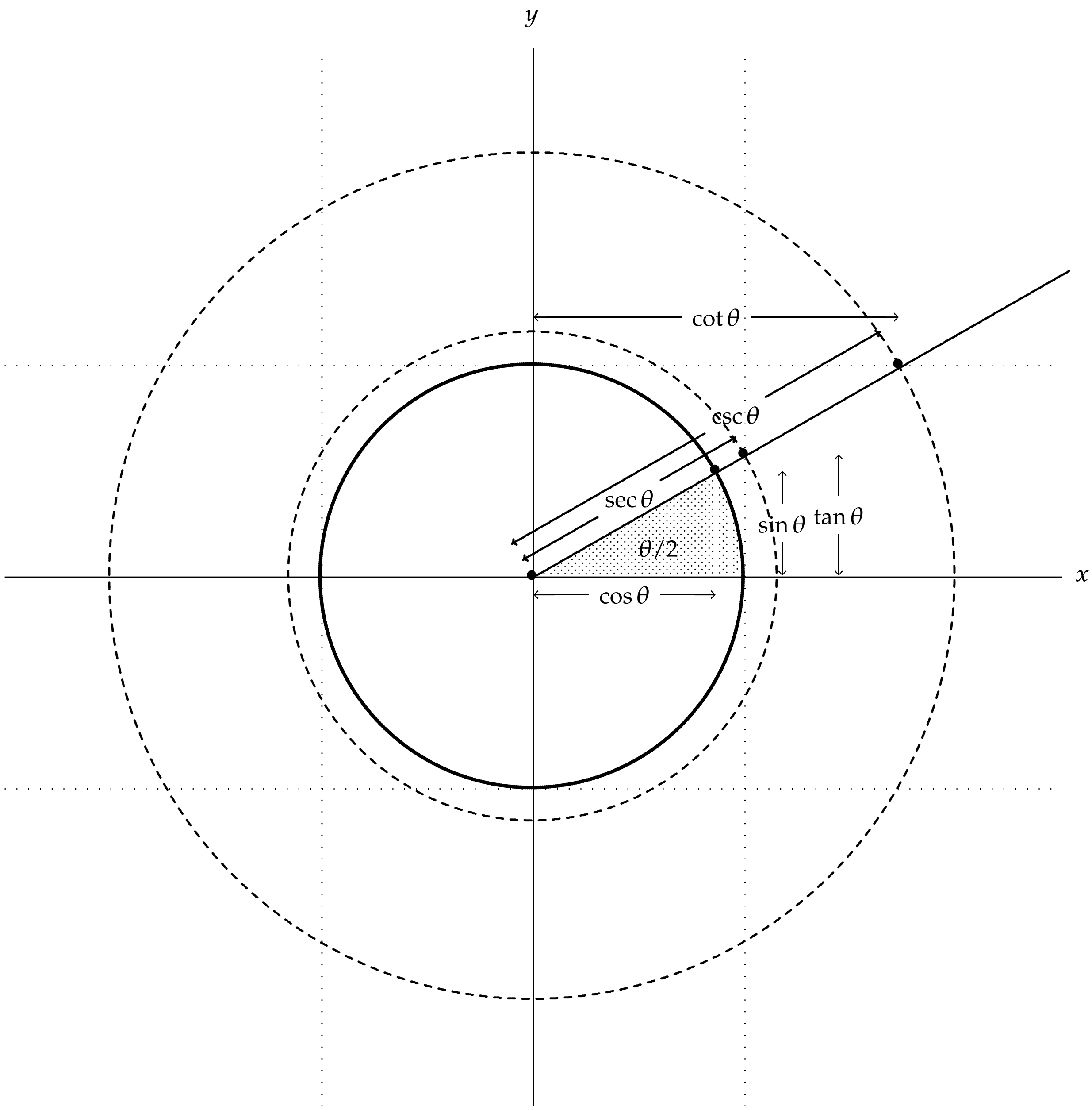
Beautiful Trigonometry

The unit circle

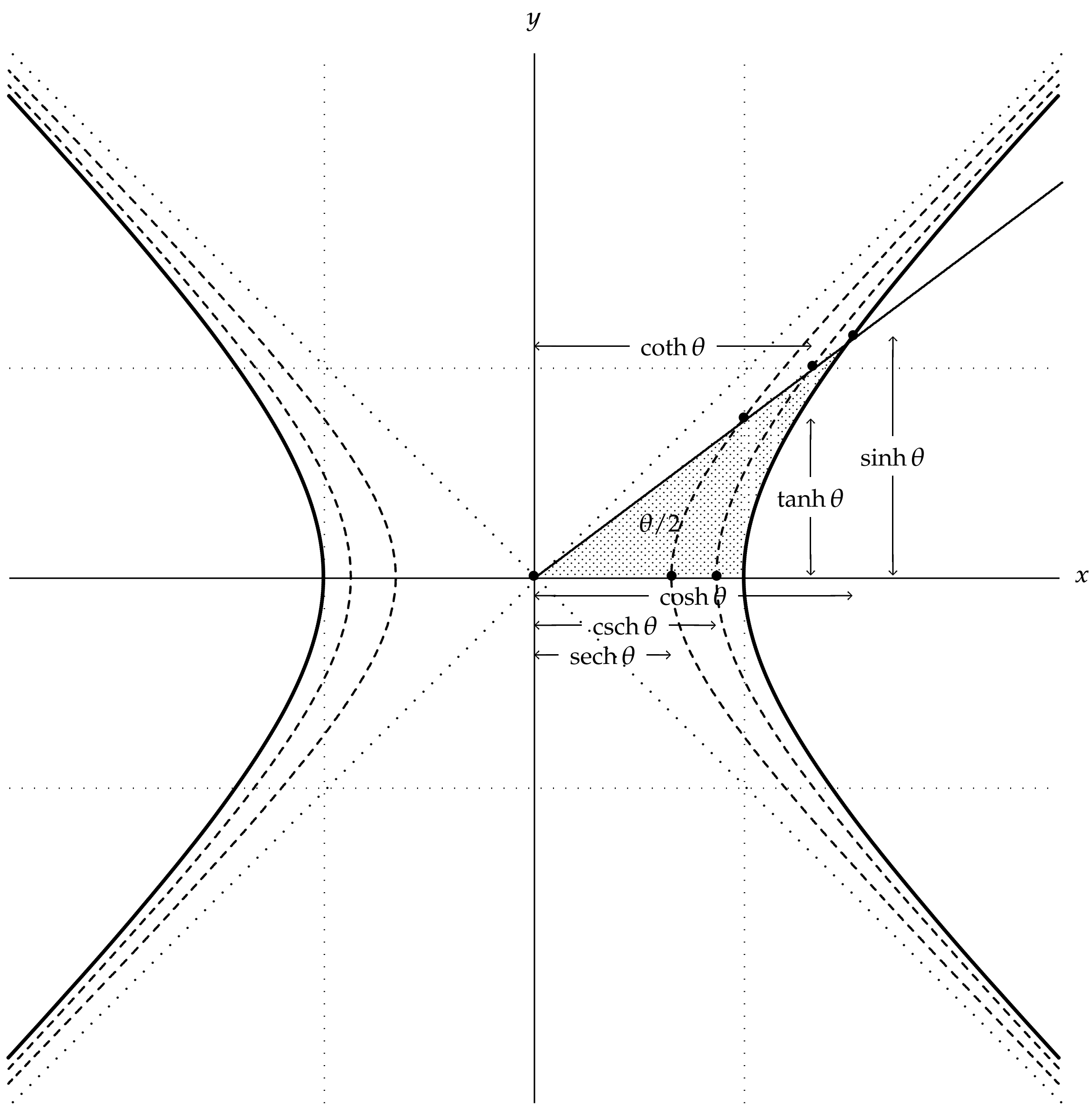


The equation $x^2 + y^2 = 1$ is a circle of radius 1. This circle is called the unit circle.

For any point on the circle, twice the shaded area is the circular angle θ.

The indicated lengths are the circular trigonometric functions of θ.

The unit hyperbola

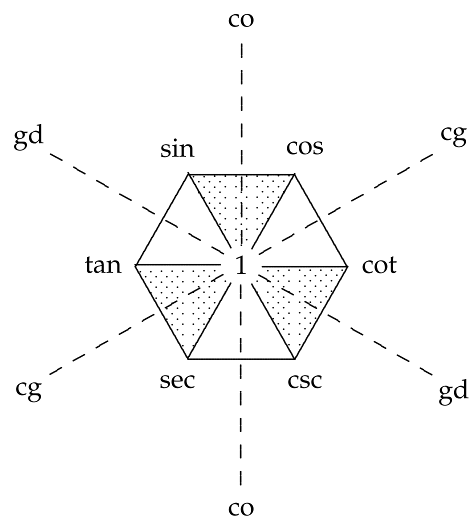


The equation $x^2 - y^2 = 1$ is a hyperbola bounded by a square and asymptotes on the corners of the square. This hyperbola is called the unit hyperbola.

For any point on the hyperbola, twice the shaded area is the hyperbolic angle θ.

The indicated lengths are the hyperbolic trigonometric functions of θ.

The trigonometric hexagon



The trigonometric hexagon relates elementary properties of the six circular and hyperbolic trigonometric functions. The points labelled sin, cos, tan, etc. are valid also for sinh, cosh, tanh, etc. The constant 1 at the center is treated as a function.

Each function is the product of its two neighbors: e.g. $\sin\theta = \tan\theta\cos\theta$ and $\sinh\theta = \tanh\theta\cosh\theta$.

Each function is the quotient of the two functions to the left and also of the two functions to the right: e.g. $\sin\theta = \dfrac{\cos\theta}{\cot\theta} = \dfrac{\tan\theta}{\sec\theta}$.

The product of two opposite functions is 1, which means that opposite functions are reciprocals: e.g. $\sin\theta\csc\theta = 1$, or $\sin\theta = \dfrac{1}{\csc\theta}$.

The three shaded triangles give the Pythagorean identities. For circular functions, the sum of the squares of the top two functions is equal to the square of the bottom function; for hyperbolic functions, the sum of the squares of the left and the bottom functions is equal to the square of the right function. For example, $\sin^2\theta + \cos^2\theta = 1$; $\sinh^2\theta + 1 = \cosh^2\theta$.

The three dashed axes give the inner transformations. Inner transformations reflect functions from one side of the axis to the other as follows:

The circular functions, when reflected across the co axis, become the circular functions on the other side, e.g. $\sin \operatorname{co}\theta = \cos\theta$.

The hyperbolic functions, when reflected across the cg axis, become the hyperbolic functions on the other side, e.g. $\sinh \operatorname{cg}\theta = \operatorname{csch}\theta$.

Either circular or hyperbolic functions, when reflected across the gd axis, become the function on the other side of the opposite type, i.e. circular becomes hyperbolic, and hyperbolic becomes circular. Examples: $\sinh \operatorname{gd}\theta = \tanh\theta$; $\sin \operatorname{gd}^{-1}\theta = \tan\theta$.

Euler identities

$$e^{\theta} = \cosh\theta + \sinh\theta = \exp\theta$$
$$e^{-\theta} = \cosh\theta - \sinh\theta = \frac{1}{\exp\theta}$$

$$e^{i\theta} = \cos\theta + i\sin\theta = \operatorname{cis}\theta$$
$$e^{-i\theta} = \cos\theta - i\sin\theta = \frac{1}{\operatorname{cis}\theta}$$

Exponential definitions

$$\cosh\theta = \frac{e^{\theta} + e^{-\theta}}{2}$$
$$\sinh\theta = \frac{e^{\theta} - e^{-\theta}}{2}$$
$$\tanh\theta = \frac{e^{\theta} - e^{-\theta}}{e^{\theta} + e^{-\theta}}$$
$$\coth\theta = \frac{e^{\theta} + e^{-\theta}}{e^{\theta} - e^{-\theta}}$$
$$\operatorname{sech}\theta = \frac{2}{e^{\theta} + e^{-\theta}}$$
$$\operatorname{csch}\theta = \frac{2}{e^{\theta} - e^{-\theta}}$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$
$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$
$$\tan\theta = -i\frac{e^{i\theta} - e^{-i\theta}}{e^{i\theta} + e^{-i\theta}}$$
$$\cot\theta = i\frac{e^{i\theta} + e^{-i\theta}}{e^{i\theta} - e^{-i\theta}}$$
$$\sec\theta = \frac{2}{e^{i\theta} + e^{-i\theta}}$$
$$\csc\theta = \frac{2i}{e^{i\theta} - e^{-i\theta}}$$

Inverse function log formulae

$$\cosh^{-1} u = \ln\left(u + \sqrt{u^2 - 1}\right)$$
$$\sinh^{-1} u = \ln\left(u + \sqrt{u^2 + 1}\right)$$
$$\tanh^{-1} u = \frac{1}{2}\ln\left(\frac{1+u}{1-u}\right)$$
$$\coth^{-1} u = \frac{1}{2}\ln\left(\frac{u+1}{u-1}\right)$$
$$\operatorname{sech}^{-1} u = \ln\left(\frac{1 + \sqrt{1-u^2}}{u}\right)$$
$$\operatorname{csch}^{-1} u = \ln\left(\frac{1 + \sqrt{1+u^2}}{u}\right)$$
$$\exp^{-1} u = \ln u$$

$$\cos^{-1} u = i\ln\left(u - i\sqrt{1-u^2}\right)$$
$$\sin^{-1} u = i\ln\left(\sqrt{1-u^2} - iu\right)$$
$$\tan^{-1} u = \frac{i}{2}\ln\left(\frac{1-iu}{1+iu}\right)$$
$$\cot^{-1} u = \frac{i}{2}\ln\left(\frac{iu+1}{iu-1}\right)$$
$$\sec^{-1} u = i\ln\left(\frac{1 - i\sqrt{u^2-1}}{u}\right)$$
$$\csc^{-1} u = i\ln\left(\frac{\sqrt{u^2-1} - i}{u}\right)$$
$$\operatorname{cis}^{-1} u = -i\ln u$$

Special values

θ	$-\infty$	0	$+\infty$
$\cosh\theta$	$+\infty$	$+1$	$+\infty$
$\sinh\theta$	$-\infty$	0	$+\infty$
$\tanh\theta$	-1	0	$+1$
$\coth\theta$	-1	$\pm\infty$	$+1$
$\operatorname{sech}\theta$	0	$+1$	0
$\operatorname{csch}\theta$	0	$\pm\infty$	0
$\exp\theta$	-1	$+1$	$+\infty$

θ	0	$\frac{\pi}{2}$	$\frac{\pi}{4}$	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{10}$	$\frac{\pi}{5}$	$\frac{3\pi}{10}$	$\frac{2\pi}{5}$
$\cos\theta$	1	0	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\frac{\sqrt{2+\phi}}{2}$	$\frac{\phi}{2}$	$\frac{\sqrt{3-\phi}}{2}$	$\frac{\phi^{-1}}{2}$
$\sin\theta$	0	1	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\phi^{-1}}{2}$	$\frac{\sqrt{3-\phi}}{2}$	$\frac{\phi}{2}$	$\frac{\sqrt{2+\phi}}{2}$
$\tan\theta$	0	$\pm\infty$	1	$\frac{1}{\sqrt{3}}$	$\sqrt{3}$	$\sqrt{7-4\phi}$	$\sqrt{7+4\phi}$	$\sqrt{\frac{3+4\phi}{5}}$	$\sqrt{3+4\phi}$
$\cot\theta$	$\pm\infty$	0	1	$\sqrt{3}$	$\frac{1}{\sqrt{3}}$	$\sqrt{3+4\phi}$	$\sqrt{\frac{3+4\phi}{5}}$	$\sqrt{7+4\phi}$	$\sqrt{7-4\phi}$
$\sec\theta$	1	$\pm\infty$	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	2	$2\sqrt{3-\phi}$	$2\phi^{-1}$	$2\sqrt{\frac{2+\phi}{5}}$	2ϕ
$\csc\theta$	$\pm\infty$	1	$\sqrt{2}$	2	$\frac{2}{\sqrt{3}}$	2ϕ	$2\sqrt{\frac{2+\phi}{5}}$	$2\phi^{-1}$	$2\sqrt{3-\phi}$

Signs

$\theta \in$	$(-\infty, 0)$	$(0, +\infty)$
$\cosh\theta$	$+$	$+$
$\sinh\theta$	$-$	$+$
$\tanh\theta$	$-$	$+$
$\coth\theta$	$-$	$+$
$\operatorname{sech}\theta$	$+$	$+$
$\operatorname{csch}\theta$	$-$	$+$
$\exp\theta$	$+$	$+$

$\theta \in$	$(0, \frac{\pi}{2})$	$(\frac{\pi}{2}, \pi)$	$(\pi, \frac{3\pi}{2})$	$(\frac{3\pi}{2}, 2\pi)$
$\cos\theta$	$+$	$-$	$-$	$+$
$\sin\theta$	$+$	$+$	$-$	$-$
$\tan\theta$	$+$	$-$	$+$	$-$
$\cot\theta$	$+$	$-$	$+$	$-$
$\sec\theta$	$+$	$-$	$-$	$+$
$\csc\theta$	$+$	$+$	$-$	$-$

Ranges

$\theta \in R \Rightarrow$
$+1 \leq \cosh\theta < +\infty$
$-\infty < \sinh\theta < +\infty$
$-1 < \tanh\theta < +1$
$\coth\theta < -1$ or $+1 < \coth\theta$
$0 < \operatorname{sech}\theta \leq +1$
$-\infty < \operatorname{csch}\theta < +\infty$, but $\operatorname{csch}\theta \neq 0$
$0 < \exp\theta$

$\theta \in R \Rightarrow$
$-1 \leq \cos\theta \leq +1$
$-1 \leq \sin\theta \leq +1$
$-\infty < \tan\theta < +\infty$
$-\infty < \cot\theta < +\infty$
$\sec\theta \leq -1$ or $+1 \leq \sec\theta$
$\csc\theta \leq -1$ or $+1 \leq \csc\theta$
$|\operatorname{cis}\theta| = 1$

Product formulae

$\cosh\theta = \sinh\theta\coth\theta$
$\sinh\theta = \cosh\theta\tanh\theta$
$\tanh\theta = \sinh\theta\,\mathrm{sech}\,\theta$
$\coth\theta = \cosh\theta\,\mathrm{csch}\,\theta$
$\mathrm{sech}\,\theta = \tanh\theta\,\mathrm{csch}\,\theta$
$\mathrm{csch}\,\theta = \coth\theta\,\mathrm{sech}\,\theta$
$\cosh\theta\,\mathrm{sech}\,\theta = \sinh\theta\,\mathrm{csch}\,\theta$
$\quad = \tanh\theta\coth\theta = \exp\theta\exp(-\theta) = 1$

$\cos\theta = \sin\theta\cot\theta$
$\sin\theta = \cos\theta\tan\theta$
$\tan\theta = \sin\theta\sec\theta$
$\cot\theta = \cos\theta\csc\theta$
$\sec\theta = \tan\theta\csc\theta$
$\csc\theta = \cot\theta\sec\theta$
$\cos\theta\sec\theta = \sin\theta\csc\theta$
$\quad = \tan\theta\cot\theta = \mathrm{cis}\,\theta\,\mathrm{cis}(-\theta) = 1$

Ratio formulae

$$\cosh\theta = \frac{1}{\mathrm{sech}\,\theta} = \frac{\sinh\theta}{\tanh\theta} = \frac{\coth\theta}{\mathrm{csch}\,\theta}$$
$$\sinh\theta = \frac{1}{\mathrm{csch}\,\theta} = \frac{\cosh\theta}{\coth\theta} = \frac{\tanh\theta}{\mathrm{sech}\,\theta}$$
$$\tanh\theta = \frac{1}{\coth\theta} = \frac{\sinh\theta}{\cosh\theta} = \frac{\mathrm{sech}\,\theta}{\mathrm{csch}\,\theta}$$
$$\coth\theta = \frac{1}{\tanh\theta} = \frac{\cosh\theta}{\sinh\theta} = \frac{\mathrm{csch}\,\theta}{\mathrm{sech}\,\theta}$$
$$\mathrm{sech}\,\theta = \frac{1}{\cosh\theta} = \frac{\tanh\theta}{\sinh\theta} = \frac{\mathrm{csch}\,\theta}{\coth\theta}$$
$$\mathrm{csch}\,\theta = \frac{1}{\sinh\theta} = \frac{\coth\theta}{\cosh\theta} = \frac{\mathrm{sech}\,\theta}{\tanh\theta}$$
$$\exp(-\theta) = \frac{1}{\exp\theta}$$

$$\cos\theta = \frac{1}{\sec\theta} = \frac{\sin\theta}{\tan\theta} = \frac{\cot\theta}{\csc\theta}$$
$$\sin\theta = \frac{1}{\csc\theta} = \frac{\cos\theta}{\cot\theta} = \frac{\tan\theta}{\sec\theta}$$
$$\tan\theta = \frac{1}{\cot\theta} = \frac{\sin\theta}{\cos\theta} = \frac{\sec\theta}{\csc\theta}$$
$$\cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta} = \frac{\csc\theta}{\sec\theta}$$
$$\sec\theta = \frac{1}{\cos\theta} = \frac{\tan\theta}{\sin\theta} = \frac{\csc\theta}{\cot\theta}$$
$$\csc\theta = \frac{1}{\sin\theta} = \frac{\cot\theta}{\cos\theta} = \frac{\sec\theta}{\tan\theta}$$
$$\mathrm{cis}(-\theta) = \frac{1}{\mathrm{cis}\,\theta}$$

Inverse function reciprocal formulae

$$\cosh^{-1} u = \mathrm{sech}^{-1}\frac{1}{u}$$
$$\sinh^{-1} u = \mathrm{csch}^{-1}\frac{1}{u}$$
$$\tanh^{-1} u = \coth^{-1}\frac{1}{u}$$
$$\coth^{-1} u = \tanh^{-1}\frac{1}{u}$$
$$\mathrm{sech}^{-1} u = \cosh^{-1}\frac{1}{u}$$
$$\mathrm{csch}^{-1} u = \sinh^{-1}\frac{1}{u}$$
$$\exp^{-1} u = -\exp^{-1}\frac{1}{u}$$

$$\cos^{-1} u = \sec^{-1}\frac{1}{u}$$
$$\sin^{-1} u = \csc^{-1}\frac{1}{u}$$
$$\tan^{-1} u = \cot^{-1}\frac{1}{u}$$
$$\cot^{-1} u = \tan^{-1}\frac{1}{u}$$
$$\sec^{-1} u = \cos^{-1}\frac{1}{u}$$
$$\csc^{-1} u = \sin^{-1}\frac{1}{u}$$
$$\mathrm{cis}^{-1} u = -\,\mathrm{cis}^{-1}\frac{1}{u}$$

Pythagorean identities

$$\begin{aligned}1 &= \cosh^2\theta - \sinh^2\theta \\ &= \tanh^2\theta + \operatorname{sech}^2\theta = \coth^2\theta - \operatorname{csch}^2\theta \\ \cosh^2\theta &= \sinh^2\theta + 1 \\ \sinh^2\theta &= \cosh^2\theta - 1 \\ \tanh^2\theta &= 1 - \operatorname{sech}^2\theta \\ \coth^2\theta &= \operatorname{csch}^2\theta + 1 \\ \operatorname{sech}^2\theta &= 1 - \tanh^2\theta \\ \operatorname{csch}^2\theta &= \coth^2\theta - 1 \\ (\coth\theta - \tanh\theta)^2 &= \operatorname{sech}^2\theta - \operatorname{csch}^2\theta \\ \coth\theta - \tanh\theta &= \operatorname{sech}\theta\operatorname{csch}\theta \\ \operatorname{sech}^2\theta - \operatorname{csch}^2\theta &= \operatorname{sech}^2\theta\operatorname{csch}^2\theta\end{aligned}$$

$$\begin{aligned}1 &= \cos^2\theta + \sin^2\theta \\ &= \sec^2\theta - \tanh^2\theta = \csc^2\theta - \cot^2\theta \\ \cos^2\theta &= 1 - \sin^2\theta \\ \sin^2\theta &= 1 - \cos^2\theta \\ \tan^2\theta &= \sec^2\theta - 1 \\ \cot^2\theta &= \csc^2\theta - 1 \\ \sec^2\theta &= 1 + \tan^2\theta \\ \csc^2\theta &= \cot^2\theta + 1 \\ (\cot\theta + \tan\theta)^2 &= \sec^2\theta + \csc^2\theta \\ \cot\theta + \tan\theta &= \sec\theta\csc\theta \\ \sec^2\theta + \csc^2\theta &= \sec^2\theta\csc^2\theta\end{aligned}$$

Pythagorean conversions

$$\begin{aligned}\cosh\theta &= \sqrt{1+\sinh^2\theta} = \frac{1}{\sqrt{1-\tanh^2\theta}} \\ &= \frac{\coth\theta}{\sqrt{\coth^2\theta-1}} = \frac{\sqrt{\operatorname{csch}^2\theta-1}}{\operatorname{csch}\theta} \\ &= \frac{\exp^2\theta+1}{2\exp\theta} \\ \sinh\theta &= \sqrt{\cosh^2\theta-1} = \frac{\tanh\theta}{\sqrt{1-\tanh^2\theta}} \\ &= \frac{1}{\sqrt{\coth^2\theta-1}} = \frac{\sqrt{1-\operatorname{sech}^2\theta}}{\operatorname{sech}\theta} \\ &= \frac{\exp^2\theta-1}{2\exp\theta} \\ \tanh\theta &= \frac{\sqrt{\cosh^2\theta-1}}{\cosh\theta} = \frac{\sinh\theta}{\sqrt{1+\sinh^2\theta}} \\ &= \sqrt{1-\operatorname{sech}^2\theta} = \frac{1}{\sqrt{\operatorname{csch}^2\theta+1}} \\ &= \frac{\exp^2\theta-1}{\exp^2\theta+1} \\ \coth\theta &= \frac{\cosh\theta}{\sqrt{\cosh^2\theta-1}} = \frac{\sqrt{1+\sinh^2\theta}}{\sinh\theta} \\ &= \frac{1}{\sqrt{1-\operatorname{sech}^2\theta}} = \sqrt{\operatorname{csch}^2\theta+1} \\ &= \frac{\exp^2\theta+1}{\exp^2\theta-1}\end{aligned}$$

$$\begin{aligned}\cos\theta &= \sqrt{1-\sin^2\theta} = \frac{1}{\sqrt{1+\tan^2\theta}} \\ &= \frac{\cot\theta}{\sqrt{1+\cot^2\theta}} = \frac{\sqrt{\csc^2\theta-1}}{\csc\theta} \\ &= \frac{\operatorname{cis}^2\theta+1}{2\operatorname{cis}\theta} \\ \sin\theta &= \sqrt{1-\cos^2\theta} = \frac{\tan\theta}{\sqrt{1+\tan^2\theta}} \\ &= \frac{1}{\sqrt{1+\cot^2\theta}} = \frac{\sqrt{\sec^2\theta-1}}{\sec\theta} \\ &= \frac{\operatorname{cis}^2\theta-1}{2i\operatorname{cis}\theta} \\ \tan\theta &= \frac{\sqrt{1-\cos^2\theta}}{\cos\theta} = \frac{\sin\theta}{\sqrt{1-\sin^2\theta}} \\ &= \sqrt{\sec^2\theta-1} = \frac{1}{\sqrt{\csc^2\theta-1}} \\ &= -i\frac{\operatorname{cis}^2\theta-1}{\operatorname{cis}^2\theta+1} \\ \cot\theta &= \frac{\cos\theta}{\sqrt{1-\cos^2\theta}} = \frac{\sqrt{1-\sin^2\theta}}{\sin\theta} \\ &= \frac{1}{\sqrt{\sec^2\theta-1}} = \sqrt{\csc^2\theta-1} \\ &= i\frac{\operatorname{cis}^2\theta+1}{\operatorname{cis}^2\theta-1}\end{aligned}$$

$$\begin{aligned}\operatorname{sech}\theta &= \frac{1}{\sqrt{1+\sinh^2\theta}} = \sqrt{1-\tanh^2\theta}\\ &= \frac{\sqrt{\coth^2\theta-1}}{\coth\theta} = \frac{\operatorname{csch}\theta}{\sqrt{\operatorname{csch}^2\theta-1}}\\ &= \frac{2\exp\theta}{\exp^2\theta+1}\end{aligned}$$

$$\begin{aligned}\sec\theta &= \frac{1}{\sqrt{1-\sin^2\theta}} = \sqrt{1+\tan^2\theta}\\ &= \frac{\sqrt{1+\cot^2\theta}}{\cot\theta} = \frac{\csc\theta}{\sqrt{\csc^2\theta-1}}\\ &= \frac{2\operatorname{cis}\theta}{\operatorname{cis}^2\theta+1}\end{aligned}$$

$$\begin{aligned}\operatorname{csch}\theta &= \frac{1}{\sqrt{\cosh^2\theta-1}} = \frac{\sqrt{1-\tanh^2\theta}}{\tanh\theta}\\ &= \frac{1}{\sqrt{\coth^2\theta-1}} = \frac{\operatorname{sech}\theta}{\sqrt{1-\operatorname{sech}^2\theta}}\\ &= \frac{2\exp\theta}{\exp^2\theta-1}\end{aligned}$$

$$\begin{aligned}\csc\theta &= \frac{1}{\sqrt{1-\cos^2\theta}} = \frac{\sqrt{1+\tan^2\theta}}{\tan\theta}\\ &= \sqrt{1+\cot^2\theta} = \frac{\sec\theta}{\sqrt{\sec^2\theta-1}}\\ &= \frac{2i\operatorname{cis}\theta}{\operatorname{cis}^2\theta-1}\end{aligned}$$

$$\begin{aligned}\exp\theta &= \cosh\theta + \sqrt{\cosh^2\theta-1}\\ &= \sqrt{\sinh^2\theta+1}+\sinh\theta\\ &= \frac{1+\tanh\theta}{\sqrt{1-\tanh^2\theta}} = \frac{\coth\theta+1}{\sqrt{\coth^2\theta-1}}\\ &= \frac{1+\sqrt{1-\operatorname{sech}^2\theta}}{\operatorname{sech}\theta} = \frac{\sqrt{\operatorname{csch}^2\theta-1}+1}{\operatorname{csch}\theta}\end{aligned}$$

$$\begin{aligned}\operatorname{cis}\theta &= \cos\theta + i\sqrt{1-\cos^2\theta}\\ &= \sqrt{1-\sin^2\theta}+i\sin\theta\\ &= \frac{1+i\tan\theta}{\sqrt{1+\tan^2\theta}} = \frac{\cot\theta+i}{\sqrt{1+\cot^2\theta}}\\ &= \frac{1+i\sqrt{\sec^2\theta-1}}{\sec\theta} = \frac{\sqrt{\csc^2\theta-1}+i}{\csc\theta}\end{aligned}$$

Inverse function Pythagorean conversions

$$\begin{aligned}\cosh^{-1}u &= \sinh^{-1}\sqrt{u^2-1} = \tanh^{-1}\frac{\sqrt{u^2-1}}{u}\\ &= \coth^{-1}\frac{u}{\sqrt{u^2-1}} = \operatorname{csch}^{-1}\frac{1}{\sqrt{u^2-1}}\\ &= \exp^{-1}\left(u+\sqrt{u^2-1}\right)\end{aligned}$$

$$\begin{aligned}\cos^{-1}u &= \sin^{-1}\sqrt{1-u^2} = \tan^{-1}\frac{\sqrt{1-u^2}}{u}\\ &= \cot^{-1}\frac{u}{\sqrt{1-u^2}} = \csc^{-1}\frac{1}{\sqrt{1-u^2}}\\ &= \operatorname{cis}^{-1}\left(u+i\sqrt{1-u^2}\right)\end{aligned}$$

$$\begin{aligned}\sinh^{-1}u &= \cosh^{-1}\sqrt{1+u^2} = \tanh^{-1}\frac{u}{\sqrt{1+u^2}}\\ &= \coth^{-1}\frac{\sqrt{1+u^2}}{u} = \operatorname{sech}^{-1}\frac{1}{\sqrt{1+u^2}}\\ &= \exp^{-1}\left(u+\sqrt{1+u^2}\right)\end{aligned}$$

$$\begin{aligned}\sin^{-1}u &= \cos^{-1}\sqrt{1-u^2} = \tan^{-1}\frac{u}{\sqrt{1-u^2}}\\ &= \cot^{-1}\frac{\sqrt{1-u^2}}{u} = \sec^{-1}\frac{1}{\sqrt{1-u^2}}\\ &= \operatorname{cis}^{-1}\left(\sqrt{1-u^2}+iu\right)\end{aligned}$$

$$\begin{aligned}\tanh^{-1}u &= \cosh^{-1}\frac{1}{\sqrt{1-u^2}} = \sinh^{-1}\frac{u}{\sqrt{1-u^2}}\\ &= \operatorname{sech}^{-1}\sqrt{1-u^2} = \operatorname{csch}^{-1}\frac{\sqrt{1-u^2}}{u}\\ &= \exp^{-1}\sqrt{\frac{1+u}{1-u}}\end{aligned}$$

$$\begin{aligned}\tan^{-1}u &= \cos^{-1}\frac{1}{\sqrt{1+u^2}} = \sin^{-1}\frac{u}{\sqrt{1+u^2}}\\ &= \sec^{-1}\sqrt{1+u^2} = \csc^{-1}\frac{\sqrt{1+u^2}}{u}\\ &= \operatorname{cis}^{-1}\sqrt{\frac{1+iu}{1-iu}}\end{aligned}$$

$$\begin{aligned}\coth^{-1}u &= \cosh^{-1}\frac{u}{\sqrt{u^2-1}} = \sinh^{-1}\frac{1}{\sqrt{u^2-1}}\\ &= \operatorname{sech}^{-1}\frac{\sqrt{u^2-1}}{u} = \operatorname{csch}^{-1}\sqrt{u^2-1}\\ &= \exp^{-1}\sqrt{\frac{u-1}{u+1}}\end{aligned}$$

$$\begin{aligned}\cot^{-1}u &= \cos^{-1}\frac{u}{\sqrt{1+u^2}} = \sin^{-1}\frac{1}{\sqrt{1+u^2}}\\ &= \sec^{-1}\frac{\sqrt{1+u^2}}{u} = \csc^{-1}\sqrt{1+u^2}\\ &= \operatorname{cis}^{-1}\sqrt{\frac{iu-1}{iu+1}}\end{aligned}$$

$$
\begin{aligned}
\operatorname{sech}^{-1} u &= \sinh^{-1} \frac{\sqrt{1-u^2}}{u} = \tanh^{-1} \sqrt{1-u^2} \\
&= \coth^{-1} \frac{1}{\sqrt{1-u^2}} = \operatorname{csch}^{-1} \frac{u}{\sqrt{1-u^2}} \\
&= \exp^{-1} \frac{1+\sqrt{u^2-1}}{u}
\end{aligned}
$$

$$
\begin{aligned}
\operatorname{csch}^{-1} u &= \cosh^{-1} \frac{\sqrt{u^2+1}}{u} = \tanh^{-1} \frac{1}{\sqrt{u^2+1}} \\
&= \coth^{-1} \sqrt{u^2+1} = \operatorname{sech}^{-1} \frac{u}{\sqrt{u^2+1}} \\
&= \exp^{-1} \frac{1+\sqrt{1+u^2}}{u}
\end{aligned}
$$

$$
\begin{aligned}
\exp^{-1} u &= \cosh^{-1} \frac{u^2+1}{2u} = \sinh^{-1} \frac{u^2-1}{2u} \\
&= \tanh^{-1} \frac{u^2-1}{u^2+1} = \coth^{-1} \frac{u^2+1}{u^2-1} \\
&= \operatorname{sech}^{-1} \frac{2u}{u^2+1} = \operatorname{csch}^{-1} \frac{2u}{u^2-1}
\end{aligned}
$$

$$
\begin{aligned}
\sec^{-1} u &= \sin^{-1} \frac{\sqrt{u^2-1}}{u} = \tan^{-1} \sqrt{u^2-1} \\
&= \cot^{-1} \frac{1}{\sqrt{u^2-1}} = \csc^{-1} \frac{u}{\sqrt{u^2-1}} \\
&= \operatorname{cis}^{-1} \frac{1+i\sqrt{u^2-1}}{u}
\end{aligned}
$$

$$
\begin{aligned}
\csc^{-1} u &= \cos^{-1} \frac{\sqrt{u^2-1}}{u} = \tan^{-1} \frac{1}{\sqrt{u^2-1}} \\
&= \cot^{-1} \sqrt{u^2-1} = \sec^{-1} \frac{u}{\sqrt{u^2-1}} \\
&= \operatorname{cis}^{-1} \frac{\sqrt{u^2-1}+i}{u}
\end{aligned}
$$

$$
\begin{aligned}
\operatorname{cis}^{-1} u &= \cos^{-1} \frac{u^2+1}{2u} = \sin^{-1} \frac{u^2-1}{2iu} \\
&= \tan^{-1} \frac{u^2-1}{i(u^2+1)} = \cot^{-1} \frac{i(u^2+1)}{u^2-1} \\
&= \sec^{-1} \frac{2u}{u^2+1} = \csc^{-1} \frac{2iu}{u^2-1}
\end{aligned}
$$

Imaginary angle formulae

$\cosh i\theta = \cos\theta$
$\sinh i\theta = i\sin\theta$
$\tanh i\theta = i\tan\theta$
$\coth i\theta = -i\cot\theta$
$\operatorname{sech} i\theta = \sec\theta$
$\operatorname{csch} i\theta = -i\csc\theta$
$\exp i\theta = \operatorname{cis}\theta$
$\exp\theta = (\operatorname{cis}\theta)^{-i}$

$\cos i\theta = \cosh\theta$
$\sin i\theta = i\sinh\theta$
$\tan i\theta = i\tanh\theta$
$\cot i\theta = -i\coth\theta$
$\sec i\theta = \operatorname{sech}\theta$
$\csc i\theta = -i\operatorname{csch}\theta$
$\operatorname{cis} i\theta = \dfrac{1}{\exp\theta}$
$\operatorname{cis}\theta = \left(\exp\theta\right)^{i}$

Inverse function imaginary angle formulae

$i\cosh^{-1} u = \cos^{-1} u$
$i\sinh^{-1} u = \sin^{-1} iu$
$i\tanh^{-1} u = \tan^{-1} iu$
$i\coth^{-1} u = -\cot^{-1} iu$
$i\operatorname{sech}^{-1} u = \sec^{-1} u$
$i\operatorname{csch}^{-1} u = -\csc^{-1} iu$

$i\cos^{-1} u = \cosh^{-1} u$
$i\sin^{-1} u = \sinh^{-1} iu$
$i\tan^{-1} u = \tanh^{-1} iu$
$i\cot^{-1} u = -\coth^{-1} iu$
$i\sec^{-1} u = \operatorname{sech}^{-1} u$
$i\csc^{-1} u = -\operatorname{csch}^{-1} iu$

Negative angle formulae

$$\cosh(-\theta) = \cosh\theta$$
$$\sinh(-\theta) = -\sinh\theta$$
$$\tanh(-\theta) = -\tanh\theta$$
$$\coth(-\theta) = -\coth\theta$$
$$\operatorname{sech}(-\theta) = \operatorname{sech}\theta$$
$$\operatorname{csch}(-\theta) = -\operatorname{csch}\theta$$
$$\exp(-\theta) = \frac{1}{\exp\theta}$$

$$\cos(-\theta) = \cos\theta$$
$$\sin(-\theta) = -\sin\theta$$
$$\tan(-\theta) = -\tan\theta$$
$$\cot(-\theta) = -\cot\theta$$
$$\sec(-\theta) = \sec\theta$$
$$\csc(-\theta) = -\csc\theta$$
$$\operatorname{cis}(-\theta) = \frac{1}{\operatorname{cis}\theta}$$

Inverse function negative angle formulae

$$\cosh^{-1}(-u) = \pi i - \cosh^{-1} u$$
$$\sinh^{-1}(-u) = -\sinh^{-1} u$$
$$\tanh^{-1}(-u) = -\tanh^{-1} u$$
$$\coth^{-1}(-u) = -\coth^{-1} u$$
$$\operatorname{sech}^{-1}(-u) = \pi i - \operatorname{sech}^{-1} u$$
$$\operatorname{csch}^{-1}(-u) = -\operatorname{csch}^{-1} u$$
$$\exp^{-1}(-u) = \pi i + \exp^{-1} u$$

$$\cos^{-1}(-u) = \pi - \cos^{-1} u$$
$$\sin^{-1}(-u) = -\sin^{-1} u$$
$$\tan^{-1}(-u) = -\tan^{-1} u$$
$$\cot^{-1}(-u) = -\cot^{-1} u$$
$$\sec^{-1}(-u) = \pi - \sec^{-1} u$$
$$\csc^{-1}(-u) = -\csc^{-1} u$$
$$\operatorname{cis}^{-1}(-u) = \pi + \operatorname{cis}^{-1} u$$

Double angles

$$\begin{aligned}\cosh 2\theta &= \cosh^2\theta + \sinh^2\theta \\ &= 2\cosh^2\theta - 1 = 1 + 2\sinh^2\theta\end{aligned}$$
$$\begin{aligned}\sinh 2\theta &= 2\cosh\theta\sinh\theta \\ &= i\left(1 - (\cosh\theta + i\sinh\theta)^2\right)\end{aligned}$$
$$\tanh 2\theta = \frac{2\tanh\theta}{1 + \tanh^2\theta}$$
$$\coth 2\theta = \frac{\coth^2\theta + 1}{2\coth\theta}$$
$$\operatorname{sech} 2\theta = \frac{\operatorname{csch}^2\theta\operatorname{sech}^2\theta}{\operatorname{csch}^2\theta + \operatorname{sech}^2\theta}$$
$$\operatorname{csch} 2\theta = \frac{\operatorname{csch}\theta\operatorname{sech}\theta}{2}$$
$$\exp 2\theta = \exp^2\theta$$

$$\begin{aligned}\cos 2\theta &= \cos^2\theta - \sin^2\theta \\ &= 2\cos^2\theta - 1 = 1 - 2\sin^2\theta\end{aligned}$$
$$\begin{aligned}\sin 2\theta &= 2\cos\theta\sin\theta \\ &= (\cos\theta + \sin\theta)^2 - 1\end{aligned}$$
$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$$
$$\cot 2\theta = \frac{\cot^2\theta - 1}{2\cot\theta}$$
$$\sec 2\theta = \frac{\csc^2\theta\sec^2\theta}{\csc^2\theta - \sec^2\theta}$$
$$\csc 2\theta = \frac{\csc\theta\sec\theta}{2}$$
$$\operatorname{cis} 2\theta = \operatorname{cis}^2\theta$$

Half angles

$$\cosh\frac{\theta}{2} = \sqrt{\frac{\cosh\theta + 1}{2}}$$
$$\sinh\frac{\theta}{2} = \sqrt{\frac{\cosh\theta - 1}{2}}$$
$$\tanh\frac{\theta}{2} = \frac{\sinh\theta}{\cosh\theta + 1} = \frac{\cosh\theta - 1}{\sinh\theta} = \coth\theta - \operatorname{csch}\theta$$
$$\coth\frac{\theta}{2} = \frac{\sinh\theta}{\cosh\theta - 1} = \frac{\cosh\theta + 1}{\sinh\theta} = \coth\theta + \operatorname{csch}\theta$$
$$\operatorname{sech}\frac{\theta}{2} = \frac{\sqrt{2}\operatorname{sech}\theta}{1 + \operatorname{sech}\theta}$$
$$\operatorname{csch}\frac{\theta}{2} = \frac{\sqrt{2}\operatorname{sech}\theta}{1 - \operatorname{sech}\theta}$$
$$\exp\frac{\theta}{2} = \sqrt{\exp\theta}$$

$$\cos\frac{\theta}{2} = \sqrt{\frac{1 + \cos\theta}{2}}$$
$$\sin\frac{\theta}{2} = \sqrt{\frac{1 - \cos\theta}{2}}$$
$$\tan\frac{\theta}{2} = \frac{\sin\theta}{1 + \cos\theta} = \frac{1 - \cos\theta}{\sin\theta} = \csc\theta - \cot\theta$$
$$\cot\frac{\theta}{2} = \frac{\sin\theta}{1 - \cos\theta} = \frac{1 + \cos\theta}{\sin\theta} = \csc\theta + \cot\theta$$
$$\sec\frac{\theta}{2} = \frac{\sqrt{2}\sec\theta}{\sec\theta + 1}$$
$$\csc\frac{\theta}{2} = \frac{\sqrt{2}\sec\theta}{\sec\theta - 1}$$
$$\operatorname{cis}\frac{\theta}{2} = \sqrt{\operatorname{cis}\theta}$$

Multiple-value identities

$$\cosh(\theta + 2\pi i) = \cosh\theta$$
$$\sinh(\theta + 2\pi i) = \sinh\theta$$
$$\tanh(\theta + 2\pi i) = \tanh\theta$$
$$\coth(\theta + 2\pi i) = \coth\theta$$
$$\operatorname{sech}(\theta + 2\pi i) = \operatorname{sech}\theta$$
$$\operatorname{csch}(\theta + 2\pi i) = \operatorname{csch}\theta$$
$$\exp(\theta + 2\pi i) = \exp\theta$$

$$\cos(\theta + 2\pi) = \cos\theta$$
$$\sin(\theta + 2\pi) = \sin\theta$$
$$\tan(\theta + 2\pi) = \tan\theta$$
$$\cot(\theta + 2\pi) = \cot\theta$$
$$\sec(\theta + 2\pi) = \sec\theta$$
$$\csc(\theta + 2\pi) = \csc\theta$$
$$\operatorname{cis}(\theta + 2\pi) = \operatorname{cis}\theta$$

Straight-angle translations

$$\cosh(\theta + \pi i) = \cosh(\theta - \pi i) = -\cosh\theta$$
$$\sinh(\theta + \pi i) = \sinh(\theta - \pi i) = -\sinh\theta$$
$$\tanh(\theta + \pi i) = \tanh(\theta - \pi i) = \tanh\theta$$
$$\coth(\theta + \pi i) = \coth(\theta - \pi i) = \coth\theta$$
$$\operatorname{sech}(\theta + \pi i) = \operatorname{sech}(\theta - \pi i) = -\operatorname{sech}\theta$$
$$\operatorname{csch}(\theta + \pi i) = \operatorname{csch}(\theta - \pi i) = -\operatorname{csch}\theta$$
$$\exp(\theta + \pi i) = \exp(\theta - \pi i) = -\exp\theta$$

$$\cos(\theta + \pi) = \cos(\theta - \pi) = -\cos\theta$$
$$\sin(\theta + \pi) = \sin(\theta - \pi) = -\sin\theta$$
$$\tan(\theta + \pi) = \tan(\theta - \pi) = \tan\theta$$
$$\cot(\theta + \pi) = \cot(\theta - \pi) = \cot\theta$$
$$\sec(\theta + \pi) = \sec(\theta - \pi) = -\sec\theta$$
$$\csc(\theta + \pi) = \csc(\theta - \pi) = -\csc\theta$$
$$\operatorname{cis}(\theta + \pi) = \operatorname{cis}(\theta - \pi) = -\operatorname{cis}\theta$$

Straight-angle reflections

$$\cosh(\pi i - \theta) = -\cosh\theta$$
$$\sinh(\pi i - \theta) = \sinh\theta$$
$$\tanh(\pi i - \theta) = -\tanh\theta$$
$$\coth(\pi i - \theta) = -\coth\theta$$
$$\operatorname{sech}(\pi i - \theta) = -\operatorname{sech}\theta$$
$$\operatorname{csch}(\pi i - \theta) = \operatorname{csch}\theta$$
$$\exp(\pi i - \theta) = \frac{-1}{\exp\theta}$$

$$\cos(\pi - \theta) = -\cos\theta$$
$$\sin(\pi - \theta) = \sin\theta$$
$$\tan(\pi - \theta) = -\tan\theta$$
$$\cot(\pi - \theta) = -\cot\theta$$
$$\sec(\pi - \theta) = -\sec\theta$$
$$\csc(\pi - \theta) = \csc\theta$$
$$\operatorname{cis}(\pi - \theta) = \frac{-1}{\operatorname{cis}\theta}$$

Right-angle translations

$$\cosh(\theta + \tfrac{\pi i}{2}) = i\sinh\theta$$
$$\sinh(\theta + \tfrac{\pi i}{2}) = i\cosh\theta$$
$$\tanh(\theta + \tfrac{\pi i}{2}) = \coth\theta$$
$$\coth(\theta + \tfrac{\pi i}{2}) = \tanh\theta$$
$$\operatorname{sech}(\theta + \tfrac{\pi i}{2}) = -i\operatorname{csch}\theta$$
$$\operatorname{csch}(\theta + \tfrac{\pi i}{2}) = -i\operatorname{sech}\theta$$
$$\exp(\theta + \tfrac{\pi i}{2}) = i\exp\theta$$

$$\cos(\theta + \tfrac{\pi}{2}) = -\sin\theta$$
$$\sin(\theta + \tfrac{\pi}{2}) = \cos\theta$$
$$\tan(\theta + \tfrac{\pi}{2}) = -\cot\theta$$
$$\cot(\theta + \tfrac{\pi}{2}) = -\tan\theta$$
$$\sec(\theta + \tfrac{\pi}{2}) = -\csc\theta$$
$$\csc(\theta + \tfrac{\pi}{2}) = \sec\theta$$
$$\operatorname{cis}(\theta + \tfrac{\pi}{2}) = i\operatorname{cis}\theta$$

$$\cosh(\theta - \tfrac{\pi i}{2}) = -i\sinh\theta$$
$$\sinh(\theta - \tfrac{\pi i}{2}) = -i\cosh\theta$$
$$\tanh(\theta - \tfrac{\pi i}{2}) = \coth\theta$$
$$\coth(\theta - \tfrac{\pi i}{2}) = \tanh\theta$$
$$\operatorname{sech}(\theta - \tfrac{\pi i}{2}) = i\operatorname{csch}\theta$$
$$\operatorname{csch}(\theta - \tfrac{\pi i}{2}) = i\operatorname{sech}\theta$$
$$\exp(\theta - \tfrac{\pi i}{2}) = i\exp\theta$$

$$\cos(\theta - \tfrac{\pi}{2}) = \sin\theta$$
$$\sin(\theta - \tfrac{\pi}{2}) = -\cos\theta$$
$$\tan(\theta - \tfrac{\pi}{2}) = -\cot\theta$$
$$\cot(\theta - \tfrac{\pi}{2}) = -\tan\theta$$
$$\sec(\theta - \tfrac{\pi}{2}) = \csc\theta$$
$$\csc(\theta - \tfrac{\pi}{2}) = -\sec\theta$$
$$\operatorname{cis}(\theta - \tfrac{\pi}{2}) = i\operatorname{cis}\theta$$

Right-angle reflections

$$\cosh(\tfrac{\pi i}{2} - \theta) = -i\sinh\theta$$
$$\sinh(\tfrac{\pi i}{2} - \theta) = i\cosh\theta$$
$$\tanh(\tfrac{\pi i}{2} - \theta) = -\coth\theta$$
$$\coth(\tfrac{\pi i}{2} - \theta) = -\tanh\theta$$
$$\operatorname{sech}(\tfrac{\pi i}{2} - \theta) = i\operatorname{csch}\theta$$
$$\operatorname{csch}(\tfrac{\pi i}{2} - \theta) = -i\operatorname{sech}\theta$$
$$\exp(\tfrac{\pi i}{2} - \theta) = \frac{i}{\exp\theta}$$

$$\cos(\tfrac{\pi}{2} - \theta) = \sin\theta$$
$$\sin(\tfrac{\pi}{2} - \theta) = \cos\theta$$
$$\tan(\tfrac{\pi}{2} - \theta) = \cot\theta$$
$$\cot(\tfrac{\pi}{2} - \theta) = \tan\theta$$
$$\sec(\tfrac{\pi}{2} - \theta) = \csc\theta$$
$$\csc(\tfrac{\pi}{2} - \theta) = \sec\theta$$
$$\operatorname{cis}(\tfrac{\pi}{2} - \theta) = \frac{i}{\operatorname{cis}\theta}$$

Half-right-angle translations

$$\cosh\left(\theta + \frac{\pi i}{4}\right) = -i\sinh\left(\frac{\pi i}{4} - \theta\right) = \frac{\cosh\theta + i\sinh\theta}{\sqrt{2}}$$

$$\sinh\left(\theta + \frac{\pi i}{4}\right) = -i\cosh\left(\frac{\pi i}{4} - \theta\right) = \frac{i\cosh\theta + \sinh\theta}{\sqrt{2}}$$

$$\tanh\left(\theta + \frac{\pi i}{4}\right) = \coth\left(\frac{\pi i}{4} - \theta\right) = \frac{i + \tanh\theta}{1 + i\tanh\theta}$$

$$\coth\left(\theta + \frac{\pi i}{4}\right) = \tanh\left(\frac{\pi i}{4} - \theta\right) = \frac{\coth\theta + i}{i\coth\theta + 1}$$

$$\operatorname{sech}\left(\theta + \frac{\pi i}{4}\right) = i\operatorname{csch}\left(\frac{\pi i}{4} - \theta\right) = \frac{2\sqrt{2}\operatorname{csch}2\theta}{\operatorname{csch}\theta + i\operatorname{sech}\theta}$$

$$\operatorname{csch}\left(\theta + \frac{\pi i}{4}\right) = i\operatorname{sech}\left(\frac{\pi i}{4} - \theta\right) = \frac{2\sqrt{2}\operatorname{csch}2\theta}{i\operatorname{csch}\theta + \operatorname{sech}\theta}$$

$$\exp\left(\theta + \frac{\pi i}{4}\right) = \frac{1}{\exp\left(\frac{\pi i}{4} - \theta\right)} = \frac{(1+i)\exp\theta}{\sqrt{2}}$$

$$\cos\left(\theta + \frac{\pi}{4}\right) = \sin\left(\frac{\pi}{4} - \theta\right) = \frac{\cos\theta - \sin\theta}{\sqrt{2}}$$

$$\sin\left(\theta + \frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4} - \theta\right) = \frac{\cos\theta + \sin\theta}{\sqrt{2}}$$

$$\tan\left(\theta + \frac{\pi}{4}\right) = \cot\left(\frac{\pi}{4} - \theta\right) = \frac{1 + \tan\theta}{1 - \tan\theta}$$

$$\cot\left(\theta + \frac{\pi}{4}\right) = \tan\left(\frac{\pi}{4} - \theta\right) = \frac{\cot\theta - 1}{\cot\theta + 1}$$

$$\sec\left(\theta + \frac{\pi}{4}\right) = \csc\left(\frac{\pi}{4} - \theta\right) = \frac{2\sqrt{2}\csc 2\theta}{\csc\theta - \sec\theta}$$

$$\csc\left(\theta + \frac{\pi}{4}\right) = \sec\left(\frac{\pi}{4} - \theta\right) = \frac{2\sqrt{2}\csc 2\theta}{\csc\theta + \sec\theta}$$

$$\operatorname{cis}\left(\theta + \frac{\pi}{4}\right) = \frac{i}{\operatorname{cis}\left(\frac{\pi}{4} - \theta\right)} = \frac{(1+i)\operatorname{cis}\theta}{\sqrt{2}}$$

Inverse function multiple-value identities

$\cosh^{-1} u = 2\pi i + i\cosh^{-1} u$
$\sinh^{-1} u = 2\pi i + i\sinh^{-1} u$
$\tanh^{-1} u = \pi i + i\tanh^{-1} u$
$\coth^{-1} u = \pi i + i\coth^{-1} u$
$\operatorname{sech}^{-1} u = 2\pi i + i\operatorname{sech}^{-1} u$
$\operatorname{csch}^{-1} u = 2\pi i + i\operatorname{csch}^{-1} u$

$\cos^{-1} u = 2\pi + \cos^{-1} u$
$\sin^{-1} u = 2\pi + \sin^{-1} u$
$\tan^{-1} u = \pi + \tan^{-1} u$
$\cot^{-1} u = \pi + \cot^{-1} u$
$\sec^{-1} u = 2\pi + \sec^{-1} u$
$\csc^{-1} u = 2\pi + \csc^{-1} u$

Inverse function straight-angle identities

$$\cosh^{-1} u = \pi i - i\cosh^{-1} u$$
$$\sinh^{-1} u = \pi i - i\sinh^{-1} u$$
$$\tanh^{-1} u = \pi i + i\tanh^{-1} u$$
$$\coth^{-1} u = \pi i + i\coth^{-1} u$$
$$\operatorname{sech}^{-1} u = \pi i - i\operatorname{sech}^{-1} u$$
$$\operatorname{csch}^{-1} u = \pi i - i\operatorname{csch}^{-1} u$$

$$\cos^{-1} u = \pi - \cos^{-1} u$$
$$\sin^{-1} u = \pi - \sin^{-1} u$$
$$\tan^{-1} u = \pi + \tan^{-1} u$$
$$\cot^{-1} u = \pi + \cot^{-1} u$$
$$\sec^{-1} u = \pi - \sec^{-1} u$$
$$\csc^{-1} u = \pi - \csc^{-1} u$$

Inverse function right-angle identities

$$\cosh^{-1} u = \frac{\pi i}{2} - i\sinh^{-1} u$$
$$\sinh^{-1} u = \frac{\pi i}{2} + i\cosh^{-1} u$$
$$\tanh^{-1} u = \frac{\pi i}{2} + \coth^{-1} u$$
$$\coth^{-1} u = \frac{\pi i}{2} + \tanh^{-1} u$$
$$\operatorname{sech}^{-1} u = \frac{\pi i}{2} - i\operatorname{csch}^{-1} u$$
$$\operatorname{csch}^{-1} u = \frac{\pi i}{2} - i\operatorname{sech}^{-1} u$$

$$\cos^{-1} u = \frac{\pi}{2} - \sin^{-1} u$$
$$\sin^{-1} u = \frac{\pi}{2} + \cos^{-1} u$$
$$\tan^{-1} u = \frac{\pi}{2} - \cot^{-1} u$$
$$\cot^{-1} u = \frac{\pi}{2} - \tan^{-1} u$$
$$\sec^{-1} u = \frac{\pi}{2} - \csc^{-1} u$$
$$\csc^{-1} u = \frac{\pi}{2} + \sec^{-1} u$$

Angle sums and differences

$$\cosh(a+b) = \cosh a\cosh b + \sinh a\sinh b$$
$$\cosh(a-b) = \cosh a\cosh b - \sinh a\sinh b$$
$$\sinh(a+b) = \sinh a\cosh b + \cosh a\sinh b$$
$$\sinh(a-b) = \sinh a\cosh b - \cosh a\sinh b$$
$$\tanh(a+b) = \frac{\tanh a + \tanh b}{1 + \tanh a\tanh b}$$
$$\tanh(a-b) = \frac{\tanh a - \tanh b}{1 - \tanh a\tanh b}$$
$$\coth(a+b) = \frac{\coth a\coth b + 1}{\coth a + \coth b}$$
$$\coth(a-b) = \frac{\coth a\coth b - 1}{\coth b - \coth a}$$
$$\operatorname{sech}(a+b) = \frac{\operatorname{csch} a\operatorname{csch} b\operatorname{sech} a\operatorname{sech} b}{\operatorname{csch} a\operatorname{csch} b + \operatorname{sech} a\operatorname{sech} b}$$
$$\operatorname{sech}(a-b) = \frac{\operatorname{csch} a\operatorname{csch} b\operatorname{sech} a\operatorname{sech} b}{\operatorname{csch} a\operatorname{csch} b - \operatorname{sech} a\operatorname{sech} b}$$
$$\operatorname{csch}(a+b) = \frac{\operatorname{sech} a\operatorname{csch} b\operatorname{csch} a\operatorname{sech} b}{\operatorname{sech} a\operatorname{csch} b + \operatorname{csch} a\operatorname{sech} b}$$
$$\operatorname{csch}(a-b) = \frac{\operatorname{sech} a\operatorname{csch} b\operatorname{csch} a\operatorname{sech} b}{\operatorname{sech} a\operatorname{csch} b - \operatorname{csch} a\operatorname{sech} b}$$
$$\exp(a+b) = \exp a\exp b$$
$$\exp(a-b) = \frac{\exp a}{\exp b}$$

$$\cos(a+b) = \cos a\cos b - \sin a\sin b$$
$$\cos(a-b) = \cos a\cos b + \sin a\sin b$$
$$\sin(a+b) = \sin a\cos b + \cos a\sin b$$
$$\sin(a-b) = \sin a\cos b - \cos a\sin b$$
$$\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a\tan b}$$
$$\tan(a-b) = \frac{\tan a - \tan b}{1 + \tan a\tan b}$$
$$\cot(a+b) = \frac{\cot a\cot b - 1}{\cot a + \cot b}$$
$$\cot(a-b) = \frac{\cot a\cot b + 1}{\cot b - \cot a}$$
$$\sec(a+b) = \frac{\csc a\csc b\sec a\sec b}{\csc a\csc b - \sec a\sec b}$$
$$\sec(a-b) = \frac{\csc a\csc b\sec a\sec b}{\csc a\csc b + \sec a\sec b}$$
$$\csc(a+b) = \frac{\sec a\csc b\csc a\sec b}{\sec a\csc b + \csc a\sec b}$$
$$\csc(a-b) = \frac{\sec a\csc b\csc a\sec b}{\sec a\csc b - \csc a\sec b}$$
$$\operatorname{cis}(a+b) = \operatorname{cis} a\operatorname{cis} b$$
$$\operatorname{cis}(a-b) = \frac{\operatorname{cis} a}{\operatorname{cis} b}$$

Function sums and differences

$$\cosh a + \cosh b = 2\cosh\frac{a+b}{2}\cosh\frac{a-b}{2}$$
$$\cosh a - \cosh b = 2\sinh\frac{a+b}{2}\sinh\frac{a-b}{2}$$
$$\sinh a + \sinh b = 2\sinh\frac{a+b}{2}\cosh\frac{a-b}{2}$$
$$\sinh a - \sinh b = 2\cosh\frac{a+b}{2}\sinh\frac{a-b}{2}$$
$$\tanh a + \tanh b = \sinh(a+b)\operatorname{sech} a\operatorname{sech} b$$
$$\tanh a - \tanh b = \sinh(a-b)\operatorname{sech} a\operatorname{sech} b$$
$$\coth a + \coth b = \sinh(a+b)\operatorname{csch} a\operatorname{csch} b$$
$$\coth a - \coth b = \sinh(b-a)\operatorname{csch} a\operatorname{csch} b$$
$$\operatorname{sech} a + \operatorname{sech} b = 2\cosh\frac{a+b}{2}\cosh\frac{a-b}{2}\operatorname{sech} a\operatorname{sech} b$$
$$\operatorname{sech} a - \operatorname{sech} b = 2\sinh\frac{b+a}{2}\sinh\frac{b-a}{2}\operatorname{sech} a\operatorname{sech} b$$
$$\operatorname{csch} a + \operatorname{csch} b = 2\sinh\frac{a+b}{2}\cosh\frac{a-b}{2}\operatorname{csch} a\operatorname{csch} b$$
$$\operatorname{csch} a - \operatorname{csch} b = 2\cosh\frac{b+a}{2}\sinh\frac{b-a}{2}\operatorname{csch} a\operatorname{csch} b$$
$$\exp a + \exp b = 2\cosh\frac{a-b}{2}\exp\frac{a+b}{2}$$
$$\exp a - \exp b = 2\sinh\frac{a-b}{2}\exp\frac{a+b}{2}$$

$$\cos a + \cos b = 2\cos\frac{a+b}{2}\cos\frac{a-b}{2}$$
$$\cos a - \cos b = 2\sin\frac{b+a}{2}\sin\frac{b-a}{2}$$
$$\sin a + \sin b = 2\sin\frac{a+b}{2}\cos\frac{a-b}{2}$$
$$\sin a - \sin b = 2\cos\frac{a+b}{2}\sin\frac{a-b}{2}$$
$$\tan a + \tan b = \sin(a+b)\sec a\sec b$$
$$\tan a - \tan b = \sin(a-b)\sec a\sec b$$
$$\cot a + \cot b = \sin(a+b)\csc a\csc b$$
$$\cot a - \cot b = \sin(b-a)\csc a\csc b$$
$$\sec a + \sec b = 2\cos\frac{a+b}{2}\cos\frac{a-b}{2}\sec a\sec b$$
$$\sec a - \sec b = 2\sin\frac{a+b}{2}\sin\frac{a-b}{2}\sec a\sec b$$
$$\csc a + \csc b = 2\sin\frac{a+b}{2}\cos\frac{a-b}{2}\csc a\csc b$$
$$\csc a - \csc b = 2\cos\frac{b+a}{2}\sin\frac{b-a}{2}\csc a\csc b$$
$$\operatorname{cis} a + \operatorname{cis} b = 2\cos\frac{a-b}{2}\operatorname{cis}\frac{a+b}{2}$$
$$\operatorname{cis} a - \operatorname{cis} b = 2i\sin\frac{b-a}{2}\operatorname{cis}\frac{a+b}{2}$$

$$\cosh a + i\sinh b = 2\sinh\left(\frac{a-b}{2}-\frac{\pi i}{4}\right)\sinh\left(\frac{a+b}{2}+\frac{\pi i}{4}\right)$$
$$\cosh a - i\sinh b = 2\sinh\left(\frac{a+b}{2}-\frac{\pi i}{4}\right)\sinh\left(\frac{a-b}{2}+\frac{\pi i}{4}\right)$$
$$\coth a + \tanh b = \cosh(a+b)\operatorname{csch} a\operatorname{sech} b$$
$$\coth a - \tanh b = \cosh(a-b)\operatorname{csch} a\operatorname{sech} b$$
$$\operatorname{csch} a + i\operatorname{sech} b = 2\operatorname{csch} a\operatorname{sech} b \times \sinh\left(\frac{b-a}{2}-\frac{\pi i}{4}\right)\sinh\left(\frac{b+a}{2}+\frac{\pi i}{4}\right)$$
$$\operatorname{csch} a - i\operatorname{sech} b = 2\operatorname{csch} a\operatorname{sech} b \times \sinh\left(\frac{b+a}{2}-\frac{\pi i}{4}\right)\sinh\left(\frac{b-a}{2}+\frac{\pi i}{4}\right)$$

$$\cos a + \sin b = 2\sin\left(\frac{\pi}{4}-\frac{a-b}{2}\right)\sin\left(\frac{\pi}{4}+\frac{a+b}{2}\right)$$
$$\cos a - \sin b = 2\sin\left(\frac{\pi}{4}-\frac{a+b}{2}\right)\sin\left(\frac{\pi}{4}+\frac{a-b}{2}\right)$$
$$\cot a + \tan b = \cos(a-b)\csc a\sec b$$
$$\cot a - \tan b = \cos(a+b)\csc a\sec b$$
$$\csc a + \sec b = 2\csc a\sec b \times \sin\left(\frac{\pi}{4}-\frac{b-a}{2}\right)\sin\left(\frac{\pi}{4}+\frac{b+a}{2}\right)$$
$$\csc a + \sec b = 2\csc a\sec b \times \sin\left(\frac{\pi}{4}-\frac{b+a}{2}\right)\sin\left(\frac{\pi}{4}+\frac{b-a}{2}\right)$$

$$\frac{\sinh a + \sinh b}{\cosh a + \cosh b} = \frac{\cosh a - \cosh b}{\sinh a - \sinh b} = \tanh\frac{a+b}{2}$$
$$\frac{\sinh a - \sinh b}{\cosh a + \cosh b} = \frac{\cosh a - \cosh b}{\sinh a + \sinh b} = \tanh\frac{a-b}{2}$$
$$\frac{\tanh a + \tanh b}{\tanh a - \tanh b} = \frac{\coth a + \coth b}{\coth b - \coth a} = \sinh(a+b)\operatorname{csch}(a-b)$$

$$\frac{\sin a + \sin b}{\cos a + \cos b} = \frac{\cos b - \cos a}{\sin a - \sin b} = \tan\frac{a+b}{2}$$
$$\frac{\sin a - \sin b}{\cos a + \cos b} = \frac{\cos b - \cos a}{\sin a + \sin b} = \tan\frac{a-b}{2}$$
$$\frac{\tan a + \tan b}{\tan a - \tan b} = \frac{\cot b + \cot a}{\cot b - \cot a} = \sin(a+b)\csc(a-b)$$

$$\frac{\operatorname{sech} a - \operatorname{sech} b}{\operatorname{sech} a + \operatorname{sech} b} = \tanh\frac{a+b}{2}\tanh\frac{b-a}{2}$$
$$\frac{\operatorname{csch} a + \operatorname{csch} b}{\operatorname{csch} a - \operatorname{csch} b} = \tanh\frac{a+b}{2}\coth\frac{b-a}{2}$$
$$\frac{\exp a - \exp b}{\exp a + \exp b} = 2\tanh\frac{a-b}{2}$$

$$\frac{\sec a - \sec b}{\sec a + \sec b} = \tan\frac{a+b}{2}\tan\frac{a-b}{2}$$
$$\frac{\csc a + \csc b}{\csc a - \csc b} = \tan\frac{a+b}{2}\cot\frac{b-a}{2}$$
$$\frac{\operatorname{cis} a - \operatorname{cis} b}{\operatorname{cis} a + \operatorname{cis} b} = 2i\tan\frac{b-a}{2}$$

Function products

$$\cosh a\cosh b = \frac{\cosh(a+b)+\cosh(a-b)}{2}$$
$$\sinh a\cosh b = \frac{\sinh(a+b)+\sinh(a-b)}{2}$$
$$\sinh a\sinh b = \frac{\cosh(a+b)-\cosh(a-b)}{2}$$
$$\tanh a\tanh b = \coth(a+b)(\tanh a+\tanh b)-1$$
$$\tanh a\coth b = \tanh(a+b)(\coth a+\tanh b)-1$$
$$\coth a\coth b = \coth(a+b)(\coth a+\coth b)-1$$
$$\begin{aligned}\operatorname{sech} a\operatorname{sech} b &= \operatorname{sech}(a+b)(1+\tanh a\tanh b)\\ &= \operatorname{csch}(a+b)(\tanh a+\tanh b)\end{aligned}$$
$$\begin{aligned}\operatorname{sech} a\operatorname{csch} b &= \operatorname{sech}(a+b)(\coth b+\tanh a)\\ &= \operatorname{csch}(a+b)(\tanh a\coth b+1)\end{aligned}$$
$$\begin{aligned}\operatorname{csch} a\operatorname{csch} b &= \operatorname{sech}(a+b)(\coth a\coth b-1)\\ &= \operatorname{csch}(a+b)(\coth a+\coth b)\end{aligned}$$
$$\exp a\exp b = \exp(a+b)$$

$$\cos a\cos b = \frac{\cos(a-b)+\cos(a+b)}{2}$$
$$\sin a\cos b = \frac{\sin(a-b)+\sin(a+b)}{2}$$
$$\sin a\sin b = \frac{\cos(a-b)-\cos(a+b)}{2}$$
$$\tan a\tan b = 1-\cot(a+b)(\tan a+\tan b)$$
$$\tan a\cot b = \tan(a+b)(\cot b-\tan a)-1$$
$$\cot a\cot b = \cot(a+b)(\cot a+\cot b)+1$$
$$\begin{aligned}\sec a\sec b &= \sec(a+b)(1-\tan a\tan b)\\ &= \operatorname{csch}(a+b)(\tan a+\tan b)\end{aligned}$$
$$\begin{aligned}\sec a\csc b &= \sec(a+b)(\cot b-\tan a)\\ &= \operatorname{csch}(a+b)(\tan a\cot b+1)\end{aligned}$$
$$\begin{aligned}\csc a\csc b &= \sec(a+b)(\cot a\cot b-1)\\ &= \operatorname{csch}(a+b)(\cot a+\cot b)\end{aligned}$$
$$\operatorname{cis} a\operatorname{cis} b = \operatorname{cis}(a+b)$$

Inverse function sums and differences

$$\begin{aligned}&\cosh^{-1} u + \cosh^{-1} v\\ &\quad= \cosh^{-1}\left(uv+\sqrt{u^2-1}\sqrt{v^2-1}\right)\\ &\quad= \sinh^{-1}\left(u\sqrt{v^2-1}+v\sqrt{u^2-1}\right)\end{aligned}$$
$$\begin{aligned}&\cosh^{-1} u - \cosh^{-1} v\\ &\quad= \cosh^{-1}\left(uv-\sqrt{u^2-1}\sqrt{v^2-1}\right)\\ &\quad= \sinh^{-1}\left(v\sqrt{u^2-1}-u\sqrt{v^2-1}\right)\end{aligned}$$
$$\begin{aligned}&\sinh^{-1} u + \sinh^{-1} v\\ &\quad= \sinh^{-1}\left(u\sqrt{v^2+1}+v\sqrt{u^2+1}\right)\\ &\quad= \cosh^{-1}\left(\sqrt{u^2+1}\sqrt{v^2+1}+uv\right)\end{aligned}$$
$$\begin{aligned}&\sinh^{-1} u - \sinh^{-1} v\\ &\quad= \sinh^{-1}\left(u\sqrt{v^2+1}-v\sqrt{u^2+1}\right)\\ &\quad= \cosh^{-1}\left(\sqrt{u^2+1}\sqrt{v^2+1}-uv\right)\end{aligned}$$
$$\tanh^{-1} u + \tanh^{-1} v = \tanh^{-1}\frac{u+v}{1+uv}$$
$$\tanh^{-1} u - \tanh^{-1} v = \tanh^{-1}\frac{u-v}{1-uv}$$

$$\begin{aligned}&\cos^{-1} u + \cos^{-1} v\\ &\quad= \cos^{-1}\left(uv-\sqrt{1-u^2}\sqrt{1-v^2}\right)\\ &\quad= \sin^{-1}\left(u\sqrt{1-v^2}+v\sqrt{1-u^2}\right)\end{aligned}$$
$$\begin{aligned}&\cos^{-1} u - \cos^{-1} v\\ &\quad= \cos^{-1}\left(uv+\sqrt{1-u^2}\sqrt{1-v^2}\right)\\ &\quad= \sin^{-1}\left(v\sqrt{1-u^2}-u\sqrt{1-v^2}\right)\end{aligned}$$
$$\begin{aligned}&\sin^{-1} u + \sin^{-1} v\\ &\quad= \sin^{-1}\left(u\sqrt{1-v^2}+v\sqrt{1-u^2}\right)\\ &\quad= \cos^{-1}\left(\sqrt{1-u^2}\sqrt{1-v^2}-uv\right)\end{aligned}$$
$$\begin{aligned}&\sin^{-1} u - \sin^{-1} v\\ &\quad= \sin^{-1}\left(u\sqrt{1-v^2}-v\sqrt{1-u^2}\right)\\ &\quad= \cos^{-1}\left(\sqrt{1-u^2}\sqrt{1-v^2}+uv\right)\end{aligned}$$
$$\tan^{-1} u + \tan^{-1} v = \tan^{-1}\frac{u+v}{1-uv}$$
$$\tan^{-1} u - \tan^{-1} v = \tan^{-1}\frac{u-v}{1+uv}$$

$$\coth^{-1} u + \coth^{-1} v = \coth^{-1} \frac{1+uv}{u+v}$$

$$\coth^{-1} u - \coth^{-1} v = \coth^{-1} \frac{1-uv}{u-v}$$

$$\begin{aligned}\operatorname{sech}^{-1} u &+ \operatorname{sech}^{-1} v \\ &= \operatorname{sech}^{-1} \frac{uv\sqrt{1-u^2}\sqrt{1-v^2}}{uv+\sqrt{1-u^2}\sqrt{1-v^2}} \\ &= \operatorname{csch}^{-1} \frac{uv\sqrt{1-u^2}\sqrt{1-v^2}}{u\sqrt{1-v^2}+v\sqrt{1-u^2}}\end{aligned}$$

$$\begin{aligned}\operatorname{sech}^{-1} u &- \operatorname{sech}^{-1} v \\ &= \operatorname{sech}^{-1} \frac{uv\sqrt{1-u^2}\sqrt{1-v^2}}{\sqrt{1-u^2}\sqrt{1-v^2}-uv} \\ &= \operatorname{csch}^{-1} \frac{uv\sqrt{1-u^2}\sqrt{1-v^2}}{u\sqrt{1-v^2}-v\sqrt{1-u^2}}\end{aligned}$$

$$\begin{aligned}\operatorname{csch}^{-1} u &+ \operatorname{csch}^{-1} v \\ &= \operatorname{csch}^{-1} \frac{uv\sqrt{u^2+1}\sqrt{v^2+1}}{u\sqrt{v^2+1}+v\sqrt{u^2+1}} \\ &= \operatorname{sech}^{-1} \frac{uv\sqrt{u^2+1}\sqrt{v^2+1}}{uv+\sqrt{u^2+1}\sqrt{v^2+1}}\end{aligned}$$

$$\begin{aligned}\operatorname{csch}^{-1} u &- \operatorname{csch}^{-1} v \\ &= \operatorname{csch}^{-1} \frac{uv\sqrt{u^2+1}\sqrt{v^2+1}}{v\sqrt{u^2+1}-u\sqrt{v^2+1}} \\ &= \operatorname{sech}^{-1} \frac{uv\sqrt{u^2+1}\sqrt{v^2+1}}{uv-\sqrt{u^2+1}\sqrt{v^2+1}}\end{aligned}$$

$$\exp^{-1} u + \exp^{-1} v = \exp^{-1} uv$$

$$\exp^{-1} u - \exp^{-1} v = \exp^{-1} \frac{u}{v}$$

$$\cot^{-1} u + \cot^{-1} v = \cot^{-1} \frac{uv-1}{u+v}$$

$$\cot^{-1} u - \cot^{-1} v = \cot^{-1} \frac{vu-1}{v-u}$$

$$\begin{aligned}\sec^{-1} u &+ \sec^{-1} v \\ &= \sec^{-1} \frac{uv\sqrt{u^2-1}\sqrt{v^2-1}}{\sqrt{u^2-1}\sqrt{v^2-1}-uv} \\ &= \csc^{-1} \frac{uv\sqrt{u^2-1}\sqrt{v^2-1}}{u\sqrt{v^2-1}+v\sqrt{u^2-1}}\end{aligned}$$

$$\begin{aligned}\sec^{-1} u &- \sec^{-1} v \\ &= \sec^{-1} \frac{uv\sqrt{u^2-1}\sqrt{v^2-1}}{uv+\sqrt{u^2-1}\sqrt{v^2-1}} \\ &= \csc^{-1} \frac{uv\sqrt{u^2-1}\sqrt{v^2-1}}{u\sqrt{v^2-1}-v\sqrt{u^2-1}}\end{aligned}$$

$$\begin{aligned}\csc^{-1} u &+ \csc^{-1} v \\ &= \csc^{-1} \frac{uv\sqrt{u^2-1}\sqrt{v^2-1}}{u\sqrt{v^2-1}+v\sqrt{u^2-1}} \\ &= \sec^{-1} \frac{uv\sqrt{u^2-1}\sqrt{v^2-1}}{uv-\sqrt{u^2-1}\sqrt{v^2-1}}\end{aligned}$$

$$\begin{aligned}\csc^{-1} u &- \csc^{-1} v \\ &= \csc^{-1} \frac{uv\sqrt{u^2-1}\sqrt{v^2-1}}{v\sqrt{u^2-1}-u\sqrt{v^2-1}} \\ &= \sec^{-1} \frac{uv\sqrt{u^2-1}\sqrt{v^2-1}}{uv+\sqrt{u^2-1}\sqrt{v^2-1}}\end{aligned}$$

$$\operatorname{cis}^{-1} u + \operatorname{cis}^{-1} v = \operatorname{cis}^{-1} uv$$

$$\operatorname{cis}^{-1} u - \operatorname{cis}^{-1} v = \operatorname{cis}^{-1} \frac{u}{v}$$

Same-angle function sums and differences

$$\begin{aligned}\cosh\theta + i\sinh\theta &= \frac{\sqrt{2}}{i}\sinh\left(\theta + \frac{\pi i}{4}\right) \\ &= \sqrt{2}\cosh\left(\frac{\pi i}{4} - \theta\right)\end{aligned}$$

$$\begin{aligned}\cosh\theta - i\sinh\theta &= \sqrt{2}\cosh\left(\theta + \frac{\pi i}{4}\right) \\ &= \frac{\sqrt{2}}{i}\sinh\left(\frac{\pi i}{4} - \theta\right)\end{aligned}$$

$$\coth\theta + \tanh\theta = 2\coth 2\theta$$

$$\coth\theta - \tanh\theta = 2\operatorname{csch} 2\theta$$

$$\begin{aligned}\cos\theta + \sin\theta &= \sqrt{2}\sin\left(\theta + \frac{\pi}{4}\right) \\ &= \sqrt{2}\cos\left(\frac{\pi}{4} - \theta\right)\end{aligned}$$

$$\begin{aligned}\cos\theta - \sin\theta &= \sqrt{2}\cos\left(\theta + \frac{\pi}{4}\right) \\ &= \sqrt{2}\sin\left(\frac{\pi}{4} - \theta\right)\end{aligned}$$

$$\cot\theta + \tan\theta = 2\csc 2\theta$$

$$\cot\theta - \tan\theta = 2\cot 2\theta$$

$$\operatorname{csch}\theta + i\operatorname{sech}\theta = \frac{\sqrt{2}\operatorname{csch}\left(\theta + \frac{\pi i}{4}\right)}{i\operatorname{csch}2\theta} = \frac{\sqrt{2}\operatorname{sech}\left(\frac{\pi i}{4} - \theta\right)}{\operatorname{csch}2\theta}$$

$$\operatorname{csch}\theta - i\operatorname{sech}\theta = \frac{\sqrt{2}\operatorname{sech}\left(\theta + \frac{\pi i}{4}\right)}{i\operatorname{csch}2\theta} = \frac{\sqrt{2}\operatorname{csch}\left(\frac{\pi i}{4} - \theta\right)}{\operatorname{csch}2\theta}$$

$$\csc\theta + \sec\theta = \frac{\sqrt{2}\csc\left(\theta + \frac{\pi}{4}\right)}{\csc 2\theta} = \frac{\sqrt{2}\sec\left(\frac{\pi}{4} - \theta\right)}{\csc 2\theta}$$

$$\csc\theta - \sec\theta = \frac{\sqrt{2}\sec\left(\theta + \frac{\pi}{4}\right)}{\csc 2\theta} = \frac{\sqrt{2}\csc\left(\frac{\pi}{4} - \theta\right)}{\csc 2\theta}$$

Linear combinations of functions

$A \neq \pm B \Rightarrow$

$$A\sinh\theta + B\cosh\theta = \sqrt{A^2 - B^2}\sinh\left(\theta + \tanh^{-1}\frac{B}{A}\right) = \sqrt{B^2 - A^2}\cosh\left(\theta + \tanh^{-1}\frac{A}{B}\right)$$

$$A\tanh\theta + B\coth\theta = \sqrt{A^2\operatorname{sech}^2\theta - B^2\operatorname{csch}^2\theta}\times \sinh\left(\theta + \tanh^{-1}\left(\frac{B}{A}\coth\theta\right)\right) = \sqrt{B^2\operatorname{csch}^2\theta - A^2\operatorname{sech}^2\theta}\times \cosh\left(\theta + \tanh^{-1}\left(\frac{A}{B}\tanh\theta\right)\right)$$

$$A\operatorname{sech}\theta + B\operatorname{csch}\theta = \sqrt{A^2 - B^2}\operatorname{csch}2\theta\sinh\left(\theta + \tanh^{-1}\frac{B}{A}\right) = \sqrt{B^2 - A^2}\operatorname{csch}2\theta\cosh\left(\theta + \tanh^{-1}\frac{A}{B}\right)$$

$A \neq \pm B$ and $A, B \neq 0 \Rightarrow$

$$A\exp\theta + \frac{B}{\exp\theta} = 2\sqrt{AB}\sinh\left(\theta + \tanh^{-1}\frac{A-B}{A+B}\right) = 2i\sqrt{AB}\cosh\left(\theta + \tanh^{-1}\frac{A+B}{A-B}\right)$$

$A \neq \pm B \Rightarrow$

$$A\sin\theta + B\cos\theta = \sqrt{A^2 + B^2}\sin\left(\theta + \tan^{-1}\frac{B}{A}\right) = \sqrt{A^2 + B^2}\cos\left(\theta - \tan^{-1}\frac{A}{B}\right)$$

$$A\tan\theta + B\cot\theta = \sqrt{A^2\sec^2\theta + B^2\csc^2\theta}\times \sin\left(\theta + \tan^{-1}\left(\frac{B}{A}\cot\theta\right)\right) = \sqrt{A^2\sec^2\theta + B^2\csc^2\theta}\times \cos\left(\theta - \tan^{-1}\left(\frac{A}{B}\tan\theta\right)\right)$$

$$A\sec\theta + B\csc\theta = \sqrt{A^2 + B^2}\csc 2\theta\sin\left(\theta + \tan^{-1}\frac{B}{A}\right) = \sqrt{A^2 + B^2}\csc 2\theta\cos\left(\theta - \tan^{-1}\frac{A}{B}\right)$$

$A \neq \pm B$ and $A, B \neq 0 \Rightarrow$

$$A\operatorname{cis}\theta + \frac{B}{\operatorname{cis}\theta} = 2\sqrt{AB}\sin\left(\theta + \tan^{-1}\frac{A-B}{A+B}\right) = 2\sqrt{AB}\cos\left(\theta - \tan^{-1}\frac{A+B}{A-B}\right)$$

Multiple angles

$$\cosh n\theta = \sum_{k=0}^{\frac{n}{2}}\binom{n}{2k}\cosh^{n-2k}\theta\sinh^{2k}\theta$$

$$\sinh n\theta = \sum_{k=0}^{\frac{n-1}{2}}\binom{n}{2k+1}\cosh^{n-2k-1}\theta\sinh^{2k+1}\theta$$

$$\cos n\theta = \sum_{k=0}^{\frac{n}{2}}(-1)^k\binom{n}{2k}\cos^{n-2k}\theta\sin^{2k}\theta$$

$$\sin n\theta = \sum_{k=0}^{\frac{n-1}{2}}(-1)^k\binom{n}{2k+1}\cos^{n-2k-1}\theta\sin^{2k+1}\theta$$

Powers of two angles

$$\sinh^2 a - \sinh^2 b = \cosh^2 a - \cosh^2 b$$
$$= \sinh(a+b)\sinh(a-b)$$
$$\cosh^2 a + \sinh^2 b = \cosh^2 b + \sinh^2 a$$
$$= \cosh(a+b)\cosh(a-b)$$
$$\tanh^2 a - \tanh^2 b$$
$$= \tanh(a+b)\tanh(a-b)(1-\tanh^2 a\tanh^2 b)$$
$$\coth^2 a - \coth^2 b$$
$$= \tanh(a+b)\tanh(a-b)(1-\coth^2 a\coth^2 b)$$
$$\operatorname{sech}^2 a - \operatorname{sech}^2 b$$
$$= \sinh(a+b)\sinh(a-b)\operatorname{sech}^2 a\operatorname{sech}^2 b$$
$$\operatorname{csch}^2 a - \operatorname{csch}^2 b$$
$$= \sinh(b+a)\sinh(b-a)\operatorname{csch}^2 a\operatorname{csch}^2 b$$

$$\sin^2 a - \sin^2 b = \cos^2 b - \cos^2 a$$
$$= \sin(a+b)\sin(a-b)$$
$$\cos^2 a - \sin^2 b = \cos^2 b - \sin^2 a$$
$$= \cos(a+b)\cos(a-b)$$
$$\tan^2 a - \tan^2 b$$
$$= \tan(a+b)\tan(a-b)(1-\tan^2 a\tan^2 b)$$
$$\cot^2 a - \cot^2 b$$
$$= \tan(a+b)\tan(a-b)(1-\cot^2 a\cot^2 b)$$
$$\sec^2 a - \sec^2 b$$
$$= \sin(a+b)\sin(a-b)\sec^2 a\sec^2 b$$
$$\csc^2 a - \csc^2 b$$
$$= \sin(b+a)\sin(b-a)\csc^2 a\csc^2 b$$

Powers of one angle

$$\cosh^{2n}\theta$$
$$= \frac{1}{4^n}\binom{2n}{n}$$
$$+\frac{1}{2^{2n-1}}\sum_{k=1}^{n}\binom{2n}{n+k}\cosh[2k\theta]$$
$$\sinh^{2n}\theta$$
$$= \frac{1}{4^n}(-1)^n\binom{2n}{n}$$
$$+\frac{1}{2^{2n-1}}\sum_{k=1}^{n}(-1)^{n+k}\binom{2n}{n+k}\cosh[2k\theta]$$
$$\cosh^{2n+1}\theta$$
$$= \frac{1}{4^{n-1}}\sum_{k=0}^{n}\binom{2n+1}{n+k+1}\cosh[(2k+1)\theta]$$
$$\sinh^{2n+1}\theta$$
$$= \frac{1}{4^{n-1}}\sum_{k=0}^{n}(-1)^{k+1}\binom{2n+1}{n+k+1}\sinh[(2k+1)\theta]$$
$$\cosh^n\theta - \sinh^n\theta$$
$$= \prod_{k=1}^{n}\sqrt{1-\exp\frac{4k\pi i}{n}}\times$$
$$\sinh\left(\theta - \tanh^{-1}\exp\frac{4k\pi i}{n}\right)$$

$$\cos^{2n}\theta$$
$$= \frac{1}{4^n}\binom{2n}{n}$$
$$+\frac{1}{2^{2n-1}}\sum_{k=1}^{n}\binom{2n}{n+k}\cos[2k\theta]$$
$$\sin^{2n}\theta$$
$$= \frac{1}{4^n}\binom{2n}{n}$$
$$+\frac{1}{2^{2n-1}}\sum_{k=1}^{n}(-1)^{k}\binom{2n}{n+k}\cos[2k\theta]$$
$$\cos^{2n+1}\theta$$
$$= \frac{1}{4^{n-1}}\sum_{k=0}^{n}\binom{2n+1}{n+k+1}\cos[(2k+1)\theta]$$
$$\sin^{2n+1}\theta$$
$$= \frac{1}{4^{n-1}}\sum_{k=0}^{n}(-1)^{k}\binom{2n+1}{n+k+1}\sin[(2k+1)\theta]$$
$$\cos^n\theta - \sin^n\theta$$
$$= \prod_{k=1}^{n}\sqrt{1+\operatorname{cis}\frac{4k\pi}{n}}\times$$
$$\sin\left(\theta - \tan^{-1}\operatorname{cis}\frac{4k\pi}{n}\right)$$

$$\cosh^n\theta + \sinh^n\theta = \prod_{k=1}^{n}\sqrt{1-\exp\frac{2(2k+1)\pi i}{n}}\times \sinh\left(\theta - \tanh^{-1}\exp\frac{2(2k+1)\pi i}{n}\right)$$

$$\cosh^{2n}\theta - \sinh^{2n}\theta = \frac{1}{2^{2n-1}}\left(1-(-1)^n\right)\binom{2n}{n} + \frac{1}{4^{n-1}}\times \sum_{k=1}^{\frac{n}{2}}\binom{2n}{n+2k+(-1)^n}\cosh\left[\left(2k+(-1)^n\right)2\theta\right]$$

$$\cosh^{2n}\theta + \sinh^{2n}\theta = \frac{1}{2^{2n-1}}\left(1+(-1)^n\right)\binom{2n}{n} + \frac{1}{4^{n-1}}\times \sum_{k=1}^{\frac{n+1}{2}}\binom{2n}{n+2k-(-1)^n}\cosh\left[\left(2k-(-1)^n\right)2\theta\right]$$

$$\cosh^{2n+1}\theta - i\sinh^{2n+1}\theta = \frac{1}{2^{2n-3}}\times \sum_{k=0}^{n}\binom{2n+1}{n+k+1}\cosh\left[\frac{\pi i}{4}+(-1)^k(2k+1)\theta\right]$$

$$\cosh^{2n+1}\theta + i\sinh^{2n+1}\theta = i\frac{1}{2^{2n-3}}\times \sum_{k=0}^{n}\binom{2n+1}{n+k+1}\sinh\left[\frac{\pi i}{4}+(-1)^k(2k+1)\theta\right]$$

$$\cosh^{2n+1}\theta + \sinh^{2n+1}\theta = \frac{1}{4^{n-1}}\times \sum_{k=0}^{n}\binom{2n+1}{n+k+1}\exp\left[(-1)^{k+1}(2k+1)\theta\right]$$

$$\cos^n\theta + \sin^n\theta = \prod_{k=1}^{n}\sqrt{1+\operatorname{cis}\frac{2(2k+1)\pi}{n}}\times \sin\left(\theta - \tanh^{-1}\operatorname{cis}\frac{2(2k+1)\pi}{n}\right)$$

$$\cos^{2n}\theta - \sin^{2n}\theta = \frac{1}{4^{n-1}}\times \sum_{k=1}^{\frac{n+1}{2}}\binom{2n}{n+2k-1}\cos\left[(2k-1)\,2\theta\right]$$

$$\cos^{2n}\theta + \sin^{2n}\theta = \frac{1}{2^{2n-1}}\binom{2n}{n} + \frac{1}{4^{n-1}}\times \sum_{k=1}^{\frac{n}{2}}\binom{2n}{n+2k}\cos\left[4k\theta\right]$$

$$\cos^{2n+1}\theta - \sin^{2n+1}\theta = \frac{1}{2^{2n-3}}\times \sum_{k=0}^{n}\binom{2n+1}{n+k+1}\cos\left[\frac{\pi}{4}+(-1)^k(2k+1)\theta\right]$$

$$\cos^{2n+1}\theta + \sin^{2n+1}\theta = \frac{1}{2^{2n-3}}\times \sum_{k=0}^{n}\binom{2n+1}{n+k+1}\sin\left[\frac{\pi}{4}+(-1)^k(2k+1)\theta\right]$$

$$\cos^{2n+1}\theta + i\sin^{2n+1}\theta = \frac{1}{4^{n-1}}\times \sum_{k=0}^{n}\binom{2n+1}{n+k+1}\operatorname{cis}\left[(-1)^k(2k+1)\theta\right]$$

Differential equations

$$f''(\theta) = f(\theta) \quad\Rightarrow\quad f(\theta) = A\cosh\theta + B\sinh\theta$$

$$f''(\theta) = -f(\theta) \quad\Rightarrow\quad f(\theta) = A\cos\theta + B\sin\theta$$

Derivatives

$(\cosh\theta)' = \sinh\theta$
$(\sinh\theta)' = \cosh\theta$
$(\tanh\theta)' = \operatorname{sech}^2\theta$
$(\coth\theta)' = -\operatorname{csch}^2\theta$
$(\operatorname{sech}\theta)' = -\tanh\theta\operatorname{sech}\theta$
$(\operatorname{csch}\theta)' = -\coth\theta\operatorname{csch}\theta$
$(\exp\theta)' = \exp\theta$

$(\cos\theta)' = -\sin\theta$
$(\sin\theta)' = \cos\theta$
$(\tan\theta)' = \sec^2\theta$
$(\cot\theta)' = -\csc^2\theta$
$(\sec\theta)' = \tan\theta\sec\theta$
$(\csc\theta)' = -\cot\theta\csc\theta$
$(\operatorname{cis}\theta)' = i\operatorname{cis}\theta$

Inverse function derivatives

$$\left(\cosh^{-1} u\right)' = \frac{1}{\sqrt{u^2-1}}$$
$$\left(\sinh^{-1} u\right)' = \frac{1}{\sqrt{u^2+1}}$$
$$\left(\tanh^{-1} u\right)' = \frac{1}{1-u^2}$$
$$\left(\coth^{-1} u\right)' = \frac{1}{1-u^2}$$
$$\left(\operatorname{sech}^{-1} u\right)' = \frac{1}{u\sqrt{1-u^2}}$$
$$\left(\operatorname{csch}^{-1} u\right)' = \frac{1}{u\sqrt{1+u^2}}$$
$$(\exp^{-1} u)' = \frac{1}{u}$$

$$(\cos^{-1} u)' = \frac{-1}{\sqrt{1-u^2}}$$
$$\left(\sin^{-1} u\right)' = \frac{1}{\sqrt{1-u^2}}$$
$$(\tan^{-1} u)' = \frac{1}{1+u^2}$$
$$(\cot^{-1} u)' = \frac{-1}{1+u^2}$$
$$(\sec^{-1} u)' = \frac{1}{u\sqrt{u^2-1}}$$
$$(\csc^{-1} u)' = \frac{-1}{u\sqrt{u^2-1}}$$
$$\left(\operatorname{cis}^{-1} u\right)' = \frac{-i}{u}$$

Power series

$$\exp\theta = \sum_{k=0}^{\infty}\frac{\theta^k}{k!}$$
$$\cosh\theta = \sum_{k=0}^{\infty}\frac{\theta^{2k}}{(2k)!}$$
$$\sinh\theta = \sum_{k=0}^{\infty}\frac{\theta^{2k+1}}{(2k+1)!}$$
$$B_k = \frac{2(2k)!}{(4^k-1)\pi^{2k}}\sum_{j=0}^{\infty}\frac{1}{(2j+1)^{2k}}$$
$$\tanh\theta = \sum_{k=1}^{\infty}\frac{(-1)^k 4^k(4^k-1)B_k}{(2k)!}\theta^{2k-1}, \quad |\theta| < \tfrac{\pi}{2}$$
$$\coth\theta = \frac{1}{\theta} - \sum_{k=1}^{\infty}\frac{(-1)^{k+1}4^k B_k}{(2k)!}\theta^{2k-1}, \quad 0 < |\theta| < \pi$$
$$\operatorname{sech}\theta = \sum_{k=0}^{\infty}\frac{(-1)^k E_k}{(2k)!}\theta^{2k}, \quad |\theta| < \tfrac{\pi}{2}$$
$$\operatorname{csch}\theta = \frac{1}{\theta} + \sum_{k=1}^{\infty}\frac{(-1)^k 2(2^{2k-1}-1)B_k}{(2k)!}\theta^{2k-1}, \quad |\theta| < \pi$$

$$\operatorname{cis}\theta = \sum_{k=0}^{\infty}\frac{(i\theta)^k}{k!}$$
$$\cos\theta = \sum_{k=0}^{\infty}\frac{(-1)^k\theta^{2k}}{(2k)!}$$
$$\sin\theta = \sum_{k=0}^{\infty}\frac{(-1)^k\theta^{2k+1}}{(2k+1)!}$$
$$E_k = \frac{4^{k+1}(2k)!}{\pi^{2k+1}}\sum_{j=0}^{\infty}\frac{(-1)^{j+1}}{(2j-1)^{2k+1}}$$
$$\tan\theta = \sum_{k=1}^{\infty}\frac{4^k(4^k-1)B_k}{(2k)!}\theta^{2k-1}, \quad |\theta| < \tfrac{\pi}{2}$$
$$\cot\theta = \frac{1}{\theta} - \sum_{k=1}^{\infty}\frac{4^k B_k}{(2k)!}\theta^{2k-1}, \quad 0 < |\theta| < \pi$$
$$\sec\theta = \sum_{k=0}^{\infty}\frac{E_k}{(2k)!}\theta^{2k}, \quad |\theta| < \tfrac{\pi}{2}$$
$$\csc\theta = \frac{1}{\theta} + \sum_{k=1}^{\infty}\frac{2(2^{2k}-1)B_k}{(2k)!}\theta^{2k-1}, \quad |\theta| < \pi$$

Inverse function power series

$$\sinh^{-1} u = \sum_{k=0}^{\infty} \frac{(-1)^k (2k)!}{4^k (k!)^2 (2k+1)} u^{2k+1}, \quad |u| < 1$$

$$\cosh^{-1} u = \ln 2u - \sum_{k=1}^{\infty} \frac{(2k)!}{4^k (k!)^2 2k} u^{-2k}, \quad |u| > 1$$

$$\tanh^{-1} u = \sum_{k=0}^{\infty} \frac{u^{2k+1}}{2k+1}, \ |u| < 1$$

$$\coth^{-1} u = \sum_{k=0}^{\infty} \frac{u^{-2k-1}}{2k+1}, \ |u| > 1$$

$$\operatorname{sech}^{-1} u = \ln \frac{2}{u} - \sum_{k=1}^{\infty} \frac{(2k)!}{4^k (k!)^2 2k} u^{2k}, \quad |u| < 1$$

$$\operatorname{csch}^{-1} u = \sum_{k=0}^{\infty} \frac{(-1)^k (2k)!}{4^k (k!)^2 (2k+1)} u^{-2k-1}, \quad |u| > 1$$

$$\sin^{-1} u = \sum_{k=0}^{\infty} \frac{(2k)!}{4^k (k!)^2 (2k+1)} u^{2k+1}, \quad |u| < 1$$

$$\cos^{-1} u = \frac{\pi}{2} - \sum_{k=0}^{\infty} \frac{(2k)!}{4^k (k!)^2 (2k+1)} u^{2k+1}, \quad |u| < 1$$

$$\tan^{-1} u = \sum_{k=0}^{\infty} \frac{(-1)^k u^{2k+1}}{2k+1}, \ |u| \le 1$$

$$\cot^{-1} u = \sum_{k=0}^{\infty} \frac{(-1)^k u^{-2k-1}}{2k+1}, \ |u| \ge 1$$

$$\sec^{-1} u = \frac{\pi}{2} - \sum_{k=0}^{\infty} \frac{(2k)!}{4^k (k!)^2 (2k+1)} u^{-2k-1}, \quad |u| > 1$$

$$\csc^{-1} u = \sum_{k=0}^{\infty} \frac{(2k)!}{4^k (k!)^2 (2k+1)} u^{-2k-1}, \quad |u| > 1$$

Infinite products

$$\cosh\theta = \prod_{k=1}^{\infty} 1 + \left(\frac{\theta}{\pi(k-\frac{1}{2})}\right)^2$$

$$\sinh\theta = \theta \prod_{k=1}^{\infty} 1 + \left(\frac{\theta}{\pi k}\right)^2$$

$$\tanh\theta = \theta \prod_{k=1}^{\infty} \left(1 - \frac{1}{2k}\right)^2 \frac{k^2 + \left(\frac{\theta}{\pi}\right)^2}{(k-\frac{1}{2})^2 + \left(\frac{\theta}{\pi}\right)^2}$$

$$\coth\theta = \frac{1}{\theta} \prod_{k=1}^{\infty} \left(\frac{2k}{2k-1}\right)^2 \frac{(k-\frac{1}{2})^2 + \left(\frac{\theta}{\pi}\right)^2}{k^2 + \left(\frac{\theta}{\pi}\right)^2}$$

$$\operatorname{sech}\theta = \theta \prod_{k=1}^{\infty} \frac{(k-\frac{1}{2})^2}{(k-\frac{1}{2})^2 + \left(\frac{\theta}{\pi}\right)^2}$$

$$\operatorname{csch}\theta = \theta \prod_{k=1}^{\infty} \frac{k^2}{k^2 + \left(\frac{\theta}{\pi}\right)^2}$$

$$\frac{2}{e} = \prod_{k=1}^{\infty} \exp\left(\frac{-1}{2k(2k+1)}\right)$$

$$\cos\theta = \prod_{k=1}^{\infty} 1 - \left(\frac{\theta}{\pi(k-\frac{1}{2})}\right)^2$$

$$\sin\theta = \theta \prod_{k=1}^{\infty} 1 - \left(\frac{\theta}{\pi k}\right)^2$$

$$\tan\theta = \theta \prod_{k=1}^{\infty} \left(1 - \frac{1}{2k}\right)^2 \frac{k^2 - \left(\frac{\theta}{\pi}\right)^2}{(k-\frac{1}{2})^2 - \left(\frac{\theta}{\pi}\right)^2}$$

$$\cot\theta = \frac{1}{\theta} \prod_{k=1}^{\infty} \left(\frac{2k}{2k-1}\right)^2 \frac{(k-\frac{1}{2})^2 - \left(\frac{\theta}{\pi}\right)^2}{k^2 - \left(\frac{\theta}{\pi}\right)^2}$$

$$\sec\theta = \theta \prod_{k=1}^{\infty} \frac{(k-\frac{1}{2})^2}{(k-\frac{1}{2})^2 - \left(\frac{\theta}{\pi}\right)^2}$$

$$\csc\theta = \theta \prod_{k=1}^{\infty} \frac{k^2}{k^2 - \left(\frac{\theta}{\pi}\right)^2}$$

$$\frac{2}{\pi} = \prod_{k=2}^{\infty} \cos\left(\frac{\pi}{2^k}\right)$$

Continued fractions

$$\exp\theta = \cfrac{1}{1 - \cfrac{\theta}{1 + \cfrac{\theta}{2 - \cfrac{\theta}{3 + \cfrac{\theta}{2 - \cfrac{\theta}{5 + \cfrac{\theta}{2 - \cfrac{\theta}{\ddots}}}}}}}}$$

$$\operatorname{cis}\theta = \cfrac{1}{i + \cfrac{\theta}{i - \cfrac{\theta}{2i + \cfrac{\theta}{3i - \cfrac{\theta}{2i + \cfrac{\theta}{5i - \cfrac{\theta}{2i + \cfrac{\theta}{\ddots}}}}}}}}$$

$$\tanh\theta = \cfrac{\theta}{1 + \cfrac{\theta^2}{3 + \cfrac{\theta^2}{5 + \cfrac{\theta^2}{7 + \cfrac{\theta^2}{9 + \cfrac{\theta^2}{11 + \cfrac{\theta^2}{13 + \cfrac{\theta^2}{\ddots}}}}}}}}$$

$$\tan\theta = \cfrac{\theta}{1 - \cfrac{\theta^2}{3 - \cfrac{\theta^2}{5 - \cfrac{\theta^2}{7 - \cfrac{\theta^2}{9 - \cfrac{\theta^2}{11 - \cfrac{\theta^2}{13 - \cfrac{\theta^2}{\ddots}}}}}}}}$$

$$\coth\theta = \frac{1}{\theta} + \cfrac{\theta}{3 + \cfrac{\theta^2}{5 + \cfrac{\theta^2}{7 + \cfrac{\theta^2}{9 + \cfrac{\theta^2}{11 + \cfrac{\theta^2}{13 + \cfrac{\theta^2}{\ddots}}}}}}}$$

$$\cot\theta = \frac{1}{\theta} - \cfrac{\theta}{3 - \cfrac{\theta^2}{5 - \cfrac{\theta^2}{7 - \cfrac{\theta^2}{9 - \cfrac{\theta^2}{11 - \cfrac{\theta^2}{13 - \cfrac{\theta^2}{\ddots}}}}}}}$$

$$\exp^{-1} u = \cfrac{u}{1 + \cfrac{u}{2 + \cfrac{u}{3 + \cfrac{4u}{4 + \cfrac{4u}{5 + \cfrac{9u}{6 + \cfrac{9u}{7 + \cfrac{16u}{\ddots}}}}}}}}$$

$$\operatorname{cis}^{-1} u = \cfrac{u}{i - \cfrac{u}{2i - \cfrac{u}{3i - \cfrac{4u}{4i - \cfrac{4u}{5i - \cfrac{9u}{6i - \cfrac{9u}{7i - \cfrac{16u}{\ddots}}}}}}}}$$

$$\tanh^{-1} u = \cfrac{u}{1 - \cfrac{u^2}{3 - \cfrac{4u^2}{5 - \cfrac{9u^2}{7 - \cfrac{16u^2}{9 - \cfrac{25u^2}{11 - \cfrac{36u^2}{\ddots}}}}}}}$$

$$\tan^{-1} u = \cfrac{u}{1 + \cfrac{u^2}{3 + \cfrac{4u^2}{5 + \cfrac{9u^2}{7 + \cfrac{16u^2}{9 + \cfrac{25u^2}{11 + \cfrac{36u^2}{\ddots}}}}}}}$$

$$\coth^{-1} u = \cfrac{1}{u - \cfrac{1}{3u - \cfrac{4}{5u - \cfrac{9}{7u - \cfrac{16}{9u - \cfrac{25}{11u - \cfrac{36}{\ddots}}}}}}}$$

$$\cot^{-1} u = \cfrac{1}{u + \cfrac{1}{3u + \cfrac{4}{5u + \cfrac{9}{7u + \cfrac{16}{9u + \cfrac{25}{11u + \cfrac{36}{\ddots}}}}}}}$$

Inner transformation function definitions

$$\operatorname{gd}\theta = 2\tan^{-1} e^{\theta} - \frac{\pi}{2} = \frac{\pi}{2} - 2\cot^{-1} e^{\theta}$$

$$\operatorname{gd}^{-1}\theta = \ln\tan\left(\frac{\theta}{2} + \frac{\pi}{4}\right) = \ln\cot\left(\frac{\pi}{4} - \frac{\theta}{2}\right) = \ln(\sec\theta + \tan\theta)$$

$$\operatorname{cg}\theta = 2\coth^{-1} e^{\theta} = -2\tanh^{-1} e^{-\theta} = \ln(\operatorname{csch}\theta + \coth\theta)$$

$$\operatorname{co}\theta = \frac{\pi}{2} - \theta = \frac{\pi}{2} - \theta + 2n\pi,\ n \in N$$

$$\operatorname{gs}\theta = \frac{\operatorname{gd} 2\theta}{2}$$

$$\operatorname{gs}^{-1}\theta = \frac{\operatorname{gd}^{-1} 2\theta}{2}$$

Gudermannian transformations

$\cosh\operatorname{gd}^{-1}\theta = \sec\theta$
$\sinh\operatorname{gd}^{-1}\theta = \tan\theta$
$\tanh\operatorname{gd}^{-1}\theta = \sin\theta$
$\coth\operatorname{gd}^{-1}\theta = \csc\theta$
$\operatorname{sech}\operatorname{gd}^{-1}\theta = \cos\theta$
$\operatorname{csch}\operatorname{gd}^{-1}\theta = \cot\theta$

$\cos\operatorname{gd}\theta = \operatorname{sech}\theta$
$\sin\operatorname{gd}\theta = \tanh\theta$
$\tan\operatorname{gd}\theta = \sinh\theta$
$\cot\operatorname{gd}\theta = \operatorname{csch}\theta$
$\sec\operatorname{gd}\theta = \cosh\theta$
$\csc\operatorname{gd}\theta = \coth\theta$

Gudermannian transformations of inverses

$\operatorname{gd}\cosh^{-1} u = \sec^{-1} u$
$\operatorname{gd}\sinh^{-1} u = \tan^{-1} u$
$\operatorname{gd}\tanh^{-1} u = \sin^{-1} u$
$\operatorname{gd}\coth^{-1} u = \csc^{-1} u$
$\operatorname{gd}\operatorname{sech}^{-1} u = \cos^{-1} u$
$\operatorname{gd}\operatorname{csch}^{-1} u = \cot^{-1} u$

$\operatorname{gd}^{-1}\cos^{-1} u = \operatorname{sech}^{-1} u$
$\operatorname{gd}^{-1}\sin^{-1} u = \tanh^{-1} u$
$\operatorname{gd}^{-1}\tan^{-1} u = \sinh^{-1} u$
$\operatorname{gd}^{-1}\cot^{-1} u = \operatorname{csch}^{-1} u$
$\operatorname{gd}^{-1}\sec^{-1} u = \cosh^{-1} u$
$\operatorname{gd}^{-1}\csc^{-1} u = \coth^{-1} u$

Gudermannian special transformations

$\tanh \operatorname{gs}^{-1}\theta = \tan\theta$
$\coth \operatorname{gs}^{-1}\theta = \cot\theta$

$\tan \operatorname{gs}\theta = \tanh\theta$
$\cot \operatorname{gs}\theta = \coth\theta$

Gudermannian special transformations of inverses

$\operatorname{gs}\tanh^{-1} u = \tan^{-1} u$
$\operatorname{gs}\coth^{-1} u = \cot^{-1} u$

$\operatorname{gs}^{-1}\tan^{-1} u = \tanh^{-1} u$
$\operatorname{gs}^{-1}\cot^{-1} u = \coth^{-1} u$

Cogudermannian and coangle transformations

$\cosh \operatorname{cg}\theta = \coth\theta$
$\sinh \operatorname{cg}\theta = \operatorname{csch}\theta$
$\tanh \operatorname{cg}\theta = \operatorname{sech}\theta$
$\coth \operatorname{cg}\theta = \cosh\theta$
$\operatorname{sech} \operatorname{cg}\theta = \tanh\theta$
$\operatorname{csch} \operatorname{cg}\theta = \sinh\theta$

$\cos \operatorname{co}\theta = \sin\theta$
$\sin \operatorname{co}\theta = \cos\theta$
$\tan \operatorname{co}\theta = \cot\theta$
$\cot \operatorname{co}\theta = \tan\theta$
$\sec \operatorname{co}\theta = \csc\theta$
$\csc \operatorname{co}\theta = \sec\theta$

Cogudermannian and coangle transformations of inverses

$\operatorname{cg}\cosh^{-1} u = \coth^{-1} u$
$\operatorname{cg}\sinh^{-1} u = \operatorname{csch}^{-1} u$
$\operatorname{cg}\tanh^{-1} u = \operatorname{sech}^{-1} u$
$\operatorname{cg}\coth^{-1} u = \cosh^{-1} u$
$\operatorname{cg}\operatorname{sech}^{-1} u = \tanh^{-1} u$
$\operatorname{cg}\operatorname{csch}^{-1} u = \sinh^{-1} u$

$\operatorname{co}\cos^{-1} u = \sin^{-1} u$
$\operatorname{co}\sin^{-1} u = \cos^{-1} u$
$\operatorname{co}\tan^{-1} u = \cot^{-1} u$
$\operatorname{co}\cot^{-1} u = \tan^{-1} u$
$\operatorname{co}\sec^{-1} u = \csc^{-1} u$
$\operatorname{co}\csc^{-1} u = \sec^{-1} u$

Properties of inner transformation functions

$\mathrm{gd}(-\theta) = -\mathrm{gd}\,\theta$
$\mathrm{cg}(-\theta) = \mathrm{cg}\,\theta - \frac{\pi i}{2}$
$\mathrm{gs}(-\theta) = -\mathrm{gs}\,\theta$
$(\mathrm{gd}\,\theta)' = \operatorname{sech}\theta$
$(\mathrm{cg}\,\theta)' = \operatorname{csch}\theta$
$(\mathrm{gs}\,\theta)' = \operatorname{sech}2\theta$
$\mathrm{gd}(i\theta) = i\,\mathrm{gd}^{-1}\theta$
$\mathrm{cg}^{-1}\theta = \mathrm{cg}\,\theta$
$\mathrm{gs}(i\theta) = i\,\mathrm{gs}^{-1}\theta$

$$\mathrm{gd}(a+b) = \tan^{-1}\frac{\sinh a}{\cosh b} + \tan^{-1}\frac{\sinh b}{\cosh a}$$
$$\mathrm{gd}(a-b) = \tan^{-1}\frac{\sinh a}{\cosh b} - \tan^{-1}\frac{\sinh b}{\cosh a}$$
$$\mathrm{cg}(a+b) = \tanh^{-1}\frac{\cosh b - \sinh a}{\cosh b + \sinh a} + \tanh^{-1}\frac{\cosh a - \sinh b}{\cosh a + \sinh b}$$
$$\mathrm{cg}(a-b) = \tanh^{-1}\frac{\cosh b - \sinh a}{\cosh b + \sinh a} + \tanh^{-1}\frac{\cosh a + \sinh b}{\cosh a - \sinh b}$$
$$\mathrm{gs}(a+b) = \tfrac{1}{2}\tan^{-1}\frac{2\cosh a\sinh a}{\cosh^2 b + \sinh^2 b} + \tfrac{1}{2}\tan^{-1}\frac{2\cosh b\sinh b}{\cosh^2 a + \sinh^2 a}$$
$$\mathrm{gs}(a-b) = \tfrac{1}{2}\tan^{-1}\frac{2\cosh a\sinh a}{\cosh^2 b + \sinh^2 b} - \tfrac{1}{2}\tan^{-1}\frac{2\cosh b\sinh b}{\cosh^2 a + \sinh^2 a}$$
$\mathrm{gd}\,2\theta = 2\tan^{-1}\tanh\theta$
$\mathrm{cg}\,2\theta = 2\coth^{-1}\exp 2\theta$
$$\mathrm{gs}\,2\theta = \tan^{-1}\frac{2\tanh\theta}{1+\tanh^2\theta}$$
$$\mathrm{gd}\,\frac{\theta}{2} = \csc^{-1}(\operatorname{csch}\theta + \coth\theta)$$
$$\mathrm{cg}\,\frac{\theta}{2} = \cosh^{-1}(\operatorname{csch}\theta + \coth\theta)$$
$$\mathrm{gs}\,\frac{\theta}{2} = \frac{\tan^{-1}\sin\theta}{2}$$
$$\mathrm{gd}\,a + \mathrm{gd}\,b = 2\tan^{-1}\left(\frac{e^a + e^b}{1 - e^{a+b}}\right) - \pi$$
$$\mathrm{gd}\,a - \mathrm{gd}\,b = 2\tan^{-1}\left(\frac{e^{a+b} + 1}{e^b - e^a}\right) - \pi$$
$$\mathrm{cg}\,a + \mathrm{cg}\,b = 2\tanh^{-1}\left(\frac{e^a + e^b}{1 - e^{a+b}}\right)$$
$$\mathrm{cg}\,a - \mathrm{cg}\,b = 2\tanh^{-1}\left(\frac{e^a - e^b}{1 - e^{a+b}}\right)$$

$\mathrm{gd}^{-1}(-\theta) = -\mathrm{gd}^{-1}\theta$
$\mathrm{co}(-\theta) = \pi - \mathrm{co}\,\theta$
$\mathrm{gs}^{-1}(-\theta) = -\mathrm{gs}^{-1}\theta$
$(\mathrm{gd}^{-1}\theta)' = -\sec\theta$
$(\mathrm{co}\,\theta)' = -\theta$
$(\mathrm{gs}^{-1}\theta)' = -\sec 2\theta$
$\mathrm{gd}^{-1}(i\theta) = i\,\mathrm{gd}\,\theta$
$\mathrm{co}^{-1}\theta = \mathrm{co}\,\theta$
$\mathrm{gs}^{-1}(i\theta) = i\,\mathrm{gs}\,\theta$

$$\mathrm{gd}^{-1}(a+b) = \tanh^{-1}\frac{\sin a}{\cos b} + \tanh^{-1}\frac{\sin b}{\cos a}$$
$$\mathrm{gd}^{-1}(a-b) = \tanh^{-1}\frac{\sin a}{\cos b} - \tanh^{-1}\frac{\sin b}{\cos a}$$
$$\mathrm{co}(a+b) = \frac{\mathrm{co}\,2a}{2} + \frac{\mathrm{co}\,2b}{2}$$
$$\mathrm{co}(a-b) = \mathrm{co}\,a + b$$
$$\mathrm{gs}^{-1}(a+b) = \tfrac{1}{2}\tanh^{-1}\frac{2\cos a\sin a}{\cos^2 b + \sin^2 b} + \tfrac{1}{2}\tanh^{-1}\frac{2\cos b\sin b}{\cos^2 a + \sin^2 a}$$
$$\mathrm{gs}^{-1}(a-b) = \tfrac{1}{2}\tanh^{-1}\frac{2\cos a\sin a}{\cos^2 b + \sin^2 b} - \tfrac{1}{2}\tanh^{-1}\frac{2\cos b\sin b}{\cos^2 a + \sin^2 a}$$
$\mathrm{gd}^{-1}\,2\theta = 2\tanh^{-1}\tan\theta$
$\mathrm{co}\,2\theta = 2\,\mathrm{co}\,\theta - \frac{\pi}{2}$
$$\mathrm{gs}^{-1}\,2\theta = \tanh^{-1}\frac{2\tan\theta}{1-\tan^2\theta}$$
$$\mathrm{gd}^{-1}\,\frac{\theta}{2} = \operatorname{csch}^{-1}(\csc\theta + \cot\theta)$$
$$\mathrm{co}\,\frac{\theta}{2} = \tan^{-1}(\csc\theta + \cot\theta)$$
$$\mathrm{gs}^{-1}\,\frac{\theta}{2} = \frac{\tanh^{-1}\sinh\theta}{2}$$
$$\mathrm{gd}^{-1}a + \mathrm{gd}^{-1}b = 2i\tanh^{-1}\left(\frac{e^{ia} + e^{ib}}{1 - e^{ia+ib}}\right) - \pi i$$
$$\mathrm{gd}^{-1}a - \mathrm{gd}^{-1}b = 2i\tanh^{-1}\left(\frac{e^{ia+ib} + 1}{e^{ia} - e^{ib}}\right) - \pi i$$
$$\mathrm{co}\,a + \mathrm{co}\,b = 2\,\mathrm{co}\,\frac{a+b}{2}$$
$$\mathrm{co}\,a - \mathrm{co}\,b = b - a$$

$$\operatorname{gs} a + \operatorname{gs} b = \tan^{-1}\left(\frac{e^{2a}+e^{2b}}{1-e^{2a+2b}}\right) - \frac{\pi}{2}$$

$$\operatorname{gs} a - \operatorname{gs} b = \tan^{-1}\left(\frac{e^{2a+2b}+1}{e^{2b}-e^{2a}}\right) - \frac{\pi}{2}$$

$$\begin{aligned}
\operatorname{gd}\theta &= \sum_{k=0}^{\infty}\frac{E_{2k}\theta^{2k+1}}{(2k+1)!} \\
&= 2\sum_{k=0}^{\infty}\frac{(-1)^k \tanh^{2k+1}\frac{\theta}{2}}{2k+1} \\
&= \frac{\pi}{2} - \sum_{k=0}^{\infty}\frac{(2k)!\operatorname{sech}^{2k+1}\theta}{4^k(n!)^2(2k+1)}
\end{aligned}$$

$$\begin{aligned}
\operatorname{cg}\theta &= \ln\theta + \sum_{k=1}^{\infty}\frac{(-1)^{k+1}2(2^{2k-1}-1)B_k\theta^{2k}}{2k(2k)!} \\
&= \sum_{k=0}^{\infty}\frac{(-1)^k(2k)!\operatorname{csch}^{2k+1}\theta}{4^k(n!)^2(2k+1)}
\end{aligned}$$

$$\operatorname{co}\operatorname{gd}\theta = \operatorname{gd}\operatorname{cg}\theta = \operatorname{cg}\operatorname{co}(-i\theta)$$

$$\begin{aligned}
\operatorname{cg}\theta &= \operatorname{gd}^{-1}\operatorname{co}\operatorname{gd}\theta \\
&= \operatorname{co}\operatorname{gd} i\operatorname{co}\theta = \operatorname{co}(-i\operatorname{gd}^{-1}\operatorname{co}\theta)
\end{aligned}$$

$$\operatorname{gs}^{-1} a + \operatorname{gs}^{-1} b = i\tanh^{-1}\left(\frac{e^{2ia}+e^{2ib}}{1-e^{2ia+2ib}}\right) - \frac{\pi i}{2}$$

$$\operatorname{gs}^{-1} a - \operatorname{gs}^{-1} b = i\tanh^{-1}\left(\frac{e^{2ia+2ib}+1}{e^{2ia}-e^{2ib}}\right) - \frac{\pi i}{2}$$

$$\begin{aligned}
\operatorname{gd}^{-1}\theta &= \sum_{k=0}^{\infty}\frac{(-1)^k E_{2k}\theta^{2k+1}}{(2k+1)!} \\
&= 2\sum_{k=0}^{\infty}\frac{\tan^{2k+1}\frac{\theta}{2}}{2k+1}
\end{aligned}$$

$$\operatorname{gd}^{-1}\operatorname{co}\theta = \operatorname{cg}\operatorname{gd}^{-1}\theta = i\operatorname{co}\operatorname{cg}\theta$$

$$\operatorname{co}\theta = \operatorname{gd}\operatorname{cg}\operatorname{gd}^{-1}\theta$$

Inner transformation conversions

$$\begin{aligned}
\cosh\theta & && = \cos i\theta \\
&= -i\sinh i\operatorname{co} i\theta && = \sin\operatorname{co} i\theta \\
&= \tanh\operatorname{gd}^{-1}\operatorname{co} i\theta && = -i\tan i\operatorname{gd}^{-1}\operatorname{co} i\theta \\
&= \coth\operatorname{cg}\theta && = i\cot i\operatorname{cg}\theta \\
&= \operatorname{sech}\operatorname{gd}^{-1} i\theta && = \sec\operatorname{gd}\theta \\
&= i\operatorname{csch} i\operatorname{co}\operatorname{gd}\theta && = \csc\operatorname{co}\operatorname{gd}\theta \\
\sinh\theta &= -i\cosh i\operatorname{co} i\theta && = -i\cos\operatorname{co} i\theta \\
& && = -i\sin i\theta \\
&= -i\tanh\operatorname{gd}^{-1} i\theta && = \tan\operatorname{gd}\theta \\
&= i\coth i\operatorname{co}\operatorname{gd}\theta && = \cot\operatorname{co}\operatorname{gd}\theta \\
&= -i\operatorname{sech}\operatorname{gd}^{-1}\operatorname{co} i\theta && = -i\sec i\operatorname{gd}^{-1}\operatorname{co} i\theta \\
&= \operatorname{csch}\operatorname{cg}\theta && = i\csc i\operatorname{cg}\theta \\
\tanh\theta &= \cosh i\operatorname{co}\operatorname{gd}\theta && = \cos\operatorname{co}\operatorname{gd}\theta \\
&= -i\sinh\operatorname{gd}^{-1} i\theta && = \sin\operatorname{gd}\theta \\
& && = -i\tan i\theta \\
&= \coth i\operatorname{co} i\theta && = -i\cot\operatorname{co} i\theta \\
&= \operatorname{sech}\operatorname{cg}\theta && = \sec i\operatorname{cg}\theta \\
&= i\operatorname{csch}\operatorname{gd}^{-1}\operatorname{co} i\theta && = \csc i\operatorname{gd}^{-1}\operatorname{co} i\theta \\
\coth\theta &= \cosh\operatorname{cg}\theta && = \cos i\operatorname{cg}\theta \\
&= i\sinh\operatorname{gd}^{-1}\operatorname{co} i\theta && = \sin i\operatorname{gd}^{-1}\operatorname{co} i\theta \\
&= \tanh i\operatorname{co} i\theta && = i\tan\operatorname{co} i\theta \\
& && = i\cot i\theta \\
&= \operatorname{sech} i\operatorname{co}\operatorname{gd}\theta && = \sec\operatorname{co}\operatorname{gd}\theta \\
&= i\operatorname{csch}\operatorname{gd}^{-1} i\theta && = \csc\operatorname{gd}\theta
\end{aligned}$$

$$\begin{aligned}
\cos\theta &= \cosh i\theta && \\
&= -i\sinh i\operatorname{co}\theta && = \sin\operatorname{co}\theta \\
&= \tanh\operatorname{gd}^{-1}\operatorname{co}\theta && = -i\tan i\operatorname{gd}^{-1}\operatorname{co}\theta \\
&= \coth\operatorname{cg} i\theta && = i\cot i\operatorname{cg} i\theta \\
&= \operatorname{sech}\operatorname{gd}^{-1}\theta && = \sec\operatorname{gd} i\theta \\
&= i\operatorname{csch} i\operatorname{co}\operatorname{gd} i\theta && = \csc\operatorname{co}\operatorname{gd} i\theta \\
\sin\theta &= \cosh i\operatorname{co}\theta && = \cos\operatorname{co}\theta \\
&= -i\sinh i\theta && \\
&= \tanh\operatorname{gd}^{-1}\theta && = -i\tan\operatorname{gd}^{-1} i\theta \\
&= \coth i\operatorname{co}\operatorname{gd}^{-1} i\theta && = -i\cot\operatorname{co}\operatorname{gd}^{-1} i\theta \\
&= \operatorname{sech}\operatorname{gd}^{-1}\operatorname{co}\theta && = \sec i\operatorname{gd}^{-1}\operatorname{co}\theta \\
&= -i\operatorname{csch}\operatorname{cg} i\theta && = \csc i\operatorname{cg} i\theta \\
\tan\theta &= -i\cosh i\operatorname{co}\operatorname{gd} i\theta && = -i\cos\operatorname{co}\operatorname{gd}^{-1} i\theta \\
&= \sinh\operatorname{gd}^{-1}\theta && = -i\sin\operatorname{gd} i\theta \\
&= -i\tanh i\theta && \\
&= i\coth i\operatorname{co}\theta && = \cot\operatorname{co}\theta \\
&= -i\operatorname{sech}\operatorname{cg} i\theta && = -i\sec i\operatorname{cg} i\theta \\
&= \operatorname{csch}\operatorname{gd}^{-1}\operatorname{co}\theta && = i\csc i\operatorname{gd}^{-1}\operatorname{co}\theta \\
\cot\theta &= i\cosh\operatorname{cg} i\theta && = i\cos i\operatorname{gd}^{-1} i\theta \\
&= \sinh\operatorname{gd}^{-1}\operatorname{co}\theta && = -i\sin i\operatorname{gd}^{-1}\operatorname{co}\theta \\
&= -i\tanh i\operatorname{co}\theta && = \tan\operatorname{co}\theta \\
&= i\coth i\theta && \\
&= i\operatorname{sech} i\operatorname{co}\operatorname{gd} i\theta && = i\sec\operatorname{co}\operatorname{gd} i\theta \\
&= \operatorname{csch}\operatorname{gd}^{-1}\theta && = i\csc\operatorname{gd} i\theta
\end{aligned}$$

$$
\begin{aligned}
\operatorname{sech}\theta &= \cosh\operatorname{gd}^{-1} i\theta && = \cos\operatorname{gd}\theta\\
&= -i\sinh i\operatorname{co}\operatorname{gd}\theta && = \sin\operatorname{co}\operatorname{gd}\theta\\
&= \tanh\operatorname{cg}\theta && = -i\tan i\operatorname{cg}\theta\\
&= \coth\operatorname{gd}^{-1}\operatorname{co} i\theta && = -i\cot i\operatorname{gd}^{-1}\operatorname{co} i\theta\\
& && = \sec i\theta\\
&= i\operatorname{csch} i\operatorname{co} i\theta && = \csc\operatorname{co} i\theta\\
\operatorname{csch}\theta &= i\cosh\operatorname{gd}^{-1}\operatorname{co} i\theta && = i\cos i\operatorname{gd}^{-1}\operatorname{co} i\theta\\
&= \sinh\operatorname{cg}\theta && = -i\sin i\operatorname{cg}\theta\\
&= -i\tanh i\operatorname{co}\operatorname{gd}\theta && = \tan\operatorname{co}\operatorname{gd}\theta\\
&= i\coth\operatorname{gd}^{-1} i\theta && = \cot\operatorname{gd}\theta\\
&= i\operatorname{sech} i\operatorname{co} i\theta && = i\sec\operatorname{co} i\theta\\
& && = i\csc i\theta
\end{aligned}
$$

$$
\begin{aligned}
\sec\theta &= \cosh\operatorname{gd}^{-1}\theta && = \cos\operatorname{gd} i\theta\\
&= -i\sinh i\operatorname{co}\operatorname{gd} i\theta && = \sin\operatorname{co}\operatorname{gd} i\theta\\
&= \tanh\operatorname{cg} i\theta && = -i\tan i\operatorname{cg} i\theta\\
&= \coth\operatorname{gd}^{-1}\operatorname{co}\theta && = i\cot i\operatorname{gd}^{-1}\operatorname{co}\theta\\
&= \operatorname{sech} i\theta && \\
&= i\operatorname{csch} i\operatorname{co}\theta && = \csc\operatorname{co}\theta\\
\csc\theta &= \cosh\operatorname{gd}^{-1}\operatorname{co}\theta && = \cos i\operatorname{gd}^{-1}\operatorname{co}\theta\\
&= i\sinh\operatorname{cg} i\theta && = \sin i\operatorname{cg} i\theta\\
&= \tanh i\operatorname{co}\operatorname{gd} i\theta && = i\tan\operatorname{co}\operatorname{gd}^{-1} i\theta\\
&= \coth\operatorname{gd}^{-1}\theta && = i\cot\operatorname{gd} i\theta\\
&= \operatorname{sech} i\operatorname{co}\theta && = \sec\operatorname{co}\theta\\
&= i\operatorname{csch} i\theta &&
\end{aligned}
$$

Alternative inner transformation conversions

$$
\begin{aligned}
\cosh\theta &= \sinh\operatorname{gd}^{-1}\operatorname{co}\operatorname{gs}\operatorname{cg}\theta && = \tan\operatorname{co}\operatorname{gs}\operatorname{cg}\theta\\
&= \operatorname{csch}\operatorname{gd}^{-1}\operatorname{gs}\operatorname{cg}\theta && = \cot\operatorname{gs}\operatorname{cg}\theta\\
\sinh\theta &= \cosh\operatorname{cg}\operatorname{gs}^{-1}\operatorname{co}\operatorname{gd}\theta && = \cos\operatorname{co}\operatorname{gd}\operatorname{gs}^{-1}\operatorname{gd}\theta\\
&= \tanh\operatorname{gs}^{-1}\operatorname{gd}\theta && = \sin\operatorname{gd}^{-1}\operatorname{gs}^{-1}\operatorname{gd}\theta\\
&= \coth\operatorname{gs}^{-1}\operatorname{co}\operatorname{gd}\theta && = \sec\operatorname{co}\operatorname{gd}\operatorname{gs}^{-1}\operatorname{co}\operatorname{gd}\theta\\
&= \operatorname{sech}\operatorname{cg}\operatorname{gs}^{-1}\operatorname{gd}\theta && = \csc\operatorname{gd}\operatorname{gs}^{-1}\operatorname{co}\operatorname{gd}\theta\\
\tanh\theta &= \sinh\operatorname{gd}^{-1}\operatorname{gs}\theta && = \tan\operatorname{gs}\theta\\
&= \operatorname{csch}\operatorname{gd}^{-1}\operatorname{co}\operatorname{gs}\theta && = \cot\operatorname{co}\operatorname{gs}\theta\\
\coth\theta &= \sinh\operatorname{gd}^{-1}\operatorname{co}\operatorname{gs}\theta && = \tan\operatorname{co}\operatorname{gs}\theta\\
&= \operatorname{csch}\operatorname{gd}^{-1}\operatorname{gs}\theta && = \cot\operatorname{gs}\theta\\
\operatorname{sech}\theta &= \sinh\operatorname{gd}^{-1}\operatorname{gs}\operatorname{cg}\theta && = \tan\operatorname{gs}\operatorname{cg}\theta\\
&= \operatorname{csch}\operatorname{gd}^{-1}\operatorname{co}\operatorname{gs}\operatorname{cg}\theta && = \cot\operatorname{co}\operatorname{gs}\operatorname{cg}\theta\\
\operatorname{csch}\theta &= \cosh\operatorname{cg}\operatorname{gs}^{-1}\operatorname{gd}\theta && = \cos\operatorname{co}\operatorname{gd}\operatorname{gs}^{-1}\operatorname{co}\operatorname{gd}\theta\\
&= \tanh\operatorname{gs}^{-1}\operatorname{co}\operatorname{gd}\theta && = \sin\operatorname{gd}\operatorname{gs}^{-1}\operatorname{co}\operatorname{gd}\theta\\
&= \coth\operatorname{gs}^{-1}\operatorname{gd}\theta && = \sec\operatorname{co}\operatorname{gd}\operatorname{gs}^{-1}\operatorname{gd}\theta\\
&= \operatorname{sech}\operatorname{cg}\operatorname{gs}^{-1}\operatorname{co}\operatorname{gd}\theta && = \csc\operatorname{gd}\operatorname{gs}^{-1}\operatorname{gd}\theta
\end{aligned}
$$

$$
\begin{aligned}
\cos\theta &= \sinh\operatorname{gd}^{-1}\operatorname{gs}\operatorname{gd}\operatorname{co}\theta && = \tan\operatorname{gs}\operatorname{gd}^{-1}\operatorname{co}\theta\\
&= \operatorname{csch}\operatorname{gd}^{-1}\operatorname{co}\operatorname{gs}\operatorname{gd}^{-1}\operatorname{co}\theta && = \cot\operatorname{co}\operatorname{gs}\operatorname{gd}^{-1}\operatorname{co}\theta\\
\sin\theta &= \sinh\operatorname{gd}^{-1}\operatorname{gs}\operatorname{gd}\theta && = \tan\operatorname{gs}\operatorname{gd}^{-1}\theta\\
&= \operatorname{csch}\operatorname{gd}^{-1}\operatorname{co}\operatorname{gs}\operatorname{gd}^{-1}\theta && = \cot\operatorname{gs}\operatorname{cg}\theta\\
\tan\theta &= \cosh\operatorname{cg}\operatorname{gs}^{-1}\operatorname{co}\theta && = \cos\operatorname{co}\operatorname{gd}\operatorname{gs}^{-1}\theta\\
&= \tanh\operatorname{gs}^{-1}\theta && = \sin\operatorname{gd}\operatorname{gs}^{-1}\theta\\
&= \coth\operatorname{gs}^{-1}\operatorname{co}\theta && = \sec\operatorname{co}\operatorname{gd}\operatorname{gs}^{-1}\operatorname{co}\theta\\
&= \operatorname{sech}\operatorname{cg}\operatorname{gs}^{-1}\theta && = \csc\operatorname{gd}\operatorname{gs}^{-1}\operatorname{co}\theta\\
\cot\theta &= \cosh\operatorname{cg}\operatorname{gs}^{-1}\theta && = \cos\operatorname{co}\operatorname{gd}\operatorname{gs}^{-1}\operatorname{co}\theta\\
&= \tanh\operatorname{gs}^{-1}\operatorname{co}\theta && = \sin\operatorname{gd}\operatorname{gs}^{-1}\operatorname{co}\theta\\
&= \coth\operatorname{gs}^{-1}\theta && = \sec\operatorname{co}\operatorname{gd}\operatorname{gs}^{-1}\theta\\
&= \operatorname{sech}\operatorname{cg}\operatorname{gs}^{-1}\operatorname{co}\theta && = \csc\operatorname{gd}\operatorname{gs}^{-1}\theta\\
\sec\theta &= \sinh\operatorname{gd}^{-1}\operatorname{co}\operatorname{gs}\operatorname{gd}^{-1}\operatorname{co}\theta && = \tan\operatorname{co}\operatorname{gs}\operatorname{gd}^{-1}\operatorname{co}\theta\\
&= \operatorname{csch}\operatorname{gd}^{-1}\operatorname{gs}\operatorname{gd}^{-1}\operatorname{co}\theta && = \cot\operatorname{gs}\operatorname{gd}^{-1}\operatorname{co}\theta\\
\csc\theta &= \sinh\operatorname{gd}^{-1}\operatorname{co}\operatorname{gs}\operatorname{gd}^{-1}\theta && = \tan\operatorname{co}\operatorname{gs}\operatorname{gd}^{-1}\theta\\
&= \operatorname{csch}\operatorname{gd}^{-1}\operatorname{gs}\operatorname{gd}^{-1}\theta && = \cot\operatorname{gs}\operatorname{gd}^{-1}\theta
\end{aligned}
$$

DeMoivre identites

$$
\begin{aligned}
\exp^n\theta &= \exp(n\theta)\\
\frac{1}{\exp^n\theta} &= \exp(-n\theta)\\
\sqrt[n]{\exp\theta} &= \exp\frac{\theta}{n}\\
\exp^{m+in}\theta &= \exp(m\theta)\operatorname{cis}(n\theta)
\end{aligned}
$$

$$
\begin{aligned}
\operatorname{cis}^n\theta &= \operatorname{cis}(n\theta)\\
\frac{1}{\operatorname{cis}^n\theta} &= \operatorname{cis}(-n\theta)\\
\sqrt[n]{\operatorname{cis}\theta} &= \operatorname{cis}\frac{\theta}{n}\\
\operatorname{cis}^{m+in}\theta &= \operatorname{cis}(m\theta)\exp(-n\theta)
\end{aligned}
$$

$$(\exp t\operatorname{cis}\theta)^{m+in} = \exp\left((m-n)t\right)\operatorname{cis}\left((m+n)\theta\right)$$

Point on the complex plane

$$\begin{aligned} a + bi = e^{t+i\theta} &= \exp t \operatorname{cis} \theta \\ &= \cosh t \cos\theta + \sinh t \cos\theta + i \cosh t \sin\theta + i \sinh t \sin\theta \end{aligned}$$

$$\Updownarrow$$

$$\begin{aligned} a &= \exp t \cos\theta = \cosh t \cos\theta + \sinh t \cos\theta \\ b &= \exp t \sin\theta = \cosh t \sin\theta + \sinh t \sin\theta \end{aligned}$$

$$\Updownarrow$$

$$\begin{aligned} t &= \exp^{-1} \sqrt{a^2+b^2} = \exp^{-1} \sqrt{(a+bi)(a-bi)} = \tanh^{-1} \frac{a^2+b^2-1}{a^2+b^2+1} \\ \theta &= \operatorname{cis}^{-1} \frac{a+bi}{\sqrt{a^2+b^2}} = \operatorname{cis}^{-1} \sqrt{\frac{a+bi}{a-bi}} = \tan^{-1} \frac{b}{a} \end{aligned}$$

$$\Updownarrow$$

$$\cosh t = \frac{a^2+b^2+1}{\sqrt{a^2+b^2}} \qquad \cos\theta = \frac{a}{\sqrt{a^2+b^2}}$$

$$\sinh t = \frac{a^2+b^2-1}{\sqrt{a^2+b^2}} \qquad \sin\theta = \frac{b}{\sqrt{a^2+b^2}}$$

Twice-applied formulae

$$\begin{aligned} e^{e^{\theta}} &= \exp\exp\theta = (\cosh\cosh\theta + \sinh\cosh\theta)(\cosh\sinh\theta + \sinh\sinh\theta) \\ e^{e^{i\theta}} &= \exp\operatorname{cis}\theta = (\cosh\cos\theta + \sinh\cos\theta)(\cos\sin\theta + i\sin\sin\theta) \\ e^{ie^{\theta}} &= \operatorname{cis}\exp\theta = (\cos\cosh\theta + \sin\sinh\theta)(\cos\sinh\theta + i\sin\sinh\theta) \\ e^{ie^{i\theta}} &= \operatorname{cis}\operatorname{cis}\theta = (\cos\cos\theta + i\sin\cos\theta)(\cosh\sin\theta - \sinh\sin\theta) \end{aligned}$$