Beautiful Algebra

Sums and differences of powers

$$\frac{a-b}{a-b} = 1$$

$$\frac{a^2-b^2}{a-b} = a+b$$

$$\frac{a^3-b^3}{a-b} = a^2+ab+b^2$$

$$\frac{a^4-b^4}{a-b} = a^3+a^2b+ab^2+b^3$$

$$\frac{a^5-b^5}{a-b} = a^4+a^3b+a^2b^2+ab^3+b^4$$

$$\frac{a^6-b^6}{a-b} = a^5+a^4b+a^3b^2+a^2b^3+ab^4+b^5$$

$$\frac{a^7-b^7}{a-b} = a^6+a^5b+a^4b^2+a^3b^3+a^2b^4+ab^5+b^6$$

$$\frac{a^8-b^8}{a-b} = a^7+a^6b+a^5b^2+a^4b^3+a^3b^4+a^2b^5+ab^6+b^7$$

$$\frac{a^9-b^9}{a-b} = a^8+a^7b+a^6b^2+a^5b^3+a^4b^4+a^3b^5+a^2b^6+ab^7+b^8$$

$$\frac{a+b}{a+b} = 1$$

$$\frac{a^2-b^2}{a+b} = a-b$$

$$\frac{a^3+b^3}{a+b} = a^2-ab+b^2$$

$$\frac{a^4-b^4}{a+b} = a^3-a^2b+ab^2-b^3$$

$$\frac{a^5+b^5}{a+b} = a^4-a^3b+a^2b^2-ab^3+b^4$$

$$\frac{a^6-b^6}{a+b} = a^5-a^4b+a^3b^2-a^2b^3+ab^4-b^5$$

$$\frac{a^7+b^7}{a+b} = a^6-a^5b+a^4b^2-a^3b^3+a^2b^4-ab^5+b^6$$

$$\frac{a^8 - b^8}{a + b} = a^7 - a^6b + a^5b^2 - a^4b^3 + a^3b^4 - a^2b^5 + ab^6 - b^7$$

$$\frac{a^9 + b^9}{a + b} = a^8 - a^7b + a^6b^2 - a^5b^3 + a^4b^4 - a^3b^5 + a^2b^6 - ab^7 + b^8$$

Pascal's triangle

$$
\begin{aligned}
(a+b)^0 &= 1\\
(a+b)^1 &= a+b\\
(a+b)^2 &= a^2+2ab+b^2\\
(a+b)^3 &= a^3+3a^2b+3ab^2+b^3\\
(a+b)^4 &= a^4+4a^3b+6a^2b^2+4ab^3+b^4\\
(a+b)^5 &= a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5\\
(a+b)^6 &= a^6+6a^5b+15a^4b^2+20a^3b^3+15a^2b^4+6ab^5+b^6\\
(a+b)^7 &= a^7+7a^6b+21a^5b^2+35a^4b^3+35a^3b^4+21a^2b^5+7ab^6+b^7\\
(a+b)^8 &= a^8+8a^7b+28a^6b^2+56a^5b^3+70a^4b^4+56a^3b^5+28a^2b^6+8ab^7+b^8\\
(a+b)^9 &= a^9+9a^8b+36a^7b^2+84a^6b^3+126a^5b^4+126a^4b^5+84a^3b^6+36a^2b^7+9ab^8+b^9
\end{aligned}
$$

The coefficients from the right side form Pascal's triangle:

```
                  1
                1   1
              1   2   1
            1   3   3   1
          1   4   6   4   1
        1   5  10  10   5   1
      1   6  15  20  15   6   1
    1   7  21  35  35  21   7   1
  1   8  28  56  70  56  28   8   1
1   9  36  84 126 126  84  36   9   1
```

For reference, we imagine the rows being numbered 0, 1, 2, 3, etc. from the top. For example, row 4 has the numbers 1, 4, 6, 4, 1.

We also imagine the columns being numbered 0, 1, 2, 3, etc. from the left. For example, column 1 in row 4 is the number 4; column 2 in row 4 is the number 6.

We denote the number in row r, column c by the symbol $\binom{r}{c}$. For example, $\binom{4}{2} = 6$.

1. In row r, the columns are the same whether numbered from the left or the right; that is, $\binom{r}{c} = \binom{r}{r-c}$.

2. Any number is the sum of the two numbers just to the left and the right in the row above it; that is, $\binom{r}{c} = \binom{r-1}{c-1} + \binom{r-1}{c}$.

3. The number $\binom{r}{c}$ is the number of combinations of r objects taken c at a time. This means that if we have a set of r objects, there are $\binom{r}{c}$ different subsets that contain c objects.

The number $1 \times 2 \times 3 \times \cdots \times (n-1) \times n$ is called the *factorial* of n and is denoted $n!$. Since $(n-1)! = \frac{n!}{n}$, it is easy to see that $0! = 1$.

4. For any number in Pascal's triangle, $\binom{r}{c} = \frac{r!}{c!(r-c)!}$.

5. The sum of all the numbers in row r is 2^r. The sum of all the numbers in all rows *above* row r is 2^{r-1}.

Continued fractions

$$\sum_{k=0}^{\infty} c_k = c_0 + \cfrac{c_1}{1 - \cfrac{\frac{c_2}{c_1}}{1 + \frac{c_2}{c_1} - \cfrac{\frac{c_3}{c_2}}{1 + \frac{c_3}{c_2} - \cfrac{\frac{c_4}{c_3}}{1 + \frac{c_4}{c_3} - \cfrac{\frac{c_5}{c_4}}{1 + \frac{c_5}{c_4} - \cfrac{\frac{c_6}{c_5}}{1 + \frac{c_6}{c_5} - \cfrac{\frac{c_7}{c_6}}{1 + \frac{c_7}{c_6} - \cfrac{\frac{c_8}{c_7}}{\ddots}}}}}}}}$$

$$= \cfrac{c_0}{1 - \cfrac{\frac{c_1}{c_0}}{1 + \frac{c_1}{c_0} - \cfrac{\frac{c_2}{c_1}}{1 + \frac{c_2}{c_1} - \cfrac{\frac{c_3}{c_2}}{1 + \frac{c_3}{c_2} - \cfrac{\frac{c_4}{c_3}}{1 + \frac{c_4}{c_3} - \cfrac{\frac{c_5}{c_4}}{1 + \frac{c_5}{c_4} - \cfrac{\frac{c_6}{c_5}}{1 + \frac{c_6}{c_5} - \cfrac{\frac{c_7}{c_6}}{\ddots}}}}}}}}$$

$$= c_0 + \cfrac{c_1}{1 - \cfrac{c_2}{c_1 + c_2 - \cfrac{c_1 c_3}{c_2 + c_3 - \cfrac{c_2 c_4}{c_3 + c_4 - \cfrac{c_3 c_5}{c_4 + c_5 - \cfrac{c_4 c_6}{c_5 + c_6 - \cfrac{c_5 c_7}{c_6 + c_7 - \cfrac{c_6 c_8}{\ddots}}}}}}}}$$

$$\sum_{k=0}^{\infty} c_k = \cfrac{c_0}{1 - \cfrac{c_1}{c_0 + c_1 - \cfrac{c_0 c_2}{c_1 + c_2 - \cfrac{c_1 c_3}{c_2 + c_3 - \cfrac{c_2 c_4}{c_3 + c_4 - \cfrac{c_3 c_5}{c_4 + c_5 - \cfrac{c_4 c_6}{c_5 + c_6 - \cfrac{c_5 c_7}{\ddots}}}}}}}}$$

$$c_0 + \sum_{k=1}^{\infty} \prod_{n=1}^{k} \gamma_n = c_0 + \cfrac{\gamma_1}{1 - \cfrac{\gamma_2}{1 + \gamma_2 - \cfrac{\gamma_3}{1 + \gamma_3 - \cfrac{\gamma_4}{1 + \gamma_4 - \cfrac{\gamma_5}{1 + \gamma_5 - \cfrac{\gamma_6}{1 + \gamma_6 - \cfrac{\gamma_7}{1 + \gamma_7 - \cfrac{\gamma_8}{\ddots}}}}}}}}$$

$$\sum_{k=0}^{\infty} c_k x^k = c_0 + \cfrac{c_1 x}{1 - \cfrac{\frac{c_2}{c_1} x}{1 + \frac{c_2}{c_1} - \cfrac{\frac{c_3}{c_2} x}{1 + \frac{c_3}{c_2} - \cfrac{\frac{c_4}{c_3} x}{1 + \frac{c_4}{c_3} - \cfrac{\frac{c_5}{c_4} x}{1 + \frac{c_5}{c_4} - \cfrac{\frac{c_6}{c_5} x}{1 + \frac{c_6}{c_5} - \cfrac{\frac{c_7}{c_6} x}{1 + \frac{c_7}{c_6} - \cfrac{\frac{c_8}{c_7} x}{\ddots}}}}}}}}$$

$$= \cfrac{c_0}{1 - \cfrac{\frac{c_1}{c_0} x}{1 + \frac{c_1}{c_0} - \cfrac{\frac{c_2}{c_1} x}{1 + \frac{c_2}{c_1} - \cfrac{\frac{c_3}{c_2} x}{1 + \frac{c_3}{c_2} - \cfrac{\frac{c_4}{c_3} x}{1 + \frac{c_4}{c_3} - \cfrac{\frac{c_5}{c_4} x}{1 + \frac{c_5}{c_4} - \cfrac{\frac{c_6}{c_5} x}{1 + \frac{c_6}{c_5} - \cfrac{\frac{c_7}{c_6} x}{\ddots}}}}}}}}$$

$$\prod_{k=0}^{\infty}(1+\gamma_k) = 1+\gamma_0+\cfrac{(1+\gamma_0)\gamma_1}{1-\cfrac{c_2}{1+c_2-\cfrac{c_3}{1+c_3-\cfrac{c_4}{1+c_4-\cfrac{c_5}{1+c_5-\cfrac{c_6}{1+c_6-\cfrac{c_7}{1+c_7-\cfrac{c_8}{\ddots}}}}}}}}$$

$$= 1+\gamma_0+\cfrac{(1+\gamma_0)\gamma_1}{\gamma_1-\cfrac{(1+\gamma_1)\gamma_2}{\gamma_1+(1+\gamma_1)\gamma_2-\cfrac{(1+\gamma_2)\gamma_3}{\gamma_2+(1+\gamma_2)\gamma_3-\cfrac{(1+\gamma_3)\gamma_4}{\gamma_3+(1+\gamma_3)\gamma_4-\cfrac{(1+\gamma_4)\gamma_5}{\ddots}}}}},$$

$$c_k = (1+\gamma_{k-1})\frac{\gamma_k}{\gamma_{k-1}}$$

$$\prod_{k=0}^{\infty} b_k = b_0+\cfrac{b_0(b_1-1)}{b_1-1-\cfrac{b_1(b_2-1)}{b_1b_2-1-\cfrac{(b_1-1)b_2(b_3-1)}{b_2b_3-1-\cfrac{(b_2-1)b_3(b_4-1)}{b_3b_4-1-\cfrac{(b_3-1)b_4(b_5-1)}{b_4b_5-1-\cfrac{(b_4-1)b_5(b_6-1)}{b_5b_6-1-\cfrac{(b_5-1)b_6(b_7-1)}{\ddots}}}}}}}$$

$$(1+x)^k = \cfrac{1}{1-\cfrac{kx}{1+\cfrac{1\cdot 1(1+k)x}{2+\cfrac{1\cdot 2(1-k)x}{3+\cfrac{2\cdot 3(1+k)x}{4+\cfrac{2\cdot 4(1-k)x}{5+\cfrac{3\cdot 5(1+k)x}{6+\cfrac{3\cdot 6(1-k)x}{\ddots}}}}}}}}$$

Solution of the general linear equation

$$
\begin{aligned}
ax + b &= 0, \quad a \neq 0 \\
ax + b - b &= -b \\
ax &= -b \\
\frac{ax}{a} &= \frac{-b}{a}
\end{aligned}
$$

$$
\boxed{x = -\frac{b}{a}}
$$

Solution of the general quadratic equation

$$\begin{aligned}
ax^2 + bx + c &= 0, \quad a \neq 0 \\
ax^2 + bx + c - c &= -c \\
ax^2 + bx &= -c \\
x^2 + \frac{b}{a}x &= -\frac{c}{a} \\
x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 &= \left(\frac{b}{2a}\right)^2 - \frac{c}{a} \\
x^2 + 2\frac{b}{2a}x + \left(\frac{b}{2a}\right)^2 &= \frac{b^2}{4a^2} - \frac{4ac}{4a^2} \\
\left(x + \frac{b}{2a}\right)^2 &= \frac{b^2 - 4ac}{4a^2} \\
\left(x + \frac{b}{2a}\right)^2 &= \left(\pm\sqrt{\frac{b^2 - 4ac}{4a^2}}\right)^2 \\
x + \frac{b}{2a} &= \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} \\
x + \frac{b}{2a} &= \pm\frac{\sqrt{b^2 - 4ac}}{\sqrt{(2a)^2}} \\
x + \frac{b}{2a} - \frac{b}{2a} &= -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{\sqrt{(2a)^2}} \\
x &= -\frac{b \pm \sqrt{b^2 - 4ac}}{2a}
\end{aligned}$$

$$\boxed{x = -\frac{b + \sqrt{b^2 - 4ac}}{2a}}$$

$$\boxed{x = -\frac{b - \sqrt{b^2 - 4ac}}{2a}}$$

Solution of the general cubic equation

$$ax^3 + bx^2 + cx + d = 0, \quad a \neq 0$$

$$\boxed{x = y - \frac{b}{3a}}$$

$$a\left(y - \frac{b}{3a}\right)^3 + b\left(y - \frac{b}{3a}\right)^2 + c\left(y - \frac{b}{3a}\right) + d = 0$$

$$a\left(y^3 - 3y^2\left(\frac{b}{3a}\right) + 3y\left(\frac{b}{3a}\right)^2 - \left(\frac{b}{3a}\right)^3\right) + b\left(y^2 - 2y\left(\frac{b}{3a}\right) + \left(\frac{b}{3a}\right)^2\right) + c\left(y - \frac{b}{3a}\right) + d = 0$$

$$\begin{aligned}
ay^3 - by^2 + \frac{b^2}{3a}y - \frac{b^3}{27a^2} + by^2 - \frac{2b^2}{3a}y + \frac{b^3}{9a^2} + cy - \frac{bc}{3a} + d &= 0\\
ay^3 + \left(\frac{b^2}{3a} - \frac{2b^2}{3a} + c\right)y + \left(-\frac{b^3}{27a^2} + \frac{b^3}{9a^2} - \frac{bc}{3a} + d\right) &= 0\\
ay^3 + \left(\frac{b^2}{3a} - \frac{2b^2}{3a} + \frac{3ac}{3a}\right)y + \left(-\frac{b^3}{27a^2} + \frac{3b^3}{27a^2} - \frac{9abc}{27a^2} + \frac{27a^2d}{27a^2}\right) &= 0\\
ay^3 + \left(\frac{b^2 - 2b^2 + 3ac}{3a}\right)y + \frac{-b^3 + 3b^3 - 9abc + 27a^2d}{27a^2} &= 0\\
ay^3 + \left(\frac{3ac - b^2}{3a}\right)y + \frac{2b^3 - 9abc + 27a^2d}{27a^2} &= 0\\
y^3 + \left(\frac{3ac - b^2}{3a^2}\right)y + \frac{2b^3 - 9abc + 27a^2d}{27a^3} &= 0
\end{aligned}$$

$$\boxed{p = \frac{3ac - b^2}{3a^2}}$$

$$\boxed{q = \frac{2b^3 - 9abc + 27a^2d}{27a^3}}$$

$$y^3 + py + q = 0$$
$$y^3 - 3\left(-\frac{p}{3}\right)y + 2\left(\frac{q}{2}\right) = 0$$
$$y^3 - 3\sqrt[3]{-\left(\frac{p}{3}\right)^3}\,y + \frac{q}{2} + \frac{q}{2} = 0$$
$$y^3 - 3\sqrt[3]{\left(\frac{q}{2}\right)^2 - \left(\frac{q}{2}\right)^2 - \left(\frac{p}{3}\right)^3}\,y + \frac{q}{2} + \frac{q}{2} = 0$$

$$y^3 - 3\sqrt[3]{\left(\frac{q}{2}\right)^2 - \left(\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3\right)}\,y + \frac{q}{2} + \frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3} = 0$$
$$y^3 - 3\sqrt[3]{\left(\frac{q}{2}\right)^2 - \left(\sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}\right)^2}\,y + \frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3} + \frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3} = 0$$

$$y^3 - 3\sqrt[3]{\left(\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}\right)\left(\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}\right)}\,y$$
$$+ \left(\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}\right) + \left(\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}\right) = 0$$
$$y^3 - 3\sqrt[3]{\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}}\sqrt[3]{\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}}\,y$$
$$+ \left(\sqrt[3]{\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}}\right)^3 + \left(\sqrt[3]{\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}}\right)^3 = 0$$

$$\boxed{u = \sqrt[3]{\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}}}$$

$$\boxed{v = \sqrt[3]{\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}}}$$

$$
\begin{aligned}
y^3 - 3uvy + u^3 + v^3 &= 0 \\
y^3 + (-3uv)y + u^3 + v^3 &= 0 \\
y^3 + 0y^2 + (-uv - 2uv)y + u^3 + v^3 &= 0 \\
y^3 + (u - u + v - v)y^2 + (u^2 - u^2 - uv - 2uv + v^2 - v^2)y + u^3 + v^3 &= 0 \\
y^3 + (u+v)y^2 + (-u-v)y^2 + (u^2 - uv + v^2)y + (-u^2 - 2uv - v^2)y & \\
+(u^3 + u^2v - u^2v + uv^2 - uv^2 + v^3) &= 0 \\
y^3 + (u+v)y^2 - (u+v)y^2 + (u^2 - uv + v^2)y - (u^2 + 2uv + v^2)y & \\
+(u^3 - u^2v + uv^2 + u^2v - uv^2 + v^3) &= 0 \\
y^3 + (u+v)y^2 - (u+v)y^2 + (u^2 - uv + v^2)y - (u+v)^2y + (u+v)(u^2 - uv + v^2) &= 0 \\
\big(y + (u+v)\big)\big(y^2 - (u+v)y + (u^2 - uv + v^2)\big) &= 0
\end{aligned}
$$

$$y + (u + v) = 0$$

$$\boxed{y_1 = -u - v}$$

$$y^2 - (u+v)y + (u^2 - uv + v^2) = 0$$

$$
\begin{aligned}
y &= \frac{u + v \pm \sqrt{(u+v)^2 - 4(1)(u^2 - uv + v^2)}}{2(1)} \\
y &= \frac{u + v \pm \sqrt{u^2 + 2uv + v^2 - 4u^2 + 4uv - 4v^2}}{2} \\
y &= \frac{u + v \pm \sqrt{-3u^2 + 6uv - 3v^2}}{2} \\
y &= \frac{u + v \pm \sqrt{-3(u^2 - 2uv + v^2)}}{2} \\
y &= \frac{u + v \pm \sqrt{-3(u-v)^2}}{2} \\
y &= \frac{u + v \pm \sqrt{-3}\sqrt{(u-v)^2}}{2} \\
y &= \frac{u + v \pm i\sqrt{3}(u-v)}{2}
\end{aligned}
$$

$$\boxed{y_2 = \frac{u + v + i\sqrt{3}(u-v)}{2}}$$

$$\boxed{y_3 = \frac{u + v - i\sqrt{3}(u-v)}{2}}$$

Solution of the general quartic equation

$$ax^4 + bx^3 + cx^2 + dx + e = 0, \quad a \neq 0$$

$$\boxed{x = y - \frac{b}{4a}}$$

$$a\left(y - \frac{b}{4a}\right)^4 + b\left(y - \frac{b}{4a}\right)^3 + c\left(y - \frac{b}{4a}\right)^2 + d\left(y - \frac{b}{4a}\right) + e = 0$$

$$a\left(y^4 - 4y^3\left(\frac{b}{4a}\right) + 6y^2\left(\frac{b}{4a}\right)^2 - 4y\left(\frac{b}{4a}\right)^3 + \left(\frac{b}{4a}\right)^4\right) + b\left(y^3 - 3y^2\left(\frac{b}{4a}\right) + 3y\left(\frac{b}{4a}\right)^2 - \left(\frac{b}{4a}\right)^3\right)$$
$$+c\left(y^2 - 2y\left(\frac{b}{4a}\right) + \left(\frac{b}{4a}\right)^2\right) + d\left(y - \frac{b}{4a}\right) + e = 0$$

$$ay^4 - by^3 + \frac{3b^2}{8a}y^2 - \frac{b^3}{16a^2}y + \frac{b^4}{256a^3} + by^3 - \frac{3b^2}{4a}y^2 + \frac{3b^3}{16a^2}y - \frac{b^4}{64a^3}$$
$$+cy^2 - \frac{bc}{2a}y + \frac{b^2c}{16a^2} + dy - \frac{bd}{4a} + e = 0$$

$$ay^4 + \left(\frac{3b^2}{8a} - \frac{3b^2}{4a} + c\right)y^2 - \left(\frac{b^3}{16a^2} + \frac{3b^3}{16a^2} - \frac{bc}{2a} + d\right)y + \left(\frac{b^4}{256a^3} - \frac{b^4}{64a^3} + \frac{b^2c}{16a^2} - \frac{bd}{4a} + e\right) = 0$$

$$ay^4 + \left(\frac{3b^2 - 6b^2 + 8ac}{8a}\right)y^2 + \left(\frac{-b^3 + 3b^3 - 8abc + 16a^2d}{16a^2}\right)y$$
$$+\left(\frac{b^4 - 4b^4 + 16ab^2c - 64a^2bd + 256a^3e}{256a^3}\right) = 0$$

$$ay^4 + \left(\frac{8ac - 3b^2}{8a}\right)y^2 + \left(\frac{b^3 - 4abc + 8a^2d}{8a^2}\right)y + \left(\frac{b^4 - 4b^4 + 16ab^2c - 64a^2bd + 256a^3e}{256a^3}\right) = 0$$

$$y^4 + \left(\frac{8ac - 3b^2}{8a^2}\right)y^2 + \left(\frac{b^3 - 4abc + 8a^2d}{8a^3}\right)y + \left(\frac{b^4 - 4b^4 + 16ab^2c - 64a^2bd + 256a^3e}{256a^4}\right) = 0$$

$$\boxed{r = \frac{8ac - 3b^2}{8a^2}}$$

$$\boxed{s = \frac{b^3 - 4abc + 8a^2d}{8a^3}}$$

$$\boxed{t = \frac{-3b^4 + 16ab^2c - 64a^2bd + 256a^3e}{256a^4}}$$

$$y^4 + ry^2 + sy + t = 0$$

$$\begin{aligned}
t &= \frac{z^2}{4} - \frac{s^2}{4(z-r)} \\
t &= \frac{z^2(z-r)}{4(z-r)} - \frac{s^2}{4(z-r)} \\
t &= \frac{z^2(z-r) - s^2}{4(z-r)} \\
4(z-r)t &= 4(z-r)\frac{z^2(z-r) - s^2}{4(z-r)} \\
4(z-r)t &= z^2(z-r) - s^2 \\
4tz - 4rt &= z^3 - rz^2 - s^2 \\
4tz - 4rt - 4tz + 4rt &= z^3 - rz^2 - s^2 - 4tz + 4rt \\
0 &= z^3 - rz^2 - 4tz - s^2 + 4rt
\end{aligned}$$

$$\begin{aligned}
p &= \frac{3(1)(-4t) - (-r)^2}{3(1)^2} \\
&= \frac{-12t - r^2}{3} \\
q &= \frac{2(-r)^3 - 9(1)(-r)(-4t) + 27(1)^2(-s^2 + 4rt)}{27(1)^3} \\
&= \frac{-2r^3 - 36rt - 27s^2 + 108rt}{27} \\
&= \frac{-2r^3 + 72rt - 27s^2}{27} \\
u &= \sqrt[3]{\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} \\
v &= \sqrt[3]{\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} \\
z_1 &= -u - v - \frac{-r}{3(1)} = -u - v + \frac{r}{3}
\end{aligned}$$

$$\boxed{p = \frac{-12t - r^2}{3}}$$

$$\boxed{q = \frac{-2r^3 + 72rt - 27s^2}{27}}$$

$$\boxed{z = -\sqrt[3]{\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} - \sqrt[3]{\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} + \frac{r}{3}}$$

$$y^4 + ry^2 + sy + \frac{z^2}{4} - \frac{s^2}{4(z-r)} = 0$$

$$y^4 + ry^2 + sy + \frac{z^2}{4} - \frac{s^2}{4(z-r)} + zy^2 - ry^2 - sy + \frac{s^2}{4(z-r)} = +zy^2 - ry^2 - sy + \frac{s^2}{4(z-r)}$$

$$y^4 + zy^2 + \frac{z^2}{4} = +zy^2 - ry^2 - sy + \frac{s^2}{4(z-r)}$$

$$(y^2)^2 + 2\left(\frac{z}{2}\right)y^2 + \left(\frac{z}{2}\right)^2 = +(z-r)y^2 - \frac{sy(z-r)}{z-r} + \frac{s^2(z-r)}{4(z-r)^2}$$

$$\left(y^2 + \frac{z}{2}\right)^2 = (z-r)\left(y^2 - \frac{sy}{z-r} + \frac{s^2}{2^2(z-r)^2}\right)$$

$$\left(y^2 + \frac{z}{2}\right)^2 = (z-r)\left(y^2 - 2\left(\frac{s}{2(z-r)}\right)y + \left(\frac{s}{2(z-r)}\right)^2\right)$$

$$\left(y^2 + \frac{z}{2}\right)^2 = \left(\sqrt{z-r}\right)^2\left(y - \frac{s}{2(z-r)}\right)^2$$

$$y^2 + \frac{z}{2} = \pm\sqrt{z-r}\left(y - \frac{s}{2(z-r)}\right)$$

$$y^2 + \frac{z}{2} = \pm\sqrt{z-r}\,y \mp \sqrt{z-r}\frac{s}{2(z-r)}$$

$$y^2 + \frac{z}{2} = \pm\sqrt{z-r}\,y \mp \frac{s\sqrt{z-r}}{2\left(\sqrt{z-r}\right)^2}$$

$$y^2 + \frac{z}{2} = \pm\sqrt{z-r}\,y \mp \frac{s}{2\sqrt{z-r}}$$

$$y^2 + \frac{z}{2} \mp \sqrt{z-r}\,y \pm \frac{s}{2\sqrt{z-r}} = \pm\sqrt{z-r}\,y \mp \frac{s}{2\sqrt{z-r}} \mp \sqrt{z-r}\,y \pm \frac{s}{2\sqrt{z-r}}$$

$$y^2 \mp \sqrt{z-r}\,y + \frac{z}{2} \pm \frac{s}{2\sqrt{z-r}} = 0$$

$$y^2 - \sqrt{z-r}\,y + \frac{z}{2} + \frac{s}{2\sqrt{z-r}} = 0$$

$$y = \frac{-\left(-\sqrt{z-r}\right) \pm \sqrt{\left(-\sqrt{z-r}\right)^2 - 4(1)\left(\frac{z}{2} + \frac{s}{2\sqrt{z-r}}\right)}}{2(1)}$$

$$y = \frac{\sqrt{z-r} \pm \sqrt{z - r - 2z + \frac{2s}{\sqrt{z-r}}}}{2}$$

$$y = \frac{\sqrt{z-r} \pm \sqrt{-z - r + \frac{2s}{\sqrt{z-r}}}}{2}$$

$$\boxed{y_1 = \frac{\sqrt{z-r} + \sqrt{-z - r + \frac{2s}{\sqrt{z-r}}}}{2}}$$

$$y_2 = \frac{\sqrt{z-r} - \sqrt{-z-r+\frac{2s}{\sqrt{z-r}}}}{2}$$

$$y^2 + \sqrt{z-r}y + \frac{z}{2} - \frac{s}{2\sqrt{z-r}} = 0$$

$$y = \frac{-\left(\sqrt{z-r}\right) \pm \sqrt{\left(\sqrt{z-r}\right)^2 - 4(1)\left(\frac{z}{2} - \frac{s}{2\sqrt{z-r}}\right)}}{2(1)}$$

$$y = \frac{-\sqrt{z-r} \pm \sqrt{z-r-2z-\frac{2s}{\sqrt{z-r}}}}{2}$$

$$y = \frac{-\sqrt{z-r} \pm \sqrt{-z-r-\frac{2s}{\sqrt{z-r}}}}{2}$$

$$y_3 = \frac{-\sqrt{z-r} + \sqrt{-z-r-\frac{2s}{\sqrt{z-r}}}}{2}$$

$$y_4 = \frac{-\sqrt{z-r} - \sqrt{-z-r-\frac{2s}{\sqrt{z-r}}}}{2}$$