Beautiful Calculus

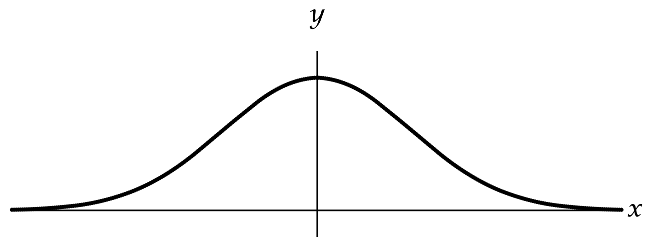
Power series

If $f^{(n)}(x)$ exist and $f(x) = \sum_{k=0}^{\infty} a_k x^k$, then

$$
\begin{aligned}
f^{(0)}(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \ldots \text{ and } f^{(0)}(0) = a_0 \\
f^{(1)}(x) &= a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + \ldots \text{ and } f^{(1)}(0) = a_1 \\
f^{(2)}(x) &= 2a_2 + 2 \cdot 3a_3 x + 3 \cdot 4a_4 x^2 + 4 \cdot 5a_5 x^3 + \ldots \text{ and } f^{(2)}(0) = 2a_2 \\
f^{(3)}(x) &= 2 \cdot 3a_3 + 2 \cdot 3 \cdot 4a_4 x + 3 \cdot 4 \cdot 5a_5 x^2 + \ldots \text{ and } f^{(3)}(0) = 2 \cdot 3a_3 \\
f^{(4)}(x) &= 2 \cdot 3 \cdot 4a_4 + 2 \cdot 3 \cdot 4 \cdot 5a_5 x + \ldots \text{ and } f^{(4)}(0) = 2 \cdot 3 \cdot 4a_4 \\
f^{(5)}(x) &= 2 \cdot 3 \cdot 4 \cdot 5a_5 + \ldots \text{ and } f^{(5)}(0) = 2 \cdot 3 \cdot 4 \cdot 5a_5
\end{aligned}
$$

Hence $a_k = \dfrac{f^{(k)}(0)}{k!}$ and $f(x) = \sum_{k=0}^{\infty} \dfrac{f^{(k)}(0)}{k!} x^k$.

Area under the bell curve



$$
\begin{aligned}
\int_{-\infty}^{+\infty} e^{-x^2}\,dx &= \sqrt{\left(\int_{-\infty}^{+\infty} e^{-x^2}\,dx\right)^2} = \sqrt{\left(\int_{-\infty}^{+\infty} e^{-x^2}\,dx\right)\left(\int_{-\infty}^{+\infty} e^{-x^2}\,dx\right)} \\
&= \sqrt{\left(\int_{-\infty}^{+\infty} e^{-x^2}\,dx\right)\left(\int_{-\infty}^{+\infty} e^{-y^2}\,dy\right)} = \sqrt{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-x^2} e^{-y^2}\,dx\,dy} \\
&= \sqrt{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-(x^2+y^2)}\,dx\,dy} = \sqrt{\int_{0}^{2\pi}\int_{0}^{\infty} e^{-r^2}\,r\,dr\,d\theta} \\
&= \sqrt{\frac{1}{2}\int_{0}^{2\pi}\int_{0}^{\infty} e^{-r^2}\,2r\,dr\,d\theta} = \sqrt{\frac{1}{2}\int_{0}^{2\pi}\int_{0}^{\infty} e^{-z}\,dz\,d\theta} \\
&= \sqrt{\frac{1}{2}\int_{0}^{2\pi} -e^{-z}\Big|_0^{\infty}\,dz\,d\theta} = \sqrt{\frac{1}{2}\int_{0}^{2\pi} (-0-(-1))\,d\theta} \\
&= \sqrt{\frac{1}{2}\int_{0}^{2\pi} d\theta} = \sqrt{\frac{1}{2}\theta\Big|_0^{2\pi}} = \sqrt{\frac{1}{2}2\pi} = \sqrt{\pi}
\end{aligned}
$$

$$
\int_{-u}^{+u} e^{-x^2}\,dx = \sqrt{\pi} - \cfrac{e^{-u^2}}{u + \cfrac{1}{2u + \cfrac{2}{u + \cfrac{3}{2u + \cfrac{4}{u + \cfrac{5}{2u + \cfrac{6}{u + \cfrac{7}{\ddots}}}}}}}}
$$