta + \pi i) &= \operatorname{sech}(\theta - \pi i) = -\operatorname{sech} \theta & \sec(\theta + \pi) &= \sec(\theta - \pi) = -\sec \theta \\
 \operatorname{csch}(\theta + \pi i) &= \operatorname{csch}(\theta - \pi i) = -\operatorname{csch} \theta & \operatorname{csc}(\theta + \pi) &= \operatorname{csc}(\theta - \pi) = -\operatorname{csc} \theta \\
 \exp(\theta + \pi i) &= \exp(\theta - \pi i) = -\exp \theta & \operatorname{cis}(\theta + \pi) &= \operatorname{cis}(\theta - \pi) = -\operatorname{cis} \theta
 \end{aligned}$$

Straight-angle reflections

$$\begin{aligned}\cosh(\pi i - \theta) &= -\cosh \theta \\ \sinh(\pi i - \theta) &= \sinh \theta \\ \tanh(\pi i - \theta) &= -\tanh \theta \\ \coth(\pi i - \theta) &= -\coth \theta \\ \operatorname{sech}(\pi i - \theta) &= -\operatorname{sech} \theta \\ \operatorname{csch}(\pi i - \theta) &= \operatorname{csch} \theta \\ \exp(\pi i - \theta) &= \frac{-1}{\exp \theta}\end{aligned}$$

$$\begin{aligned}\cos(\pi - \theta) &= -\cos \theta \\ \sin(\pi - \theta) &= \sin \theta \\ \tan(\pi - \theta) &= -\tan \theta \\ \cot(\pi - \theta) &= -\cot \theta \\ \sec(\pi - \theta) &= -\sec \theta \\ \csc(\pi - \theta) &= \csc \theta \\ \operatorname{cis}(\pi - \theta) &= \frac{-1}{\operatorname{cis} \theta}\end{aligned}$$

Right-angle translations

$$\begin{aligned}\cosh\left(\theta + \frac{\pi i}{2}\right) &= i \sinh \theta \\ \sinh\left(\theta + \frac{\pi i}{2}\right) &= i \cosh \theta \\ \tanh\left(\theta + \frac{\pi i}{2}\right) &= \coth \theta \\ \coth\left(\theta + \frac{\pi i}{2}\right) &= \tanh \theta \\ \operatorname{sech}\left(\theta + \frac{\pi i}{2}\right) &= -i \operatorname{csch} \theta \\ \operatorname{csch}\left(\theta + \frac{\pi i}{2}\right) &= -i \operatorname{sech} \theta \\ \exp\left(\theta + \frac{\pi i}{2}\right) &= i \exp \theta\end{aligned}$$

$$\begin{aligned}\cos\left(\theta + \frac{\pi}{2}\right) &= -\sin \theta \\ \sin\left(\theta + \frac{\pi}{2}\right) &= \cos \theta \\ \tan\left(\theta + \frac{\pi}{2}\right) &= -\cot \theta \\ \cot\left(\theta + \frac{\pi}{2}\right) &= -\tan \theta \\ \sec\left(\theta + \frac{\pi}{2}\right) &= -\csc \theta \\ \csc\left(\theta + \frac{\pi}{2}\right) &= \sec \theta \\ \operatorname{cis}\left(\theta + \frac{\pi}{2}\right) &= i \operatorname{cis} \theta\end{aligned}$$

$$\begin{aligned}\cosh\left(\theta - \frac{\pi i}{2}\right) &= -i \sinh \theta \\ \sinh\left(\theta - \frac{\pi i}{2}\right) &= -i \cosh \theta \\ \tanh\left(\theta - \frac{\pi i}{2}\right) &= \coth \theta \\ \coth\left(\theta - \frac{\pi i}{2}\right) &= \tanh \theta \\ \operatorname{sech}\left(\theta - \frac{\pi i}{2}\right) &= i \operatorname{csch} \theta \\ \operatorname{csch}\left(\theta - \frac{\pi i}{2}\right) &= i \operatorname{sech} \theta \\ \exp\left(\theta - \frac{\pi i}{2}\right) &= i \exp \theta\end{aligned}$$

$$\begin{aligned}\cos\left(\theta - \frac{\pi}{2}\right) &= \sin \theta \\ \sin\left(\theta - \frac{\pi}{2}\right) &= -\cos \theta \\ \tan\left(\theta - \frac{\pi}{2}\right) &= -\cot \theta \\ \cot\left(\theta - \frac{\pi}{2}\right) &= -\tan \theta \\ \sec\left(\theta - \frac{\pi}{2}\right) &= \csc \theta \\ \csc\left(\theta - \frac{\pi}{2}\right) &= -\sec \theta \\ \operatorname{cis}\left(\theta - \frac{\pi}{2}\right) &= i \operatorname{cis} \theta\end{aligned}$$

Right-angle reflections

$$\begin{aligned}\cosh\left(\frac{\pi i}{2} - \theta\right) &= -i \sinh \theta \\ \sinh\left(\frac{\pi i}{2} - \theta\right) &= i \cosh \theta \\ \tanh\left(\frac{\pi i}{2} - \theta\right) &= -\coth \theta \\ \coth\left(\frac{\pi i}{2} - \theta\right) &= -\tanh \theta \\ \operatorname{sech}\left(\frac{\pi i}{2} - \theta\right) &= i \operatorname{csch} \theta \\ \operatorname{csch}\left(\frac{\pi i}{2} - \theta\right) &= -i \operatorname{sech} \theta \\ \exp\left(\frac{\pi i}{2} - \theta\right) &= \frac{i}{\exp \theta}\end{aligned}$$

$$\begin{aligned}\cos\left(\frac{\pi}{2} - \theta\right) &= \sin \theta \\ \sin\left(\frac{\pi}{2} - \theta\right) &= \cos \theta \\ \tan\left(\frac{\pi}{2} - \theta\right) &= \cot \theta \\ \cot\left(\frac{\pi}{2} - \theta\right) &= \tan \theta \\ \sec\left(\frac{\pi}{2} - \theta\right) &= \csc \theta \\ \csc\left(\frac{\pi}{2} - \theta\right) &= \sec \theta \\ \operatorname{cis}\left(\frac{\pi}{2} - \theta\right) &= \frac{i}{\operatorname{cis} \theta}\end{aligned}$$

Half-right-angle translations

$$\begin{aligned}\cosh\left(\theta + \frac{\pi i}{4}\right) &= -i \sinh\left(\frac{\pi i}{4} - \theta\right) \\ &= \frac{\cosh\theta + i \sinh\theta}{\sqrt{2}}\end{aligned}$$

$$\begin{aligned}\sinh\left(\theta + \frac{\pi i}{4}\right) &= -i \cosh\left(\frac{\pi i}{4} - \theta\right) \\ &= \frac{i \cosh\theta + \sinh\theta}{\sqrt{2}}\end{aligned}$$

$$\begin{aligned}\tanh\left(\theta + \frac{\pi i}{4}\right) &= \coth\left(\frac{\pi i}{4} - \theta\right) \\ &= \frac{i + \tanh\theta}{1 + i \tanh\theta}\end{aligned}$$

$$\begin{aligned}\coth\left(\theta + \frac{\pi i}{4}\right) &= \tanh\left(\frac{\pi i}{4} - \theta\right) \\ &= \frac{\coth\theta + i}{i \coth\theta + 1}\end{aligned}$$

$$\begin{aligned}\operatorname{sech}\left(\theta + \frac{\pi i}{4}\right) &= i \operatorname{csch}\left(\frac{\pi i}{4} - \theta\right) \\ &= \frac{2\sqrt{2} \operatorname{csch} 2\theta}{\operatorname{csch}\theta + i \operatorname{sech}\theta}\end{aligned}$$

$$\begin{aligned}\operatorname{csch}\left(\theta + \frac{\pi i}{4}\right) &= i \operatorname{sech}\left(\frac{\pi i}{4} - \theta\right) \\ &= \frac{2\sqrt{2} \operatorname{csch} 2\theta}{i \operatorname{csch}\theta + \operatorname{sech}\theta}\end{aligned}$$

$$\begin{aligned}\exp\left(\theta + \frac{\pi i}{4}\right) &= \frac{1}{\exp\left(\frac{\pi i}{4} - \theta\right)} \\ &= \frac{(1 + i) \exp\theta}{\sqrt{2}}\end{aligned}$$

$$\begin{aligned}\cos\left(\theta + \frac{\pi}{4}\right) &= \sin\left(\frac{\pi}{4} - \theta\right) \\ &= \frac{\cos\theta - \sin\theta}{\sqrt{2}}\end{aligned}$$

$$\begin{aligned}\sin\left(\theta + \frac{\pi}{4}\right) &= \cos\left(\frac{\pi}{4} - \theta\right) \\ &= \frac{\cos\theta + \sin\theta}{\sqrt{2}}\end{aligned}$$

$$\begin{aligned}\tan\left(\theta + \frac{\pi}{4}\right) &= \cot\left(\frac{\pi}{4} - \theta\right) \\ &= \frac{1 + \tan\theta}{1 - \tan\theta}\end{aligned}$$

$$\begin{aligned}\cot\left(\theta + \frac{\pi}{4}\right) &= \tan\left(\frac{\pi}{4} - \theta\right) \\ &= \frac{\cot\theta - 1}{\cot\theta + 1}\end{aligned}$$

$$\begin{aligned}\sec\left(\theta + \frac{\pi}{4}\right) &= \csc\left(\frac{\pi}{4} - \theta\right) \\ &= \frac{2\sqrt{2} \csc 2\theta}{\csc\theta - \sec\theta}\end{aligned}$$

$$\begin{aligned}\csc\left(\theta + \frac{\pi}{4}\right) &= \sec\left(\frac{\pi}{4} - \theta\right) \\ &= \frac{2\sqrt{2} \csc 2\theta}{\csc\theta + \sec\theta}\end{aligned}$$

$$\begin{aligned}\operatorname{cis}\left(\theta + \frac{\pi}{4}\right) &= \frac{i}{\operatorname{cis}\left(\frac{\pi}{4} - \theta\right)} \\ &= \frac{(1 + i) \operatorname{cis}\theta}{\sqrt{2}}\end{aligned}$$

Inverse function multiple-value identities

$$\begin{aligned}\cosh^{-1} u &= 2\pi i + i \cosh^{-1} u \\ \sinh^{-1} u &= 2\pi i + i \sinh^{-1} u \\ \tanh^{-1} u &= \pi i + i \tanh^{-1} u \\ \coth^{-1} u &= \pi i + i \coth^{-1} u \\ \operatorname{sech}^{-1} u &= 2\pi i + i \operatorname{sech}^{-1} u \\ \operatorname{csch}^{-1} u &= 2\pi i + i \operatorname{csch}^{-1} u\end{aligned}$$

$$\begin{aligned}\cos^{-1} u &= 2\pi + \cos^{-1} u \\ \sin^{-1} u &= 2\pi + \sin^{-1} u \\ \tan^{-1} u &= \pi + \tan^{-1} u \\ \cot^{-1} u &= \pi + \cot^{-1} u \\ \sec^{-1} u &= 2\pi + \sec^{-1} u \\ \csc^{-1} u &= 2\pi + \csc^{-1} u\end{aligned}$$

Inverse function straight-angle identities

$$\begin{aligned}\cosh^{-1} u &= \pi i - i \cosh^{-1} u \\ \sinh^{-1} u &= \pi i - i \sinh^{-1} u \\ \tanh^{-1} u &= \pi i + i \tanh^{-1} u \\ \coth^{-1} u &= \pi i + i \coth^{-1} u \\ \operatorname{sech}^{-1} u &= \pi i - i \operatorname{sech}^{-1} u \\ \operatorname{csch}^{-1} u &= \pi i - i \operatorname{csch}^{-1} u\end{aligned}$$

$$\begin{aligned}\cos^{-1} u &= \pi - \cos^{-1} u \\ \sin^{-1} u &= \pi - \sin^{-1} u \\ \tan^{-1} u &= \pi + \tan^{-1} u \\ \cot^{-1} u &= \pi + \cot^{-1} u \\ \sec^{-1} u &= \pi - \sec^{-1} u \\ \csc^{-1} u &= \pi - \csc^{-1} u\end{aligned}$$

Inverse function right-angle identities

$$\begin{aligned}\cosh^{-1} u &= \frac{\pi i}{2} - i \sinh^{-1} u \\ \sinh^{-1} u &= \frac{\pi i}{2} + i \cosh^{-1} u \\ \tanh^{-1} u &= \frac{\pi i}{2} + \coth^{-1} u \\ \coth^{-1} u &= \frac{\pi i}{2} + \tanh^{-1} u \\ \operatorname{sech}^{-1} u &= \frac{\pi i}{2} - i \operatorname{csch}^{-1} u \\ \operatorname{csch}^{-1} u &= \frac{\pi i}{2} - i \operatorname{sech}^{-1} u\end{aligned}$$

$$\begin{aligned}\cos^{-1} u &= \frac{\pi}{2} - \sin^{-1} u \\ \sin^{-1} u &= \frac{\pi}{2} + \cos^{-1} u \\ \tan^{-1} u &= \frac{\pi}{2} - \cot^{-1} u \\ \cot^{-1} u &= \frac{\pi}{2} - \tan^{-1} u \\ \sec^{-1} u &= \frac{\pi}{2} - \csc^{-1} u \\ \csc^{-1} u &= \frac{\pi}{2} + \sec^{-1} u\end{aligned}$$

Angle sums and differences

$$\begin{aligned}\cosh(a+b) &= \cosh a \cosh b + \sinh a \sinh b \\ \cosh(a-b) &= \cosh a \cosh b - \sinh a \sinh b \\ \sinh(a+b) &= \sinh a \cosh b + \cosh a \sinh b \\ \sinh(a-b) &= \sinh a \cosh b - \cosh a \sinh b \\ \tanh(a+b) &= \frac{\tanh a + \tanh b}{1 + \tanh a \tanh b} \\ \tanh(a-b) &= \frac{\tanh a - \tanh b}{1 - \tanh a \tanh b} \\ \coth(a+b) &= \frac{\coth a \coth b + 1}{\coth a + \coth b} \\ \coth(a-b) &= \frac{\coth a \coth b - 1}{\coth a - \coth b} \\ \operatorname{sech}(a+b) &= \frac{\operatorname{csch} a \operatorname{csch} b \operatorname{sech} a \operatorname{sech} b}{\operatorname{csch} a \operatorname{csch} b + \operatorname{sech} a \operatorname{sech} b} \\ \operatorname{sech}(a-b) &= \frac{\operatorname{csch} a \operatorname{csch} b \operatorname{sech} a \operatorname{sech} b}{\operatorname{csch} a \operatorname{csch} b - \operatorname{sech} a \operatorname{sech} b} \\ \operatorname{csch}(a+b) &= \frac{\operatorname{sech} a \operatorname{csch} b \operatorname{csch} a \operatorname{sech} b}{\operatorname{sech} a \operatorname{csch} b + \operatorname{csch} a \operatorname{sech} b} \\ \operatorname{csch}(a-b) &= \frac{\operatorname{sech} a \operatorname{csch} b \operatorname{csch} a \operatorname{sech} b}{\operatorname{sech} a \operatorname{csch} b - \operatorname{csch} a \operatorname{sech} b} \\ \exp(a+b) &= \exp a \exp b \\ \exp(a-b) &= \frac{\exp a}{\exp b}\end{aligned}$$

$$\begin{aligned}\cos(a+b) &= \cos a \cos b - \sin a \sin b \\ \cos(a-b) &= \cos a \cos b + \sin a \sin b \\ \sin(a+b) &= \sin a \cos b + \cos a \sin b \\ \sin(a-b) &= \sin a \cos b - \cos a \sin b \\ \tan(a+b) &= \frac{\tan a + \tan b}{1 - \tan a \tan b} \\ \tan(a-b) &= \frac{\tan a - \tan b}{1 + \tan a \tan b} \\ \cot(a+b) &= \frac{\cot a \cot b - 1}{\cot a + \cot b} \\ \cot(a-b) &= \frac{\cot a \cot b + 1}{\cot a - \cot b} \\ \sec(a+b) &= \frac{\csc a \csc b \sec a \sec b}{\csc a \csc b - \sec a \sec b} \\ \sec(a-b) &= \frac{\csc a \csc b \sec a \sec b}{\csc a \csc b + \sec a \sec b} \\ \csc(a+b) &= \frac{\sec a \csc b \csc a \sec b}{\sec a \csc b + \csc a \sec b} \\ \csc(a-b) &= \frac{\sec a \csc b \csc a \sec b}{\sec a \csc b - \csc a \sec b} \\ \operatorname{cis}(a+b) &= \operatorname{cis} a \operatorname{cis} b \\ \operatorname{cis}(a-b) &= \frac{\operatorname{cis} a}{\operatorname{cis} b}\end{aligned}$$

Function sums and differences

$$\cosh a + \cosh b = 2 \cosh \frac{a+b}{2} \cosh \frac{a-b}{2}$$

$$\cosh a - \cosh b = 2 \sinh \frac{a+b}{2} \sinh \frac{a-b}{2}$$

$$\sinh a + \sinh b = 2 \sinh \frac{a+b}{2} \cosh \frac{a-b}{2}$$

$$\sinh a - \sinh b = 2 \cosh \frac{a+b}{2} \sinh \frac{a-b}{2}$$

$$\tanh a + \tanh b = \sinh(a+b) \operatorname{sech} a \operatorname{sech} b$$

$$\tanh a - \tanh b = \sinh(a-b) \operatorname{sech} a \operatorname{sech} b$$

$$\operatorname{coth} a + \operatorname{coth} b = \sinh(a+b) \operatorname{csch} a \operatorname{csch} b$$

$$\operatorname{coth} a - \operatorname{coth} b = \sinh(b-a) \operatorname{csch} a \operatorname{csch} b$$

$$\operatorname{sech} a + \operatorname{sech} b = 2 \cosh \frac{a+b}{2} \cosh \frac{a-b}{2} \operatorname{sech} a \operatorname{sech} b$$

$$\operatorname{sech} a - \operatorname{sech} b = 2 \sinh \frac{b+a}{2} \sinh \frac{b-a}{2} \operatorname{sech} a \operatorname{sech} b$$

$$\operatorname{csch} a + \operatorname{csch} b = 2 \sinh \frac{a+b}{2} \cosh \frac{a-b}{2} \operatorname{csch} a \operatorname{csch} b$$

$$\operatorname{csch} a - \operatorname{csch} b = 2 \cosh \frac{b+a}{2} \sinh \frac{b-a}{2} \operatorname{csch} a \operatorname{csch} b$$

$$\exp a + \exp b = 2 \cosh \frac{a+b}{2} \exp \frac{a-b}{2}$$

$$\exp a - \exp b = 2 \sinh \frac{a-b}{2} \exp \frac{a+b}{2}$$

$$\begin{aligned} \cosh a + i \sinh b &= 2 \sinh \left(\frac{a-b}{2} - \frac{\pi i}{4} \right) \sinh \left(\frac{a+b}{2} + \frac{\pi i}{4} \right) \end{aligned}$$

$$\begin{aligned} \cosh a - i \sinh b &= 2 \sinh \left(\frac{a+b}{2} - \frac{\pi i}{4} \right) \sinh \left(\frac{a-b}{2} + \frac{\pi i}{4} \right) \end{aligned}$$

$$\operatorname{coth} a + \tanh b = \cosh(a+b) \operatorname{csch} a \operatorname{sech} b$$

$$\operatorname{coth} a - \tanh b = \cosh(a-b) \operatorname{csch} a \operatorname{sech} b$$

$$\begin{aligned} \operatorname{csch} a + i \operatorname{sech} b &= 2 \operatorname{csch} a \operatorname{sech} b \times \\ &\quad \sinh \left(\frac{b-a}{2} - \frac{\pi i}{4} \right) \sinh \left(\frac{b+a}{2} + \frac{\pi i}{4} \right) \end{aligned}$$

$$\begin{aligned} \operatorname{csch} a - i \operatorname{sech} b &= 2 \operatorname{csch} a \operatorname{sech} b \times \\ &\quad \sinh \left(\frac{b+a}{2} - \frac{\pi i}{4} \right) \sinh \left(\frac{b-a}{2} + \frac{\pi i}{4} \right) \end{aligned}$$

$$\frac{\sinh a + \sinh b}{\cosh a + \cosh b} = \frac{\cosh a - \cosh b}{\sinh a - \sinh b} = \tanh \frac{a+b}{2}$$

$$\frac{\sinh a - \sinh b}{\cosh a - \cosh b} = \frac{\cosh a + \cosh b}{\sinh a + \sinh b} = \tanh \frac{a-b}{2}$$

$$\frac{\cosh a + \cosh b}{\tanh a + \tanh b} = \frac{\sinh a + \sinh b}{\operatorname{coth} a + \operatorname{coth} b}$$

$$\frac{\tanh a - \tanh b}{\operatorname{coth} b - \operatorname{coth} a} = \sinh(a+b) \operatorname{csch}(a-b)$$

$$\cos a + \cos b = 2 \cos \frac{a+b}{2} \cos \frac{a-b}{2}$$

$$\cos a - \cos b = 2 \sin \frac{b+a}{2} \sin \frac{b-a}{2}$$

$$\sin a + \sin b = 2 \sin \frac{a+b}{2} \cos \frac{a-b}{2}$$

$$\sin a - \sin b = 2 \cos \frac{a+b}{2} \sin \frac{a-b}{2}$$

$$\tan a + \tan b = \sin(a+b) \sec a \sec b$$

$$\tan a - \tan b = \sin(a-b) \sec a \sec b$$

$$\cot a + \cot b = \sin(a+b) \csc a \csc b$$

$$\cot a - \cot b = \sin(b-a) \csc a \csc b$$

$$\sec a + \sec b = 2 \cos \frac{a+b}{2} \cos \frac{a-b}{2} \sec a \sec b$$

$$\sec a - \sec b = 2 \sin \frac{a+b}{2} \sin \frac{a-b}{2} \sec a \sec b$$

$$\csc a + \csc b = 2 \sin \frac{a+b}{2} \cos \frac{a-b}{2} \csc a \csc b$$

$$\csc a - \csc b = 2 \cos \frac{b+a}{2} \sin \frac{b-a}{2} \csc a \csc b$$

$$\operatorname{cis} a + \operatorname{cis} b = 2 \cos \frac{a-b}{2} \operatorname{cis} \frac{a+b}{2}$$

$$\operatorname{cis} a - \operatorname{cis} b = 2i \sin \frac{b-a}{2} \operatorname{cis} \frac{a+b}{2}$$

$$\begin{aligned} \cos a + \sin b &= 2 \sin \left(\frac{\pi}{4} - \frac{a-b}{2} \right) \sin \left(\frac{\pi}{4} + \frac{a+b}{2} \right) \end{aligned}$$

$$\begin{aligned} \cos a - \sin b &= 2 \sin \left(\frac{\pi}{4} - \frac{a+b}{2} \right) \sin \left(\frac{\pi}{4} + \frac{a-b}{2} \right) \end{aligned}$$

$$\cot a + \tan b = \cos(a-b) \csc a \sec b$$

$$\cot a - \tan b = \cos(a+b) \csc a \sec b$$

$$\begin{aligned} \csc a + \sec b &= 2 \csc a \sec b \times \\ &\quad \sin \left(\frac{\pi}{4} - \frac{b-a}{2} \right) \sin \left(\frac{\pi}{4} + \frac{b+a}{2} \right) \end{aligned}$$

$$\begin{aligned} \csc a - \sec b &= 2 \csc a \sec b \times \\ &\quad \sin \left(\frac{\pi}{4} - \frac{b+a}{2} \right) \sin \left(\frac{\pi}{4} + \frac{b-a}{2} \right) \end{aligned}$$

$$\frac{\sin a + \sin b}{\cos a + \cos b} = \frac{\cos b - \cos a}{\sin a - \sin b} = \tan \frac{a+b}{2}$$

$$\frac{\sin a - \sin b}{\cos a - \cos b} = \frac{\cos b + \cos a}{\sin a + \sin b} = \tan \frac{a-b}{2}$$

$$\frac{\cos a + \cos b}{\tan a + \tan b} = \frac{\sin a + \sin b}{\cot b + \cot a}$$

$$\frac{\tan a - \tan b}{\cot b - \cot a} = \sin(a+b) \csc(a-b)$$

$$\begin{aligned}\frac{\operatorname{sech} a - \operatorname{sech} b}{\operatorname{sech} a + \operatorname{sech} b} &= \tanh \frac{a+b}{2} \tanh \frac{b-a}{2} \\ \frac{\operatorname{csch} a + \operatorname{csch} b}{\operatorname{csch} a - \operatorname{csch} b} &= \tanh \frac{a+b}{2} \coth \frac{b-a}{2} \\ \frac{\exp a - \exp b}{\exp a + \exp b} &= 2 \tanh \frac{a-b}{2}\end{aligned}$$

$$\begin{aligned}\frac{\sec a - \sec b}{\sec a + \sec b} &= \tan \frac{a+b}{2} \tan \frac{a-b}{2} \\ \frac{\csc a + \csc b}{\csc a - \csc b} &= \tan \frac{a+b}{2} \cot \frac{b-a}{2} \\ \frac{\operatorname{cis} a - \operatorname{cis} b}{\operatorname{cis} a + \operatorname{cis} b} &= 2i \tan \frac{b-a}{2}\end{aligned}$$

Function products

$$\begin{aligned}\cosh a \cosh b &= \frac{\cosh(a+b) + \cosh(a-b)}{2} \\ \sinh a \cosh b &= \frac{\sinh(a+b) + \sinh(a-b)}{2} \\ \sinh a \sinh b &= \frac{\cosh(a+b) - \cosh(a-b)}{2} \\ \tanh a \tanh b &= \coth(a+b)(\tanh a + \tanh b) - 1 \\ \tanh a \coth b &= \tanh(a+b)(\coth a + \tanh b) - 1 \\ \coth a \coth b &= \coth(a+b)(\coth a + \coth b) - 1 \\ \operatorname{sech} a \operatorname{sech} b &= \operatorname{sech}(a+b)(1 + \tanh a \tanh b) \\ &= \operatorname{csch}(a+b)(\tanh a + \tanh b) \\ \operatorname{sech} a \operatorname{csch} b &= \operatorname{sech}(a+b)(\coth b + \tanh a) \\ &= \operatorname{csch}(a+b)(\tanh a \coth b + 1) \\ \operatorname{csch} a \operatorname{csch} b &= \operatorname{sech}(a+b)(\coth a \coth b - 1) \\ &= \operatorname{csch}(a+b)(\coth a + \coth b) \\ \exp a \exp b &= \exp(a+b)\end{aligned}$$

$$\begin{aligned}\cos a \cos b &= \frac{\cos(a-b) + \cos(a+b)}{2} \\ \sin a \cos b &= \frac{\sin(a-b) + \sin(a+b)}{2} \\ \sin a \sin b &= \frac{\cos(a-b) - \cos(a+b)}{2} \\ \tan a \tan b &= 1 - \cot(a+b)(\tan a + \tan b) \\ \tan a \cot b &= \tan(a+b)(\cot b - \tan a) - 1 \\ \cot a \cot b &= \cot(a+b)(\cot a + \cot b) + 1 \\ \sec a \sec b &= \sec(a+b)(1 - \tan a \tan b) \\ &= \operatorname{csch}(a+b)(\tan a + \tan b) \\ \sec a \csc b &= \sec(a+b)(\cot b - \tan a) \\ &= \operatorname{csch}(a+b)(\tan a \cot b + 1) \\ \csc a \csc b &= \sec(a+b)(\cot a \cot b - 1) \\ &= \operatorname{csch}(a+b)(\cot a + \cot b) \\ \operatorname{cis} a \operatorname{cis} b &= \operatorname{cis}(a+b)\end{aligned}$$

Inverse function sums and differences

$$\begin{aligned}\cosh^{-1} u + \cosh^{-1} v &= \cosh^{-1} \left(uv + \sqrt{u^2 - 1} \sqrt{v^2 - 1} \right) \\ &= \sinh^{-1} \left(u \sqrt{v^2 - 1} + v \sqrt{u^2 - 1} \right) \\ \cosh^{-1} u - \cosh^{-1} v &= \cosh^{-1} \left(uv - \sqrt{u^2 - 1} \sqrt{v^2 - 1} \right) \\ &= \sinh^{-1} \left(v \sqrt{u^2 - 1} - u \sqrt{v^2 - 1} \right) \\ \sinh^{-1} u + \sinh^{-1} v &= \sinh^{-1} \left(u \sqrt{v^2 + 1} + v \sqrt{u^2 + 1} \right) \\ &= \cosh^{-1} \left(\sqrt{u^2 + 1} \sqrt{v^2 + 1} + uv \right) \\ \sinh^{-1} u - \sinh^{-1} v &= \sinh^{-1} \left(u \sqrt{v^2 + 1} - v \sqrt{u^2 + 1} \right) \\ &= \cosh^{-1} \left(\sqrt{u^2 + 1} \sqrt{v^2 + 1} - uv \right) \\ \tanh^{-1} u + \tanh^{-1} v &= \tanh^{-1} \frac{u+v}{1+uv} \\ \tanh^{-1} u - \tanh^{-1} v &= \tanh^{-1} \frac{u-v}{1-uv}\end{aligned}$$

$$\begin{aligned}\cos^{-1} u + \cos^{-1} v &= \cos^{-1} \left(uv - \sqrt{1-u^2} \sqrt{1-v^2} \right) \\ &= \sin^{-1} \left(u \sqrt{1-v^2} + v \sqrt{1-u^2} \right) \\ \cos^{-1} u - \cos^{-1} v &= \cos^{-1} \left(uv + \sqrt{1-u^2} \sqrt{1-v^2} \right) \\ &= \sin^{-1} \left(v \sqrt{1-u^2} - u \sqrt{1-v^2} \right) \\ \sin^{-1} u + \sin^{-1} v &= \sin^{-1} \left(u \sqrt{1-v^2} + v \sqrt{1-u^2} \right) \\ &= \cos^{-1} \left(\sqrt{1-u^2} \sqrt{1-v^2} - uv \right) \\ \sin^{-1} u - \sin^{-1} v &= \sin^{-1} \left(u \sqrt{1-v^2} - v \sqrt{1-u^2} \right) \\ &= \cos^{-1} \left(\sqrt{1-u^2} \sqrt{1-v^2} + uv \right) \\ \tan^{-1} u + \tan^{-1} v &= \tan^{-1} \frac{u+v}{1-uv} \\ \tan^{-1} u - \tan^{-1} v &= \tan^{-1} \frac{u-v}{1+uv}\end{aligned}$$

$$\begin{aligned}
\coth^{-1} u + \coth^{-1} v &= \coth^{-1} \frac{1 + uv}{u + v} \\
\coth^{-1} u - \coth^{-1} v &= \coth^{-1} \frac{1 - uv}{u - v} \\
\operatorname{sech}^{-1} u + \operatorname{sech}^{-1} v &= \operatorname{sech}^{-1} \frac{uv\sqrt{1-u^2}\sqrt{1-v^2}}{uv + \sqrt{1-u^2}\sqrt{1-v^2}} \\
&= \operatorname{csch}^{-1} \frac{uv\sqrt{1-u^2}\sqrt{1-v^2}}{u\sqrt{1-v^2} + v\sqrt{1-u^2}} \\
\operatorname{sech}^{-1} u - \operatorname{sech}^{-1} v &= \operatorname{sech}^{-1} \frac{uv\sqrt{1-u^2}\sqrt{1-v^2}}{\sqrt{1-u^2}\sqrt{1-v^2} - uv} \\
&= \operatorname{csch}^{-1} \frac{uv\sqrt{1-u^2}\sqrt{1-v^2}}{u\sqrt{1-v^2} - v\sqrt{1-u^2}} \\
\operatorname{csch}^{-1} u + \operatorname{csch}^{-1} v &= \operatorname{csch}^{-1} \frac{uv\sqrt{u^2+1}\sqrt{v^2+1}}{u\sqrt{v^2+1} + v\sqrt{u^2+1}} \\
&= \operatorname{sech}^{-1} \frac{uv\sqrt{u^2+1}\sqrt{v^2+1}}{uv + \sqrt{u^2+1}\sqrt{v^2+1}} \\
\operatorname{csch}^{-1} u - \operatorname{csch}^{-1} v &= \operatorname{csch}^{-1} \frac{uv\sqrt{u^2+1}\sqrt{v^2+1}}{v\sqrt{u^2+1} - u\sqrt{v^2+1}} \\
&= \operatorname{sech}^{-1} \frac{uv\sqrt{u^2+1}\sqrt{v^2+1}}{uv - \sqrt{u^2+1}\sqrt{v^2+1}} \\
\exp^{-1} u + \exp^{-1} v &= \exp^{-1} uv \\
\exp^{-1} u - \exp^{-1} v &= \exp^{-1} \frac{u}{v}
\end{aligned}$$

$$\begin{aligned}
\cot^{-1} u + \cot^{-1} v &= \cot^{-1} \frac{uv - 1}{u + v} \\
\cot^{-1} u - \cot^{-1} v &= \cot^{-1} \frac{vu - 1}{v - u} \\
\sec^{-1} u + \sec^{-1} v &= \sec^{-1} \frac{uv\sqrt{u^2-1}\sqrt{v^2-1}}{\sqrt{u^2-1}\sqrt{v^2-1} - uv} \\
&= \operatorname{csc}^{-1} \frac{uv\sqrt{u^2-1}\sqrt{v^2-1}}{u\sqrt{v^2-1} + v\sqrt{u^2-1}} \\
\sec^{-1} u - \sec^{-1} v &= \sec^{-1} \frac{uv\sqrt{u^2-1}\sqrt{v^2-1}}{uv + \sqrt{u^2-1}\sqrt{v^2-1}} \\
&= \operatorname{csc}^{-1} \frac{uv\sqrt{u^2-1}\sqrt{v^2-1}}{u\sqrt{v^2-1} - v\sqrt{u^2-1}} \\
\operatorname{csc}^{-1} u + \operatorname{csc}^{-1} v &= \operatorname{csc}^{-1} \frac{uv\sqrt{u^2-1}\sqrt{v^2-1}}{u\sqrt{v^2-1} + v\sqrt{u^2-1}} \\
&= \sec^{-1} \frac{uv\sqrt{u^2-1}\sqrt{v^2-1}}{uv - \sqrt{u^2-1}\sqrt{v^2-1}} \\
\operatorname{csc}^{-1} u - \operatorname{csc}^{-1} v &= \operatorname{csc}^{-1} \frac{uv\sqrt{u^2-1}\sqrt{v^2-1}}{v\sqrt{u^2-1} - u\sqrt{v^2-1}} \\
&= \sec^{-1} \frac{uv\sqrt{u^2-1}\sqrt{v^2-1}}{uv + \sqrt{u^2-1}\sqrt{v^2-1}} \\
\operatorname{cis}^{-1} u + \operatorname{cis}^{-1} v &= \operatorname{cis}^{-1} uv \\
\operatorname{cis}^{-1} u - \operatorname{cis}^{-1} v &= \operatorname{cis}^{-1} \frac{u}{v}
\end{aligned}$$

Same-angle function sums and differences

$$\begin{aligned}
\cosh \theta + i \sinh \theta &= \frac{\sqrt{2}}{i} \sinh \left(\theta + \frac{\pi i}{4} \right) \\
&= \sqrt{2} \cosh \left(\frac{\pi i}{4} - \theta \right) \\
\cosh \theta - i \sinh \theta &= \sqrt{2} \cosh \left(\theta + \frac{\pi i}{4} \right) \\
&= \frac{\sqrt{2}}{i} \sinh \left(\frac{\pi i}{4} - \theta \right) \\
\coth \theta + \tanh \theta &= 2 \coth 2\theta \\
\coth \theta - \tanh \theta &= 2 \operatorname{csch} 2\theta
\end{aligned}$$

$$\begin{aligned}
\cos \theta + \sin \theta &= \sqrt{2} \sin \left(\theta + \frac{\pi}{4} \right) \\
&= \sqrt{2} \cos \left(\frac{\pi}{4} - \theta \right) \\
\cos \theta - \sin \theta &= \sqrt{2} \cos \left(\theta + \frac{\pi}{4} \right) \\
&= \sqrt{2} \sin \left(\frac{\pi}{4} - \theta \right) \\
\cot \theta + \tan \theta &= 2 \operatorname{csc} 2\theta \\
\cot \theta - \tan \theta &= 2 \cot 2\theta
\end{aligned}$$

$$\begin{aligned} \operatorname{csch} \theta + i \operatorname{sech} \theta &= \frac{\sqrt{2} \operatorname{csch} \left(\theta + \frac{\pi i}{4} \right)}{i \operatorname{csch} 2\theta} \\ &= \frac{\sqrt{2} \operatorname{sech} \left(\frac{\pi i}{4} - \theta \right)}{\operatorname{csch} 2\theta} \\ \operatorname{csch} \theta - i \operatorname{sech} \theta &= \frac{\sqrt{2} \operatorname{sech} \left(\theta + \frac{\pi i}{4} \right)}{i \operatorname{csch} 2\theta} \\ &= \frac{\sqrt{2} \operatorname{csch} \left(\frac{\pi i}{4} - \theta \right)}{\operatorname{csch} 2\theta} \end{aligned}$$

$$\begin{aligned} \csc \theta + \sec \theta &= \frac{\sqrt{2} \csc \left(\theta + \frac{\pi}{4} \right)}{\csc 2\theta} \\ &= \frac{\sqrt{2} \sec \left(\frac{\pi}{4} - \theta \right)}{\csc 2\theta} \\ \csc \theta - \sec \theta &= \frac{\sqrt{2} \sec \left(\theta + \frac{\pi}{4} \right)}{\csc 2\theta} \\ &= \frac{\sqrt{2} \csc \left(\frac{\pi}{4} - \theta \right)}{\csc 2\theta} \end{aligned}$$

Linear combinations of functions

$$A \neq \pm B \Rightarrow$$

$$\begin{aligned} A \sinh \theta + B \cosh \theta &= \sqrt{A^2 - B^2} \sinh \left(\theta + \tanh^{-1} \frac{B}{A} \right) \\ &= \sqrt{B^2 - A^2} \cosh \left(\theta + \tanh^{-1} \frac{A}{B} \right) \end{aligned}$$

$$\begin{aligned} A \tanh \theta + B \coth \theta &= \sqrt{A^2 \operatorname{sech}^2 \theta - B^2 \operatorname{csch}^2 \theta} \times \\ &\quad \sinh \left(\theta + \tanh^{-1} \left(\frac{B}{A} \coth \theta \right) \right) \\ &= \sqrt{B^2 \operatorname{csch}^2 \theta - A^2 \operatorname{sech}^2 \theta} \times \\ &\quad \cosh \left(\theta + \tanh^{-1} \left(\frac{A}{B} \tanh \theta \right) \right) \end{aligned}$$

$$\begin{aligned} A \operatorname{sech} \theta + B \operatorname{csch} \theta &= \sqrt{A^2 - B^2} \operatorname{csch} 2\theta \sinh \left(\theta + \tanh^{-1} \frac{B}{A} \right) \\ &= \sqrt{B^2 - A^2} \operatorname{csch} 2\theta \cosh \left(\theta + \tanh^{-1} \frac{A}{B} \right) \end{aligned}$$

$$A \neq \pm B \text{ and } A, B \neq 0 \Rightarrow$$

$$\begin{aligned} A \exp \theta + \frac{B}{\exp \theta} &= 2\sqrt{AB} \sinh \left(\theta + \tanh^{-1} \frac{A-B}{A+B} \right) \\ &= 2i\sqrt{AB} \cosh \left(\theta + \tanh^{-1} \frac{A+B}{A-B} \right) \end{aligned}$$

$$A \neq \pm B \Rightarrow$$

$$\begin{aligned} A \sin \theta + B \cos \theta &= \sqrt{A^2 + B^2} \sin \left(\theta + \tan^{-1} \frac{B}{A} \right) \\ &= \sqrt{A^2 + B^2} \cos \left(\theta - \tan^{-1} \frac{A}{B} \right) \end{aligned}$$

$$\begin{aligned} A \tan \theta + B \cot \theta &= \sqrt{A^2 \sec^2 \theta + B^2 \csc^2 \theta} \times \\ &\quad \sin \left(\theta + \tan^{-1} \left(\frac{B}{A} \cot \theta \right) \right) \\ &= \sqrt{A^2 \sec^2 \theta + B^2 \csc^2 \theta} \times \\ &\quad \cos \left(\theta - \tan^{-1} \left(\frac{A}{B} \tan \theta \right) \right) \end{aligned}$$

$$\begin{aligned} A \sec \theta + B \csc \theta &= \sqrt{A^2 + B^2} \csc 2\theta \sin \left(\theta + \tan^{-1} \frac{B}{A} \right) \\ &= \sqrt{A^2 + B^2} \csc 2\theta \cos \left(\theta - \tan^{-1} \frac{A}{B} \right) \end{aligned}$$

$$A \neq \pm B \text{ and } A, B \neq 0 \Rightarrow$$

$$\begin{aligned} A \operatorname{cis} \theta + \frac{B}{\operatorname{cis} \theta} &= 2\sqrt{AB} \sin \left(\theta + \tan^{-1} \frac{A-B}{A+B} \right) \\ &= 2\sqrt{AB} \cos \left(\theta - \tan^{-1} \frac{A+B}{A-B} \right) \end{aligned}$$

Multiple angles

$$\cosh n\theta = \sum_{k=0}^{\frac{n}{2}} \binom{n}{2k} \cosh^{n-2k} \theta \sinh^{2k} \theta$$

$$\sinh n\theta = \sum_{k=0}^{\frac{n-1}{2}} \binom{n}{2k+1} \cosh^{n-2k-1} \theta \sinh^{2k+1} \theta$$

$$\cos n\theta = \sum_{k=0}^{\frac{n}{2}} (-1)^k \binom{n}{2k} \cos^{n-2k} \theta \sin^{2k} \theta$$

$$\sin n\theta = \sum_{k=0}^{\frac{n-1}{2}} (-1)^k \binom{n}{2k+1} \cos^{n-2k-1} \theta \sin^{2k+1} \theta$$

Powers of two angles

$$\begin{aligned}\sinh^2 a - \sinh^2 b &= \cosh^2 a - \cosh^2 b \\ &= \sinh(a+b) \sinh(a-b)\end{aligned}$$

$$\begin{aligned}\cosh^2 a + \sinh^2 b &= \cosh^2 b + \sinh^2 a \\ &= \cosh(a+b) \cosh(a-b)\end{aligned}$$

$$\begin{aligned}\tanh^2 a - \tanh^2 b &= \tanh(a+b) \tanh(a-b) (1 - \tanh^2 a \tanh^2 b)\end{aligned}$$

$$\begin{aligned}\coth^2 a - \coth^2 b &= \tanh(a+b) \tanh(a-b) (1 - \coth^2 a \coth^2 b)\end{aligned}$$

$$\begin{aligned}\operatorname{sech}^2 a - \operatorname{sech}^2 b &= \sinh(a+b) \sinh(a-b) \operatorname{sech}^2 a \operatorname{sech}^2 b\end{aligned}$$

$$\begin{aligned}\operatorname{csch}^2 a - \operatorname{csch}^2 b &= \sinh(b+a) \sinh(b-a) \operatorname{csch}^2 a \operatorname{csch}^2 b\end{aligned}$$

$$\begin{aligned}\sin^2 a - \sin^2 b &= \cos^2 b - \cos^2 a \\ &= \sin(a+b) \sin(a-b)\end{aligned}$$

$$\begin{aligned}\cos^2 a - \sin^2 b &= \cos^2 b - \sin^2 a \\ &= \cos(a+b) \cos(a-b)\end{aligned}$$

$$\begin{aligned}\tan^2 a - \tan^2 b &= \tan(a+b) \tan(a-b) (1 - \tan^2 a \tan^2 b)\end{aligned}$$

$$\begin{aligned}\cot^2 a - \cot^2 b &= \tan(a+b) \tan(a-b) (1 - \cot^2 a \cot^2 b)\end{aligned}$$

$$\begin{aligned}\sec^2 a - \sec^2 b &= \sin(a+b) \sin(a-b) \sec^2 a \sec^2 b\end{aligned}$$

$$\begin{aligned}\csc^2 a - \csc^2 b &= \sin(b+a) \sin(b-a) \csc^2 a \csc^2 b\end{aligned}$$

Powers of one angle

$$\begin{aligned}\cosh^{2n} \theta &= \frac{1}{4^n} \binom{2n}{n} \\ &+ \frac{1}{2^{2n-1}} \sum_{k=1}^n \binom{2n}{n+k} \cosh [2k\theta]\end{aligned}$$

$$\begin{aligned}\sinh^{2n} \theta &= \frac{1}{4^n} (-1)^n \binom{2n}{n} \\ &+ \frac{1}{2^{2n-1}} \sum_{k=1}^n (-1)^{n+k} \binom{2n}{n+k} \cosh [2k\theta]\end{aligned}$$

$$\begin{aligned}\cosh^{2n+1} \theta &= \frac{1}{4^{n-1}} \sum_{k=0}^n \binom{2n+1}{n+k+1} \cosh [(2k+1)\theta]\end{aligned}$$

$$\begin{aligned}\sinh^{2n+1} \theta &= \frac{1}{4^{n-1}} \sum_{k=0}^n (-1)^{k+1} \binom{2n+1}{n+k+1} \sinh [(2k+1)\theta]\end{aligned}$$

$$\begin{aligned}\frac{\cosh^n \theta - \sinh^n \theta}{\sinh \left(\theta - \tanh^{-1} \exp \frac{4k\pi i}{n} \right)} &= \prod_{k=1}^n \sqrt{1 - \exp \frac{4k\pi i}{n}} \times \\ &\sinh \left(\theta - \tanh^{-1} \exp \frac{4k\pi i}{n} \right)\end{aligned}$$

$$\begin{aligned}\cos^{2n} \theta &= \frac{1}{4^n} \binom{2n}{n} \\ &+ \frac{1}{2^{2n-1}} \sum_{k=1}^n \binom{2n}{n+k} \cos [2k\theta]\end{aligned}$$

$$\begin{aligned}\sin^{2n} \theta &= \frac{1}{4^n} \binom{2n}{n} \\ &+ \frac{1}{2^{2n-1}} \sum_{k=1}^n (-1)^k \binom{2n}{n+k} \cos [2k\theta]\end{aligned}$$

$$\begin{aligned}\cos^{2n+1} \theta &= \frac{1}{4^{n-1}} \sum_{k=0}^n \binom{2n+1}{n+k+1} \cos [(2k+1)\theta]\end{aligned}$$

$$\begin{aligned}\sin^{2n+1} \theta &= \frac{1}{4^{n-1}} \sum_{k=0}^n (-1)^k \binom{2n+1}{n+k+1} \sin [(2k+1)\theta]\end{aligned}$$

$$\begin{aligned}\frac{\cos^n \theta - \sin^n \theta}{\sin \left(\theta - \tan^{-1} \operatorname{cis} \frac{4k\pi}{n} \right)} &= \prod_{k=1}^n \sqrt{1 + \operatorname{cis} \frac{4k\pi}{n}} \times \\ &\sin \left(\theta - \tan^{-1} \operatorname{cis} \frac{4k\pi}{n} \right)\end{aligned}$$

$$\begin{aligned} & \cosh^n \theta + \sinh^n \theta \\ &= \prod_{k=1}^n \sqrt{1 - \exp \frac{2(2k+1)\pi i}{n}} \times \\ & \quad \sinh \left(\theta - \tanh^{-1} \exp \frac{2(2k+1)\pi i}{n} \right) \end{aligned}$$

$$\begin{aligned} & \cosh^{2n} \theta - \sinh^{2n} \theta \\ &= \frac{1}{2^{2n-1}} (1 - (-1)^n) \binom{2n}{n} + \frac{1}{4^{n-1}} \times \end{aligned}$$

$$\sum_{k=1}^{\frac{n}{2}} \binom{2n}{n+2k+(-1)^n} \cosh [(2k+(-1)^n) 2\theta]$$

$$\begin{aligned} & \cosh^{2n} \theta + \sinh^{2n} \theta \\ &= \frac{1}{2^{2n-1}} (1 + (-1)^n) \binom{2n}{n} + \frac{1}{4^{n-1}} \times \end{aligned}$$

$$\sum_{k=1}^{\frac{n+1}{2}} \binom{2n}{n+2k-(-1)^n} \cosh [(2k-(-1)^n) 2\theta]$$

$$\begin{aligned} & \cosh^{2n+1} \theta - i \sinh^{2n+1} \theta \\ &= \frac{1}{2^{2n-3}} \times \end{aligned}$$

$$\sum_{k=0}^n \binom{2n+1}{n+k+1} \cosh \left[\frac{\pi i}{4} + (-1)^k (2k+1)\theta \right]$$

$$\begin{aligned} & \cosh^{2n+1} \theta + i \sinh^{2n+1} \theta \\ &= i \frac{1}{2^{2n-3}} \times \end{aligned}$$

$$\sum_{k=0}^n \binom{2n+1}{n+k+1} \sinh \left[\frac{\pi i}{4} + (-1)^k (2k+1)\theta \right]$$

$$\begin{aligned} & \cosh^{2n+1} \theta + \sinh^{2n+1} \theta \\ &= \frac{1}{4^{n-1}} \times \end{aligned}$$

$$\sum_{k=0}^n \binom{2n+1}{n+k+1} \exp \left[(-1)^{k+1} (2k+1)\theta \right]$$

$$\begin{aligned} & \cos^n \theta + \sin^n \theta \\ &= \prod_{k=1}^n \sqrt{1 + \operatorname{cis} \frac{2(2k+1)\pi}{n}} \times \\ & \quad \sin \left(\theta - \tanh^{-1} \operatorname{cis} \frac{2(2k+1)\pi}{n} \right) \end{aligned}$$

$$\begin{aligned} & \cos^{2n} \theta - \sin^{2n} \theta \\ &= \frac{1}{4^{n-1}} \times \end{aligned}$$

$$\sum_{k=1}^{\frac{n+1}{2}} \binom{2n}{n+2k-1} \cos [(2k-1) 2\theta]$$

$$\begin{aligned} & \cos^{2n} \theta + \sin^{2n} \theta \\ &= \frac{1}{2^{2n-1}} \binom{2n}{n} + \frac{1}{4^{n-1}} \times \end{aligned}$$

$$\sum_{k=1}^{\frac{n}{2}} \binom{2n}{n+2k} \cos [4k\theta]$$

$$\begin{aligned} & \cos^{2n+1} \theta - \sin^{2n+1} \theta \\ &= \frac{1}{2^{2n-3}} \times \end{aligned}$$

$$\sum_{k=0}^n \binom{2n+1}{n+k+1} \cos \left[\frac{\pi}{4} + (-1)^k (2k+1)\theta \right]$$

$$\begin{aligned} & \cos^{2n+1} \theta + \sin^{2n+1} \theta \\ &= \frac{1}{2^{2n-3}} \times \end{aligned}$$

$$\sum_{k=0}^n \binom{2n+1}{n+k+1} \sin \left[\frac{\pi}{4} + (-1)^k (2k+1)\theta \right]$$

$$\begin{aligned} & \cos^{2n+1} \theta + i \sin^{2n+1} \theta \\ &= \frac{1}{4^{n-1}} \times \end{aligned}$$

$$\sum_{k=0}^n \binom{2n+1}{n+k+1} \operatorname{cis} \left[(-1)^k (2k+1)\theta \right]$$

Differential equations

$$f''(\theta) = f(\theta) \quad \Rightarrow \quad f(\theta) = A \cosh \theta + B \sinh \theta \quad f''(\theta) = -f(\theta) \quad \Rightarrow \quad f(\theta) = A \cos \theta + B \sin \theta$$

Derivatives

$$(\cosh \theta)' = \sinh \theta$$

$$(\sinh \theta)' = \cosh \theta$$

$$(\tanh \theta)' = \operatorname{sech}^2 \theta$$

$$(\coth \theta)' = -\operatorname{csch}^2 \theta$$

$$(\operatorname{sech} \theta)' = -\tanh \theta \operatorname{sech} \theta$$

$$(\operatorname{csch} \theta)' = -\coth \theta \operatorname{csch} \theta$$

$$(\exp \theta)' = \exp \theta$$

$$(\cos \theta)' = -\sin \theta$$

$$(\sin \theta)' = \cos \theta$$

$$(\tan \theta)' = \sec^2 \theta$$

$$(\cot \theta)' = -\operatorname{csc}^2 \theta$$

$$(\sec \theta)' = \tan \theta \sec \theta$$

$$(\operatorname{csc} \theta)' = -\cot \theta \operatorname{csc} \theta$$

$$(\operatorname{cis} \theta)' = i \operatorname{cis} \theta$$

Inverse function derivatives

$$(\cosh^{-1} u)' = \frac{1}{\sqrt{u^2 - 1}}$$

$$(\sinh^{-1} u)' = \frac{1}{\sqrt{u^2 + 1}}$$

$$(\tanh^{-1} u)' = \frac{1}{1 - u^2}$$

$$(\coth^{-1} u)' = \frac{1}{1 - u^2}$$

$$(\operatorname{sech}^{-1} u)' = \frac{1}{u\sqrt{1 - u^2}}$$

$$(\operatorname{csch}^{-1} u)' = \frac{1}{u\sqrt{1 + u^2}}$$

$$(\exp^{-1} u)' = \frac{1}{u}$$

$$(\cos^{-1} u)' = \frac{-1}{\sqrt{1 - u^2}}$$

$$(\sin^{-1} u)' = \frac{1}{\sqrt{1 - u^2}}$$

$$(\tan^{-1} u)' = \frac{1}{1 + u^2}$$

$$(\cot^{-1} u)' = \frac{-1}{1 + u^2}$$

$$(\sec^{-1} u)' = \frac{1}{u\sqrt{u^2 - 1}}$$

$$(\csc^{-1} u)' = \frac{-1}{u\sqrt{u^2 - 1}}$$

$$(\operatorname{cis}^{-1} u)' = \frac{-i}{u}$$

Power series

$$\exp \theta = \sum_{k=0}^{\infty} \frac{\theta^k}{k!}$$

$$\cosh \theta = \sum_{k=0}^{\infty} \frac{\theta^{2k}}{(2k)!}$$

$$\sinh \theta = \sum_{k=0}^{\infty} \frac{\theta^{2k+1}}{(2k+1)!}$$

$$B_k = \frac{2(2k)!}{(4^k - 1)\pi^{2k}} \sum_{j=0}^{\infty} \frac{1}{(2j+1)^{2k}}$$

$$\tanh \theta = \sum_{k=1}^{\infty} \frac{(-1)^k 4^k (4^k - 1) B_k}{(2k)!} \theta^{2k-1},$$

$$|\theta| < \frac{\pi}{2}$$

$$\coth \theta = \frac{1}{\theta} - \sum_{k=1}^{\infty} \frac{(-1)^{k+1} 4^k B_k}{(2k)!} \theta^{2k-1},$$

$$0 < |\theta| < \pi$$

$$\operatorname{sech} \theta = \sum_{k=0}^{\infty} \frac{(-1)^k E_k}{(2k)!} \theta^{2k},$$

$$|\theta| < \frac{\pi}{2}$$

$$\operatorname{csch} \theta = \frac{1}{\theta} + \sum_{k=1}^{\infty} \frac{(-1)^k 2(2^{2k-1} - 1) B_k}{(2k)!} \theta^{2k-1},$$

$$|\theta| < \pi$$

$$\operatorname{cis} \theta = \sum_{k=0}^{\infty} \frac{(i\theta)^k}{k!}$$

$$\cos \theta = \sum_{k=0}^{\infty} \frac{(-1)^k \theta^{2k}}{(2k)!}$$

$$\sin \theta = \sum_{k=0}^{\infty} \frac{(-1)^k \theta^{2k+1}}{(2k+1)!}$$

$$E_k = \frac{4^{k+1} (2k)!}{\pi^{2k+1}} \sum_{j=0}^{\infty} \frac{(-1)^{j+1}}{(2j-1)^{2k+1}}$$

$$\tan \theta = \sum_{k=1}^{\infty} \frac{4^k (4^k - 1) B_k}{(2k)!} \theta^{2k-1},$$

$$|\theta| < \frac{\pi}{2}$$

$$\cot \theta = \frac{1}{\theta} - \sum_{k=1}^{\infty} \frac{4^k B_k}{(2k)!} \theta^{2k-1},$$

$$0 < |\theta| < \pi$$

$$\sec \theta = \sum_{k=0}^{\infty} \frac{E_k}{(2k)!} \theta^{2k},$$

$$|\theta| < \frac{\pi}{2}$$

$$\csc \theta = \frac{1}{\theta} + \sum_{k=1}^{\infty} \frac{2(2^{2k} - 1) B_k}{(2k)!} \theta^{2k-1},$$

$$|\theta| < \pi$$

Inverse function power series

$$\sinh^{-1} u = \sum_{k=0}^{\infty} \frac{(-1)^k (2k)!}{4^k (k!)^2 (2k+1)} u^{2k+1},$$

$|u| < 1$

$$\cosh^{-1} u = \ln 2u - \sum_{k=1}^{\infty} \frac{(2k)!}{4^k (k!)^2 2k} u^{-2k},$$

$|u| > 1$

$$\tanh^{-1} u = \sum_{k=0}^{\infty} \frac{u^{2k+1}}{2k+1}, \quad |u| < 1$$

$$\coth^{-1} u = \sum_{k=0}^{\infty} \frac{u^{-2k-1}}{2k+1}, \quad |u| > 1$$

$$\operatorname{sech}^{-1} u = \ln \frac{2}{u} - \sum_{k=1}^{\infty} \frac{(2k)!}{4^k (k!)^2 2k} u^{2k},$$

$|u| < 1$

$$\operatorname{csch}^{-1} u = \sum_{k=0}^{\infty} \frac{(-1)^k (2k)!}{4^k (k!)^2 (2k+1)} u^{-2k-1},$$

$|u| > 1$

$$\sin^{-1} u = \sum_{k=0}^{\infty} \frac{(2k)!}{4^k (k!)^2 (2k+1)} u^{2k+1},$$

$|u| < 1$

$$\cos^{-1} u = \frac{\pi}{2} - \sum_{k=0}^{\infty} \frac{(2k)!}{4^k (k!)^2 (2k+1)} u^{2k+1},$$

$|u| < 1$

$$\tan^{-1} u = \sum_{k=0}^{\infty} \frac{(-1)^k u^{2k+1}}{2k+1}, \quad |u| \leq 1$$

$$\cot^{-1} u = \sum_{k=0}^{\infty} \frac{(-1)^k u^{-2k-1}}{2k+1}, \quad |u| \geq 1$$

$$\operatorname{sec}^{-1} u = \frac{\pi}{2} - \sum_{k=0}^{\infty} \frac{(2k)!}{4^k (k!)^2 (2k+1)} u^{-2k-1},$$

$|u| > 1$

$$\operatorname{csc}^{-1} u = \sum_{k=0}^{\infty} \frac{(2k)!}{4^k (k!)^2 (2k+1)} u^{-2k-1},$$

$|u| > 1$

Infinite products

$$\cosh \theta = \prod_{k=1}^{\infty} \left(1 + \left(\frac{\theta}{\pi(k - \frac{1}{2})} \right)^2 \right)$$

$$\sinh \theta = \theta \prod_{k=1}^{\infty} \left(1 + \left(\frac{\theta}{\pi k} \right)^2 \right)$$

$$\tanh \theta = \theta \prod_{k=1}^{\infty} \left(1 - \frac{1}{2k} \right)^2 \frac{k^2 + \left(\frac{\theta}{\pi} \right)^2}{\left(k - \frac{1}{2} \right)^2 + \left(\frac{\theta}{\pi} \right)^2}$$

$$\coth \theta = \frac{1}{\theta} \prod_{k=1}^{\infty} \left(\frac{2k}{2k-1} \right)^2 \frac{\left(k - \frac{1}{2} \right)^2 + \left(\frac{\theta}{\pi} \right)^2}{k^2 + \left(\frac{\theta}{\pi} \right)^2}$$

$$\operatorname{sech} \theta = \theta \prod_{k=1}^{\infty} \frac{\left(k - \frac{1}{2} \right)^2}{\left(k - \frac{1}{2} \right)^2 + \left(\frac{\theta}{\pi} \right)^2}$$

$$\operatorname{csch} \theta = \theta \prod_{k=1}^{\infty} \frac{k^2}{k^2 + \left(\frac{\theta}{\pi} \right)^2}$$

$$\frac{2}{e} = \prod_{k=1}^{\infty} \exp \left(\frac{-1}{2k(2k+1)} \right)$$

$$\cos \theta = \prod_{k=1}^{\infty} \left(1 - \left(\frac{\theta}{\pi(k - \frac{1}{2})} \right)^2 \right)$$

$$\sin \theta = \theta \prod_{k=1}^{\infty} \left(1 - \left(\frac{\theta}{\pi k} \right)^2 \right)$$

$$\tan \theta = \theta \prod_{k=1}^{\infty} \left(1 - \frac{1}{2k} \right)^2 \frac{k^2 - \left(\frac{\theta}{\pi} \right)^2}{\left(k - \frac{1}{2} \right)^2 - \left(\frac{\theta}{\pi} \right)^2}$$

$$\cot \theta = \frac{1}{\theta} \prod_{k=1}^{\infty} \left(\frac{2k}{2k-1} \right)^2 \frac{\left(k - \frac{1}{2} \right)^2 - \left(\frac{\theta}{\pi} \right)^2}{k^2 - \left(\frac{\theta}{\pi} \right)^2}$$

$$\operatorname{sec} \theta = \theta \prod_{k=1}^{\infty} \frac{\left(k - \frac{1}{2} \right)^2}{\left(k - \frac{1}{2} \right)^2 - \left(\frac{\theta}{\pi} \right)^2}$$

$$\operatorname{csc} \theta = \theta \prod_{k=1}^{\infty} \frac{k^2}{k^2 - \left(\frac{\theta}{\pi} \right)^2}$$

$$\frac{2}{\pi} = \prod_{k=2}^{\infty} \cos \left(\frac{\pi}{2^k} \right)$$

Continued fractions

$$\exp \theta = \frac{1}{1 - \frac{\theta}{1 + \frac{\theta}{2 - \frac{\theta}{3 + \frac{\theta}{2 - \frac{\theta}{5 + \frac{\theta}{2 - \frac{\theta}{\ddots}}}}}}}}$$

$$\operatorname{cis} \theta = \frac{1}{i + \frac{\theta}{i - \frac{\theta}{2i + \frac{\theta}{3i - \frac{\theta}{2i + \frac{\theta}{5i - \frac{\theta}{2i + \frac{\theta}{\ddots}}}}}}}}$$

$$\tanh \theta = \frac{\theta}{1 + \frac{\theta^2}{3 + \frac{\theta^2}{5 + \frac{\theta^2}{7 + \frac{\theta^2}{9 + \frac{\theta^2}{11 + \frac{\theta^2}{13 + \frac{\theta^2}{\ddots}}}}}}}}$$

$$\tan \theta = \frac{\theta}{1 - \frac{\theta^2}{3 - \frac{\theta^2}{5 - \frac{\theta^2}{7 - \frac{\theta^2}{9 - \frac{\theta^2}{11 - \frac{\theta^2}{13 - \frac{\theta^2}{\ddots}}}}}}}}$$

$$\operatorname{coth} \theta = \frac{1}{\theta} + \frac{\theta}{3 + \frac{\theta^2}{5 + \frac{\theta^2}{7 + \frac{\theta^2}{9 + \frac{\theta^2}{11 + \frac{\theta^2}{13 + \frac{\theta^2}{\ddots}}}}}}$$

$$\cot \theta = \frac{1}{\theta} - \frac{\theta}{3 - \frac{\theta^2}{5 - \frac{\theta^2}{7 - \frac{\theta^2}{9 - \frac{\theta^2}{11 - \frac{\theta^2}{13 - \frac{\theta^2}{\ddots}}}}}}$$

$$\exp^{-1} u = \frac{u}{1 + \frac{u}{2 + \frac{u}{3 + \frac{4u}{4 + \frac{4u}{5 + \frac{9u}{6 + \frac{9u}{7 + \frac{16u}{\ddots}}}}}}}}$$

$$\operatorname{cis}^{-1} u = \frac{u}{i - \frac{u}{2i - \frac{u}{3i - \frac{4u}{4i - \frac{4u}{5i - \frac{9u}{6i - \frac{9u}{7i - \frac{16u}{\ddots}}}}}}}}$$

$$\tanh^{-1} u = \frac{u}{1 - \frac{u^2}{3 - \frac{4u^2}{5 - \frac{9u^2}{7 - \frac{16u^2}{9 - \frac{25u^2}{11 - \frac{36u^2}{\ddots}}}}}}}$$

$$\tan^{-1} u = \frac{u}{1 + \frac{u^2}{3 + \frac{4u^2}{5 + \frac{9u^2}{7 + \frac{16u^2}{9 + \frac{25u^2}{11 + \frac{36u^2}{\ddots}}}}}}}$$

$$\coth^{-1} u = \frac{1}{u - \frac{1}{3u - \frac{4}{5u - \frac{9}{7u - \frac{16}{9u - \frac{25}{11u - \frac{36}{\ddots}}}}}}}}$$

$$\cot^{-1} u = \frac{1}{u + \frac{1}{3u + \frac{4}{5u + \frac{9}{7u + \frac{16}{9u + \frac{25}{11u + \frac{36}{\ddots}}}}}}}}$$

Inner transformation function definitions

$$\text{gd } \theta = 2 \tan^{-1} e^\theta - \frac{\pi}{2} = \frac{\pi}{2} - 2 \cot^{-1} e^\theta$$

$$\begin{aligned} \text{gd}^{-1} \theta &= \ln \tan \left(\frac{\theta}{2} + \frac{\pi}{4} \right) = \ln \cot \left(\frac{\pi}{4} - \frac{\theta}{2} \right) \\ &= \ln(\sec \theta + \tan \theta) \end{aligned}$$

$$\begin{aligned} \text{cg } \theta &= 2 \coth^{-1} e^\theta = -2 \tanh^{-1} e^{-\theta} \\ &= \ln(\text{csch } \theta + \coth \theta) \end{aligned}$$

$$\text{co } \theta = \frac{\pi}{2} - \theta = \frac{\pi}{2} - \theta + 2n\pi, n \in \mathbb{N}$$

$$\text{gs } \theta = \frac{\text{gd } 2\theta}{2}$$

$$\text{gs}^{-1} \theta = \frac{\text{gd}^{-1} 2\theta}{2}$$

Gudermannian transformations

$$\begin{aligned} \cosh \text{gd}^{-1} \theta &= \sec \theta \\ \sinh \text{gd}^{-1} \theta &= \tan \theta \\ \tanh \text{gd}^{-1} \theta &= \sin \theta \\ \coth \text{gd}^{-1} \theta &= \csc \theta \\ \text{sech } \text{gd}^{-1} \theta &= \cos \theta \\ \text{csch } \text{gd}^{-1} \theta &= \cot \theta \end{aligned}$$

$$\begin{aligned} \cos \text{gd } \theta &= \text{sech } \theta \\ \sin \text{gd } \theta &= \tanh \theta \\ \tan \text{gd } \theta &= \sinh \theta \\ \cot \text{gd } \theta &= \text{csch } \theta \\ \sec \text{gd } \theta &= \cosh \theta \\ \csc \text{gd } \theta &= \coth \theta \end{aligned}$$

Gudermannian transformations of inverses

$$\begin{aligned} \text{gd } \cosh^{-1} u &= \sec^{-1} u \\ \text{gd } \sinh^{-1} u &= \tan^{-1} u \\ \text{gd } \tanh^{-1} u &= \sin^{-1} u \\ \text{gd } \coth^{-1} u &= \csc^{-1} u \\ \text{gd } \operatorname{sech}^{-1} u &= \cos^{-1} u \\ \text{gd } \operatorname{csch}^{-1} u &= \cot^{-1} u \end{aligned}$$

$$\begin{aligned} \text{gd}^{-1} \cos^{-1} u &= \operatorname{sech}^{-1} u \\ \text{gd}^{-1} \sin^{-1} u &= \tanh^{-1} u \\ \text{gd}^{-1} \tan^{-1} u &= \sinh^{-1} u \\ \text{gd}^{-1} \cot^{-1} u &= \operatorname{csch}^{-1} u \\ \text{gd}^{-1} \sec^{-1} u &= \cosh^{-1} u \\ \text{gd}^{-1} \csc^{-1} u &= \coth^{-1} u \end{aligned}$$

Gudermannian special transformations

$$\begin{aligned} \tanh \operatorname{gs}^{-1} \theta &= \tan \theta \\ \coth \operatorname{gs}^{-1} \theta &= \cot \theta \end{aligned}$$

$$\begin{aligned} \tan \operatorname{gs} \theta &= \tanh \theta \\ \cot \operatorname{gs} \theta &= \coth \theta \end{aligned}$$

Gudermannian special transformations of inverses

$$\begin{aligned} \operatorname{gs} \tanh^{-1} u &= \tan^{-1} u \\ \operatorname{gs} \coth^{-1} u &= \cot^{-1} u \end{aligned}$$

$$\begin{aligned} \operatorname{gs}^{-1} \tan^{-1} u &= \tanh^{-1} u \\ \operatorname{gs}^{-1} \cot^{-1} u &= \coth^{-1} u \end{aligned}$$

Cogudermannian and coangle transformations

$$\begin{aligned} \cosh \operatorname{cg} \theta &= \coth \theta \\ \sinh \operatorname{cg} \theta &= \operatorname{csch} \theta \\ \tanh \operatorname{cg} \theta &= \operatorname{sech} \theta \\ \coth \operatorname{cg} \theta &= \cosh \theta \\ \operatorname{sech} \operatorname{cg} \theta &= \tanh \theta \\ \operatorname{csch} \operatorname{cg} \theta &= \sinh \theta \end{aligned}$$

$$\begin{aligned} \cos \operatorname{co} \theta &= \sin \theta \\ \sin \operatorname{co} \theta &= \cos \theta \\ \tan \operatorname{co} \theta &= \cot \theta \\ \cot \operatorname{co} \theta &= \tan \theta \\ \sec \operatorname{co} \theta &= \csc \theta \\ \csc \operatorname{co} \theta &= \sec \theta \end{aligned}$$

Cogudermannian and coangle transformations of inverses

$$\begin{aligned} \operatorname{cg} \cosh^{-1} u &= \coth^{-1} u \\ \operatorname{cg} \sinh^{-1} u &= \operatorname{csch}^{-1} u \\ \operatorname{cg} \tanh^{-1} u &= \operatorname{sech}^{-1} u \\ \operatorname{cg} \coth^{-1} u &= \cosh^{-1} u \\ \operatorname{cg} \operatorname{sech}^{-1} u &= \tanh^{-1} u \\ \operatorname{cg} \operatorname{csch}^{-1} u &= \sinh^{-1} u \end{aligned}$$

$$\begin{aligned} \operatorname{co} \cos^{-1} u &= \sin^{-1} u \\ \operatorname{co} \sin^{-1} u &= \cos^{-1} u \\ \operatorname{co} \tan^{-1} u &= \cot^{-1} u \\ \operatorname{co} \cot^{-1} u &= \tan^{-1} u \\ \operatorname{co} \sec^{-1} u &= \csc^{-1} u \\ \operatorname{co} \csc^{-1} u &= \sec^{-1} u \end{aligned}$$

Properties of inner transformation functions

$$\begin{aligned} \text{gd}(-\theta) &= -\text{gd } \theta \\ \text{cg}(-\theta) &= \text{cg } \theta - \frac{\pi i}{2} \\ \text{gs}(-\theta) &= -\text{gs } \theta \\ (\text{gd } \theta)' &= \text{sech } \theta \\ (\text{cg } \theta)' &= \text{csch } \theta \\ (\text{gs } \theta)' &= \text{sech } 2\theta \\ \text{gd}(i\theta) &= i \text{gd}^{-1} \theta \\ \text{cg}^{-1} \theta &= \text{cg } \theta \\ \text{gs}(i\theta) &= i \text{gs}^{-1} \theta \end{aligned}$$

$$\begin{aligned} \text{gd}(a+b) &= \tan^{-1} \frac{\sinh a}{\cosh b} + \tan^{-1} \frac{\sinh b}{\cosh a} \\ \text{gd}(a-b) &= \tan^{-1} \frac{\sinh a}{\cosh b} - \tan^{-1} \frac{\sinh b}{\cosh a} \\ \text{cg}(a+b) &= \tanh^{-1} \frac{\cosh b - \sinh a}{\cosh b + \sinh a} + \tanh^{-1} \frac{\cosh a - \sinh b}{\cosh a + \sinh b} \\ \text{cg}(a-b) &= \tanh^{-1} \frac{\cosh b - \sinh a}{\cosh b + \sinh a} + \tanh^{-1} \frac{\cosh a + \sinh b}{\cosh a - \sinh b} \\ \text{gs}(a+b) &= \frac{1}{2} \tan^{-1} \frac{2 \cosh a \sinh a}{\cosh^2 b + \sinh^2 b} \\ &\quad + \frac{1}{2} \tan^{-1} \frac{2 \cosh b \sinh b}{\cosh^2 a + \sinh^2 a} \\ \text{gs}(a-b) &= \frac{1}{2} \tan^{-1} \frac{2 \cosh a \sinh a}{\cosh^2 b + \sinh^2 b} \\ &\quad - \frac{1}{2} \tan^{-1} \frac{2 \cosh b \sinh b}{\cosh^2 a + \sinh^2 a} \end{aligned}$$

$$\begin{aligned} \text{gd } 2\theta &= 2 \tan^{-1} \tanh \theta \\ \text{cg } 2\theta &= 2 \coth^{-1} \exp 2\theta \\ \text{gs } 2\theta &= \tan^{-1} \frac{2 \tanh \theta}{1 + \tanh^2 \theta} \\ \text{gd } \frac{\theta}{2} &= \csc^{-1}(\text{csch } \theta + \coth \theta) \\ \text{cg } \frac{\theta}{2} &= \cosh^{-1}(\text{csch } \theta + \coth \theta) \\ \text{gs } \frac{\theta}{2} &= \frac{\tan^{-1} \sinh \theta}{2} \end{aligned}$$

$$\begin{aligned} \text{gd } a + \text{gd } b &= 2 \tan^{-1} \left(\frac{e^a + e^b}{1 - e^{a+b}} \right) - \pi \\ \text{gd } a - \text{gd } b &= 2 \tan^{-1} \left(\frac{e^{a+b} + 1}{e^b - e^a} \right) - \pi \\ \text{cg } a + \text{cg } b &= 2 \tanh^{-1} \left(\frac{e^a + e^b}{1 - e^{a+b}} \right) \\ \text{cg } a - \text{cg } b &= 2 \tanh^{-1} \left(\frac{e^a - e^b}{1 - e^{a+b}} \right) \end{aligned}$$

$$\begin{aligned} \text{gd}^{-1}(-\theta) &= -\text{gd}^{-1} \theta \\ \text{co}(-\theta) &= \pi - \text{co } \theta \\ \text{gs}^{-1}(-\theta) &= -\text{gs}^{-1} \theta \\ (\text{gd}^{-1} \theta)' &= -\sec \theta \\ (\text{co } \theta)' &= -\theta \\ (\text{gs}^{-1} \theta)' &= -\sec 2\theta \\ \text{gd}^{-1}(i\theta) &= i \text{gd } \theta \\ \text{co}^{-1} \theta &= \text{co } \theta \\ \text{gs}^{-1}(i\theta) &= i \text{gs } \theta \end{aligned}$$

$$\begin{aligned} \text{gd}^{-1}(a+b) &= \tanh^{-1} \frac{\sin a}{\cos b} + \tanh^{-1} \frac{\sin b}{\cos a} \\ \text{gd}^{-1}(a-b) &= \tanh^{-1} \frac{\sin a}{\cos b} - \tanh^{-1} \frac{\sin b}{\cos a} \\ \text{co}(a+b) &= \frac{\text{co } 2a}{2} + \frac{\text{co } 2b}{2} \\ \text{co}(a-b) &= \text{co } a + b \\ \text{gs}^{-1}(a+b) &= \frac{1}{2} \tanh^{-1} \frac{2 \cos a \sin a}{\cos^2 b + \sin^2 b} \\ &\quad + \frac{1}{2} \tanh^{-1} \frac{2 \cos b \sin b}{\cos^2 a + \sin^2 a} \\ \text{gs}^{-1}(a-b) &= \frac{1}{2} \tanh^{-1} \frac{2 \cos a \sin a}{\cos^2 b + \sin^2 b} \\ &\quad - \frac{1}{2} \tanh^{-1} \frac{2 \cos b \sin b}{\cos^2 a + \sin^2 a} \end{aligned}$$

$$\begin{aligned} \text{gd}^{-1} 2\theta &= 2 \tanh^{-1} \tan \theta \\ \text{co } 2\theta &= 2 \text{co } \theta - \frac{\pi}{2} \\ \text{gs}^{-1} 2\theta &= \tanh^{-1} \frac{2 \tan \theta}{1 - \tan^2 \theta} \\ \text{gd}^{-1} \frac{\theta}{2} &= \text{csch}^{-1}(\csc \theta + \cot \theta) \\ \text{co } \frac{\theta}{2} &= \tan^{-1}(\csc \theta + \cot \theta) \\ \text{gs}^{-1} \frac{\theta}{2} &= \frac{\tanh^{-1} \sinh \theta}{2} \end{aligned}$$

$$\begin{aligned} \text{gd}^{-1} a + \text{gd}^{-1} b &= 2i \tanh^{-1} \left(\frac{e^{ia} + e^{ib}}{1 - e^{ia+ib}} \right) - \pi i \\ \text{gd}^{-1} a - \text{gd}^{-1} b &= 2i \tanh^{-1} \left(\frac{e^{ia+ib} + 1}{e^{ia} - e^{ib}} \right) - \pi i \\ \text{co } a + \text{co } b &= 2 \text{co } \frac{a+b}{2} \\ \text{co } a - \text{co } b &= b - a \end{aligned}$$

$$gs a + gs b = \tan^{-1} \left(\frac{e^{2a} + e^{2b}}{1 - e^{2a+2b}} \right) - \frac{\pi}{2}$$

$$gs a - gs b = \tan^{-1} \left(\frac{e^{2a+2b} + 1}{e^{2b} - e^{2a}} \right) - \frac{\pi}{2}$$

$$gd \theta = \sum_{k=0}^{\infty} \frac{E_{2k} \theta^{2k+1}}{(2k+1)!}$$

$$= 2 \sum_{k=0}^{\infty} \frac{(-1)^k \tanh^{2k+1} \frac{\theta}{2}}{2k+1}$$

$$= \frac{\pi}{2} - \sum_{k=0}^{\infty} \frac{(2k)! \operatorname{sech}^{2k+1} \theta}{4^k (n!)^2 (2k+1)}$$

$$cg \theta = \ln \theta + \sum_{k=1}^{\infty} \frac{(-1)^{k+1} 2(2^{2k-1} - 1) B_k \theta^{2k}}{2k(2k)!}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k (2k)! \operatorname{csch}^{2k+1} \theta}{4^k (n!)^2 (2k+1)}$$

$$co gd \theta = gd cg \theta = cg co(-i\theta)$$

$$cg \theta = gd^{-1} co gd \theta$$

$$= co gd i co \theta = co(-i gd^{-1} co \theta)$$

$$gs^{-1} a + gs^{-1} b = i \tanh^{-1} \left(\frac{e^{2ia} + e^{2ib}}{1 - e^{2ia+2ib}} \right) - \frac{\pi i}{2}$$

$$gs^{-1} a - gs^{-1} b = i \tanh^{-1} \left(\frac{e^{2ia+2ib} + 1}{e^{2ia} - e^{2ib}} \right) - \frac{\pi i}{2}$$

$$gd^{-1} \theta = \sum_{k=0}^{\infty} \frac{(-1)^k E_{2k} \theta^{2k+1}}{(2k+1)!}$$

$$= 2 \sum_{k=0}^{\infty} \frac{\tan^{2k+1} \frac{\theta}{2}}{2k+1}$$

$$gd^{-1} co \theta = cg gd^{-1} \theta = i co cg \theta$$

$$co \theta = gd cg gd^{-1} \theta$$

Inner transformation conversions

$\cosh \theta$	$= \cos i\theta$
$= -i \sinh i co i\theta$	$= \sin co i\theta$
$= \tanh gd^{-1} co i\theta$	$= -i \tan i gd^{-1} co i\theta$
$= \coth cg \theta$	$= i \cot i cg \theta$
$= \operatorname{sech} gd^{-1} i\theta$	$= \sec gd \theta$
$= i \operatorname{csch} i co gd \theta$	$= \csc co gd \theta$
$\sinh \theta$	$= -i \cosh i co i\theta$
	$= -i \sin i\theta$
$= -i \tanh gd^{-1} i\theta$	$= \tan gd \theta$
$= i \coth i co gd \theta$	$= \cot co gd \theta$
$= -i \operatorname{sech} gd^{-1} co i\theta$	$= -i \sec i gd^{-1} co i\theta$
$= \operatorname{csch} cg \theta$	$= i \csc i cg \theta$
$\tanh \theta$	$= \cosh i co gd \theta$
$= -i \sinh gd^{-1} i\theta$	$= \sin gd \theta$
	$= -i \tan i\theta$
$= \coth i co i\theta$	$= -i \cot co i\theta$
$= \operatorname{sech} cg \theta$	$= \sec i cg \theta$
$= i \operatorname{csch} gd^{-1} co i\theta$	$= \csc i gd^{-1} co i\theta$
$\coth \theta$	$= \cosh cg i\theta$
$= i \sinh gd^{-1} co i\theta$	$= \sin i gd^{-1} co i\theta$
$= \tanh i co i\theta$	$= i \tan co i\theta$
	$= i \cot i\theta$
$= \operatorname{sech} i co gd \theta$	$= \sec co gd \theta$
$= i \operatorname{csch} gd^{-1} i\theta$	$= \csc gd \theta$

$\cos \theta$	$= \cosh i\theta$	$= \sin co \theta$
$= -i \sinh i co \theta$	$= \tanh gd^{-1} co \theta$	$= -i \tan i gd^{-1} co \theta$
$= \coth cg i\theta$	$= \operatorname{sech} gd^{-1} \theta$	$= i \cot i cg i\theta$
$= \operatorname{sech} gd^{-1} \theta$	$= i \operatorname{csch} i co gd i\theta$	$= \sec gd i\theta$
$= \csc co gd i\theta$	$= \csc co gd i\theta$	$= \csc co gd i\theta$
$\sin \theta$	$= \cosh i co \theta$	$= \cos co \theta$
$= -i \sinh i\theta$	$= -i \sinh i\theta$	
$= \tanh gd^{-1} \theta$	$= \tanh gd^{-1} \theta$	$= -i \tan gd^{-1} i\theta$
$= \coth i co gd^{-1} i\theta$	$= \coth i co gd^{-1} i\theta$	$= -i \cot co gd^{-1} i\theta$
$= \operatorname{sech} gd^{-1} co \theta$	$= \operatorname{sech} gd^{-1} co \theta$	$= \sec i gd^{-1} co \theta$
$= -i \operatorname{csch} cg i\theta$	$= -i \operatorname{csch} cg i\theta$	$= \csc i cg i\theta$
$\tan \theta$	$= -i \cosh i co gd i\theta$	$= -i \cos co gd^{-1} i\theta$
$= \sinh gd^{-1} \theta$	$= \sinh gd^{-1} \theta$	$= -i \sin gd i\theta$
$= -i \tanh i\theta$	$= -i \tanh i\theta$	
$= i \coth i co \theta$	$= i \coth i co \theta$	$= \cot co \theta$
$= -i \operatorname{sech} cg i\theta$	$= -i \operatorname{sech} cg i\theta$	$= -i \sec i cg i\theta$
$= \operatorname{csch} gd^{-1} co \theta$	$= \operatorname{csch} gd^{-1} co \theta$	$= i \csc i gd^{-1} co \theta$
$\cot \theta$	$= i \cosh cg i\theta$	$= i \cos i gd^{-1} i\theta$
$= \sinh gd^{-1} co \theta$	$= \sinh gd^{-1} co \theta$	$= -i \sin i gd^{-1} co \theta$
$= -i \tanh i co \theta$	$= -i \tanh i co \theta$	$= \tan co \theta$
$= i \coth i\theta$	$= i \coth i\theta$	
$= i \operatorname{sech} i co gd i\theta$	$= i \operatorname{sech} i co gd i\theta$	$= i \sec co gd i\theta$
$= \operatorname{csch} gd^{-1} \theta$	$= \operatorname{csch} gd^{-1} \theta$	$= i \csc gd^{-1} \theta$

$\operatorname{sech} \theta = \cosh \operatorname{gd}^{-1} i \theta$ $= -i \sinh i \operatorname{co} \operatorname{gd} \theta$ $= \tanh \operatorname{cg} \theta$ $= \operatorname{coth} \operatorname{gd}^{-1} \operatorname{co} i \theta$ $= i \operatorname{csch} i \operatorname{co} i \theta$	$= \cos \operatorname{gd} \theta$ $= \sin \operatorname{co} \operatorname{gd} \theta$ $= -i \tan i \operatorname{cg} \theta$ $= -i \cot i \operatorname{gd}^{-1} \operatorname{co} i \theta$ $= \sec i \theta$ $= \csc \operatorname{co} i \theta$	$\sec \theta = \cosh \operatorname{gd}^{-1} \theta$ $= -i \sinh i \operatorname{co} \operatorname{gd} i \theta$ $= \tanh \operatorname{cg} i \theta$ $= \operatorname{coth} \operatorname{gd}^{-1} \operatorname{co} \theta$ $= \operatorname{sech} i \theta$ $= i \operatorname{csch} i \operatorname{co} \theta$	$= \cos \operatorname{gd} i \theta$ $= \sin \operatorname{co} \operatorname{gd} i \theta$ $= -i \tan i \operatorname{cg} i \theta$ $= i \cot i \operatorname{gd}^{-1} \operatorname{co} \theta$ $= \csc \operatorname{co} \theta$
$\operatorname{csch} \theta = i \cosh \operatorname{gd}^{-1} \operatorname{co} i \theta$ $= \sinh \operatorname{cg} \theta$ $= -i \tanh i \operatorname{co} \operatorname{gd} \theta$ $= i \operatorname{coth} \operatorname{gd}^{-1} i \theta$ $= i \operatorname{sech} i \operatorname{co} i \theta$	$= i \cos i \operatorname{gd}^{-1} \operatorname{co} i \theta$ $= -i \sin i \operatorname{cg} \theta$ $= \tan \operatorname{co} \operatorname{gd} \theta$ $= \cot \operatorname{gd} \theta$ $= i \sec \operatorname{co} i \theta$ $= i \csc i \theta$	$\operatorname{csc} \theta = \cosh \operatorname{gd}^{-1} \operatorname{co} \theta$ $= i \sinh \operatorname{cg} i \theta$ $= \tanh i \operatorname{co} \operatorname{gd} i \theta$ $= \operatorname{coth} \operatorname{gd}^{-1} \theta$ $= \operatorname{sech} i \operatorname{co} \theta$ $= i \operatorname{csch} i \theta$	$= \cos i \operatorname{gd}^{-1} \operatorname{co} \theta$ $= \sin i \operatorname{cg} i \theta$ $= i \tan \operatorname{co} \operatorname{gd}^{-1} i \theta$ $= i \cot \operatorname{gd} i \theta$ $= \sec \operatorname{co} \theta$

Alternative inner transformation conversions

$\cosh \theta = \sinh \operatorname{gd}^{-1} \operatorname{co} \operatorname{gs} \operatorname{cg} \theta$ $= \operatorname{csch} \operatorname{gd}^{-1} \operatorname{gs} \operatorname{cg} \theta$	$= \tan \operatorname{co} \operatorname{gs} \operatorname{cg} \theta$ $= \cot \operatorname{gs} \operatorname{cg} \theta$	$\cos \theta = \sinh \operatorname{gd}^{-1} \operatorname{gs} \operatorname{gd} \operatorname{co} \theta$ $= \operatorname{csch} \operatorname{gd}^{-1} \operatorname{co} \operatorname{gs} \operatorname{gd}^{-1} \operatorname{co} \theta$	$= \tan \operatorname{gs} \operatorname{gd}^{-1} \operatorname{co} \theta$ $= \cot \operatorname{co} \operatorname{gs} \operatorname{gd}^{-1} \operatorname{co} \theta$
$\sinh \theta = \cosh \operatorname{cg} \operatorname{gs}^{-1} \operatorname{co} \operatorname{gd} \theta$ $= \tanh \operatorname{gs}^{-1} \operatorname{gd} \theta$ $= \operatorname{coth} \operatorname{gs}^{-1} \operatorname{co} \operatorname{gd} \theta$ $= \operatorname{sech} \operatorname{cg} \operatorname{gs}^{-1} \operatorname{gd} \theta$	$= \cos \operatorname{co} \operatorname{gd} \operatorname{gs}^{-1} \operatorname{gd} \theta$ $= \sin \operatorname{gd}^{-1} \operatorname{gs}^{-1} \operatorname{gd} \theta$ $= \sec \operatorname{co} \operatorname{gd} \operatorname{gs}^{-1} \operatorname{co} \operatorname{gd} \theta$ $= \csc \operatorname{gd} \operatorname{gs}^{-1} \operatorname{co} \operatorname{gd} \theta$	$\sin \theta = \sinh \operatorname{gd}^{-1} \operatorname{gs} \operatorname{gd} \theta$ $= \operatorname{csch} \operatorname{gd}^{-1} \operatorname{co} \operatorname{gs} \operatorname{gd}^{-1} \theta$	$= \tan \operatorname{gs} \operatorname{gd}^{-1} \theta$ $= \cot \operatorname{gs} \operatorname{cg} \theta$
$\tanh \theta = \sinh \operatorname{gd}^{-1} \operatorname{gs} \theta$ $= \operatorname{csch} \operatorname{gd}^{-1} \operatorname{co} \operatorname{gs} \theta$	$= \tan \operatorname{gs} \theta$ $= \cot \operatorname{co} \operatorname{gs} \theta$	$\tan \theta = \cosh \operatorname{cg} \operatorname{gs}^{-1} \operatorname{co} \theta$ $= \tanh \operatorname{gs}^{-1} \theta$ $= \operatorname{coth} \operatorname{gs}^{-1} \operatorname{co} \theta$ $= \operatorname{sech} \operatorname{cg} \operatorname{gs}^{-1} \theta$	$= \cos \operatorname{co} \operatorname{gd} \operatorname{gs}^{-1} \theta$ $= \sin \operatorname{gd} \operatorname{gs}^{-1} \theta$ $= \sec \operatorname{co} \operatorname{gd} \operatorname{gs}^{-1} \operatorname{co} \theta$ $= \csc \operatorname{gd} \operatorname{gs}^{-1} \operatorname{co} \theta$
$\operatorname{coth} \theta = \sinh \operatorname{gd}^{-1} \operatorname{co} \operatorname{gs} \theta$ $= \operatorname{csch} \operatorname{gd}^{-1} \operatorname{gs} \theta$	$= \tan \operatorname{co} \operatorname{gs} \theta$ $= \cot \operatorname{gs} \theta$	$\cot \theta = \cosh \operatorname{cg} \operatorname{gs}^{-1} \theta$ $= \tanh \operatorname{gs}^{-1} \operatorname{co} \theta$ $= \operatorname{coth} \operatorname{gs}^{-1} \theta$ $= \operatorname{sech} \operatorname{cg} \operatorname{gs}^{-1} \operatorname{co} \theta$	$= \cos \operatorname{co} \operatorname{gd} \operatorname{gs}^{-1} \operatorname{co} \theta$ $= \sin \operatorname{gd} \operatorname{gs}^{-1} \operatorname{co} \theta$ $= \sec \operatorname{co} \operatorname{gd} \operatorname{gs}^{-1} \theta$ $= \csc \operatorname{gd} \operatorname{gs}^{-1} \theta$
$\operatorname{sech} \theta = \sinh \operatorname{gd}^{-1} \operatorname{gs} \operatorname{cg} \theta$ $= \operatorname{csch} \operatorname{gd}^{-1} \operatorname{co} \operatorname{gs} \operatorname{cg} \theta$	$= \tan \operatorname{gs} \operatorname{cg} \theta$ $= \cot \operatorname{co} \operatorname{gs} \operatorname{cg} \theta$	$\sec \theta = \sinh \operatorname{gd}^{-1} \operatorname{co} \operatorname{gs} \operatorname{gd}^{-1} \operatorname{co} \theta$ $= \operatorname{csch} \operatorname{gd}^{-1} \operatorname{gs} \operatorname{gd}^{-1} \operatorname{co} \theta$	$= \tan \operatorname{co} \operatorname{gs} \operatorname{gd}^{-1} \operatorname{co} \theta$ $= \cot \operatorname{gs} \operatorname{gd}^{-1} \operatorname{co} \theta$
$\operatorname{csch} \theta = \cosh \operatorname{cg} \operatorname{gs}^{-1} \operatorname{gd} \theta$ $= \tanh \operatorname{gs}^{-1} \operatorname{co} \operatorname{gd} \theta$ $= \operatorname{coth} \operatorname{gs}^{-1} \operatorname{gd} \theta$ $= \operatorname{sech} \operatorname{cg} \operatorname{gs}^{-1} \operatorname{co} \operatorname{gd} \theta$	$= \cos \operatorname{co} \operatorname{gd} \operatorname{gs}^{-1} \operatorname{co} \operatorname{gd} \theta$ $= \sin \operatorname{gd} \operatorname{gs}^{-1} \operatorname{co} \operatorname{gd} \theta$ $= \sec \operatorname{co} \operatorname{gd} \operatorname{gs}^{-1} \operatorname{gd} \theta$ $= \csc \operatorname{gd} \operatorname{gs}^{-1} \operatorname{gd} \theta$	$\operatorname{csc} \theta = \sinh \operatorname{gd}^{-1} \operatorname{co} \operatorname{gs} \operatorname{gd}^{-1} \theta$ $= \operatorname{csch} \operatorname{gd}^{-1} \operatorname{gs} \operatorname{gd}^{-1} \theta$	$= \tan \operatorname{co} \operatorname{gs} \operatorname{gd}^{-1} \theta$ $= \cot \operatorname{gs} \operatorname{gd}^{-1} \theta$

DeMoivre identities

$\exp^n \theta = \exp(n\theta)$ $\frac{1}{\exp^n \theta} = \exp(-n\theta)$ $\sqrt[n]{\exp \theta} = \exp \frac{\theta}{n}$ $\exp^{m+in} \theta = \exp(m\theta) \operatorname{cis}(n\theta)$	$\operatorname{cis}^n \theta = \operatorname{cis}(n\theta)$ $\frac{1}{\operatorname{cis}^n \theta} = \operatorname{cis}(-n\theta)$ $\sqrt[n]{\operatorname{cis} \theta} = \operatorname{cis} \frac{\theta}{n}$ $\operatorname{cis}^{m+in} \theta = \operatorname{cis}(m\theta) \exp(-n\theta)$
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$$(\exp t \operatorname{cis} \theta)^{m+in} = \exp((m-n)t) \operatorname{cis}((m+n)\theta)$$

Point on the complex plane

$$\begin{aligned} a + bi &= e^{t+i\theta} = \exp t \operatorname{cis} \theta \\ &= \cosh t \cos \theta + \sinh t \cos \theta + i \cosh t \sin \theta + i \sinh t \sin \theta \end{aligned}$$

↕

$$\begin{aligned} a &= \exp t \cos \theta = \cosh t \cos \theta + \sinh t \cos \theta \\ b &= \exp t \sin \theta = \cosh t \sin \theta + \sinh t \sin \theta \end{aligned}$$

↕

$$\begin{aligned} t &= \exp^{-1} \sqrt{a^2 + b^2} = \exp^{-1} \sqrt{(a + bi)(a - bi)} = \tanh^{-1} \frac{a^2 + b^2 - 1}{a^2 + b^2 + 1} \\ \theta &= \operatorname{cis}^{-1} \frac{a + bi}{\sqrt{a^2 + b^2}} = \operatorname{cis}^{-1} \sqrt{\frac{a + bi}{a - bi}} = \tan^{-1} \frac{b}{a} \end{aligned}$$

↕

$$\begin{aligned} \cosh t &= \frac{a^2 + b^2 + 1}{\sqrt{a^2 + b^2}} \\ \sinh t &= \frac{a^2 + b^2 - 1}{\sqrt{a^2 + b^2}} \end{aligned}$$

$$\begin{aligned} \cos \theta &= \frac{a}{\sqrt{a^2 + b^2}} \\ \sin \theta &= \frac{b}{\sqrt{a^2 + b^2}} \end{aligned}$$

Twice-applied formulae

$$e^{e^\theta} = \exp \exp \theta = (\cosh \cosh \theta + \sinh \cosh \theta)(\cosh \sinh \theta + \sinh \sinh \theta)$$

$$e^{e^{i\theta}} = \exp \operatorname{cis} \theta = (\cosh \cos \theta + \sinh \cos \theta)(\cos \sin \theta + i \sin \sin \theta)$$

$$e^{ie^\theta} = \operatorname{cis} \exp \theta = (\cos \cosh \theta + \sin \sinh \theta)(\cos \sinh \theta + i \sin \sinh \theta)$$

$$e^{ie^{i\theta}} = \operatorname{cis} \operatorname{cis} \theta = (\cos \cos \theta + i \sin \cos \theta)(\cosh \sin \theta - \sinh \sin \theta)$$